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# Attouch-Wets convergence and Kuratowski convergence on compact sets

Paolo Piccione, Rosella Sampalmieri

*Abstract.* Let $X$ be a locally connected, $b$-compact metric space and $E$ a closed subset of $X$. Let $\mathbb{G}$ be the space of all continuous real-valued functions defined on some closed subsets of $E$. We prove the equivalence of the $\tau_{aw}$ and $\tau_K^c$ topologies on $\mathbb{G}$, where $\tau_{aw}$ is the so called *Attouch-Wets* topology, defined in terms of uniform convergence of distance functionals, and $\tau_K^c$ is the topology of Kuratowski convergence on compacta.

*Keywords:* function spaces, Kuratowski convergence, hyperspaces

*Classification:* 54C35, 54C99

## 0. Introduction

The study of graph spaces, and more in general hyperspaces, has been applied to different fields of mathematics, including calculus of variations, differential equations, convex analysis, optimization etc.

In particular, the problem of continuous dependence on the data for the solutions of functional differential equations leads to the problem of defining a suitable notion of convergence in the space of real continuous functions whose domain can vary in a fixed closed set [5], [6].

Several topologies have been introduced on the space $\mathbb{G}$ of such functions and most of them are defined in terms of some notion of convergence of *graphs* or *epigraphs* of functions. We recall here the graph topology of [12], [13], the Hausdorff metric topology [3], [7], the topology of Hausdorff convergence on compact sets [7], the topology of Kuratowski convergence [10], the topology of Kuratowski convergence on compact sets [14] and the Attouch-Wets topology [4], [8].

We prove the equivalence in $\mathbb{G}$ of the Attouch-Wets topology, $\tau_{aw}$, and the topology of Kuratowski convergence on compact sets, $\tau_K^c$, by showing that they define the same converging nets. In this framework, by Kuratowski convergence on compact sets, we mean Kuratowski convergence of the *restriction* of functions on a compact subset of their domain, so that equiboundedness cannot be used. Instead, we will use a result, proved in Lemma 3.4, that relates global and local convergence in the sense of Kuratowski of closed sets locally connected and locally compact spaces. Proposition 3.5 then shows that, in $\mathbb{G}$, $\tau_K^c$-convergence is equivalent to local convergence in the sense of Kuratowski.

A crucial point in the definition of $\tau_K^c$-convergence is a non trivial assumption about the relative position of the compact set and the domain of the limit function.

A detailed discussion about the relative position of two closed sets in a metric space is contained in Section 2. The results proved show the necessity of the restriction to locally connected and locally compact spaces, where any closed set can be covered by a family of compact sets without singular intersection points on their boundary.

The Attouch-Wets topology on $\mathbb{G}$ derives from the notion of convergence of closed convex sets in a normed space, introduced by U. Mosco in [11].

It is a uniform topology, with uniformity having a countable base, therefore it is metrizable. The Attouch-Wets topology is widely used to study approximation and optimization problems.

## 1. Preliminaries

Let us consider two metric spaces $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$.

Let $E \subseteq X$ be a closed, possibly unbounded subset of $X$ and let $\mathcal{C}$ be the family of all closed subsets of $E$.

For $\Omega \in \mathcal{C}$ let $\mathrm{C}(\Omega, Y)$ be the space of all continuous functions from $\Omega$ to $Y$. Let $\mathbb{G}$ be the space:

$$\mathbb{G} = \{f : \Omega \longmapsto Y;\ \Omega \in \mathcal{C},\ f \text{ continuous on } \Omega\}.$$

For $\Omega$ a fixed closed subset of $X$, we also denote by $\mathbb{G}_\Omega$ the space of all continuous functions from $\Omega$ to $Y$.

We think of an element of $\mathbb{G}$ as a pair $(f, \Omega_f)$, together with its graph $\Gamma(f, \Omega_f)$, which is a closed subset of $X \times Y$. If $\Delta$ is a closed subset of $X$, with a little abuse of notation we will write $\Gamma(f, \Omega_f \cap \Delta)$ to mean the graph of the restriction of $f$ to $\Omega_f \cap \Delta$.

We will be concerned with the three metric spaces $(X, \mathrm{d}_X)$, $(Y, \mathrm{d}_Y)$ and $(X \times Y, \mathrm{d}_X \times \mathrm{d}_Y)$, where $\mathrm{d}_X \times \mathrm{d}_Y$ is the product metric on $X \times Y$. We keep the notation uniform for all of them and in the sequel, when not confusing, we will refer to any of them as $(Z, \mathrm{d})$.

Let us denote by $B(z, \delta)$, $B[z, \delta]$, $\delta \in \mathbb{R}^+$, respectively the open and the closed ball of $Z$ with center in $z$ and radius $\delta$; and for $A$ closed subset of $Z$ and $\delta \in \mathbb{R}^+$ the closed *parallel body* $B[A, \delta]$ of $A$ with radius $\delta$ the set

$$B[A, \delta] = \{z \in Z;\ \inf_{a \in A} \mathrm{d}(z, a) \leq \delta\}.$$

If $C$ is a closed set in $Z$, we denote by $d_C$ the *distance functional* associated with $C$ the function on $Z$ defined by

$$Z \ni z \longmapsto d_C(z) = \inf_{c \in C} \mathrm{d}(z, c).$$

For any subset $S \subset Z$, we denote by $S^o$, $\overline{S}$ and $S^c$ respectively the interior, the closure and the complement of $S$ in $Z$.

Let us consider a *directed* set $A$. If $\mathcal{P}_\alpha$ is a logical proposition indexed by the elements of $A$, we will say that $\mathcal{P}_\alpha$ holds *eventually* if there exists $\overline{\alpha} \in A$ such that $\mathcal{P}_\alpha$ holds for every $\alpha \geq \overline{\alpha}$, and we will say that $\mathcal{P}_\alpha$ holds *frequently* if for every $\alpha \in A$ there exists $\beta \geq \alpha$ in $A$ such that $\mathcal{P}_\beta$ holds (see [9]).

Recall that a net (generalized sequence) of closed sets $\{C_\alpha\}_{\alpha \in A}$ in a metric space $X$, is *Kuratowski convergent* to the closed set $C_\infty \subseteq X$ if

$$K \liminf_\alpha C_\alpha = K \limsup_\alpha C_\alpha = C_\infty,$$

where

$$K \liminf_\alpha C_\alpha = \{x \in X \ : \ \text{every neighbourhood of } x \ \text{ meets } C_\alpha \ \text{ eventually}\}$$

and

$$K \limsup_\alpha C_\alpha = \overline{\bigcap_\beta \left( \bigcup_{\gamma \geq \beta} C_\gamma \right)}.$$

The latter is easily seen to be the set of points $x \in X$ that are cluster points for the $C_\alpha$'s frequently.

**Definition 1.1.** A net $\{(f_\alpha, \Omega_\alpha)\}_{\alpha \in A}$ in $\mathbb{G}$ is said to be $\tau_K^c$-convergent to $(f_0, \Omega_0)$ $\in \mathbb{G}$ if the sequence of graphs

$$\Gamma(f_\alpha, \Omega_\alpha \cap \Delta)$$

Kuratowski converges to the graph

$$\Gamma(f_0, \Omega_0 \cap \Delta)$$

for every compact set $\Delta \subset X$ such that

$$(*) \qquad \overline{\Delta^o \cap \Omega_0} = \Delta \cap \Omega_0.$$

An extensive discussion of property $(*)$ is postponed to the next section.

The Attouch-Wets topology on $\mathbb{G}$ is based on the notion of convergence of distance functionals. Namely, a net $C_\alpha$ in $\mathbb{G}$ $\tau_{aw}$ converges to $C_\infty \in \mathbb{G}$ iff the net of functions $d_{C_\alpha}$ converges to $d_{C_\infty}$ uniformly on bounded sets.

Alternatively, the Attouch-Wets topology $\tau_{aw}$ on $\mathbb{G}$ can be described as a uniform topology, with uniform structure generated by the countable family of entourages $V_l$, $l \in \mathbb{N}$,

$$V_l = \{\big(\Gamma(f, \Omega_f), \Gamma(g, \Omega_g)\big) \in \mathbb{G} \times \mathbb{G} \ : \Gamma(f, \Omega_f) \cap B\,[x_0, l] \subseteq B\left[\Gamma(g, \Omega_g), \frac{1}{l}\right] \text{ and }$$
$$\Gamma(g, \Omega_g) \cap B\,[x_0, l] \subseteq B\left[\Gamma(f, \Omega_f), \frac{1}{l}\right] \},$$

where $x_0$ is an arbitrary point in $X$.

**Definition 1.2.** A net $\{(f_\alpha, \Omega_\alpha)\}_{\alpha\in A}$ in $\mathbb{G}$ is said to be $\tau_{aw}$-convergent to $(f_0, \Omega_0) \in \mathbb{G}$ if for every bounded $B \subset X \times Y$ and every $l \in \mathbb{N}$

$$\Gamma(f_\alpha, \Omega_\alpha) \cap B \subseteq B\left[\Gamma(f_0, \Omega_0), \frac{1}{l}\right] \qquad \text{and}$$

$$\Gamma(f_0, \Omega_0) \cap B \subseteq B\left[\Gamma(f_\alpha, \Omega_\alpha), \frac{1}{l}\right]$$

eventually.

## 2. About the relative position of closed sets

In the definition of the $\tau_K^c$ topology, it is requested a certain *non triviality* property of the intersection between the domain of a function and a compact set. We now formalize this property in a more general environment, showing that, under certain conditions, given a closed set there exist enough compact sets that satisfy the property.

Let $(X, \mathrm{d})$ be a metric space and $C$, $L \subset X$ closed subsets of $X$.

**Definition 2.1.** We say that $L$ has property $(*)$ with respect to $C$ if $L \cap C = \overline{L^o \cap C}$.

Notice that in general $L \cap C \supset \overline{L^o \cap C}$. Since $L \cap C = (L^o \cap C) \cup (\partial L \cap C)$, then $L$ has property $(*)$ with respect to $C$ iff the points of intersection between the boundary of $L$ and $C$ are limits of points in $L^0 \cap C$.

Property $(*)$ is evidently preserved through homeomorphisms, but *not* through projections, as the following counter-example shows.

If $C = \Gamma(f_0, \Omega_0) \subset \mathbb{R}^2$ is the graph of the zero function on the interval $[0, 1]$ and $L$ is the square $[1, 2] \times [1, 2]$, then $L$ has the property $(*)$ with respect to $C$ since $L \cap \Gamma(f_0, \Omega_0) = \emptyset$.

Denote by $\pi$ the projection $\mathbb{R}^2 \ni (x, y) \longmapsto x \in \mathbb{R}$, then $\pi(C) = C' = [0, 1]$, $\pi(L) = L' = [1, 2]$ and $L' \cap C' \neq \overline{(L')^o \cap C'}$.

We start with two introductory lemmas.

**Lemma 2.2.** *Let $C \subset X$ be a closed set. Then*

(i) *If $L_1, L_2, \cdots, L_n$ is a finite collection of closed subsets of $X$ satisfying property $(*)$ with respect to $C$, then $L = \bigcup_{i=1}^n L_i$ has property $(*)$ with respect to $C$;*

(ii) *If $C'$ is a closed set such that $L^o \subset C'$ and $L$ has property $(*)$ with respect to $C$, then $L$ has property $(*)$ with respect to $C \cap C'$. Conversely, if $L \subset C'$ and $L$ has property $(*)$ with respect to $C \cap C'$, then $L$ has property $(*)$ with respect to $C$.*

Proof: (i) By induction, it is clearly enough to take $n = 2$. Then

$$(L_1 \cup L_2) \cap C = (L_1 \cap C) \cup (L_2 \cap C) = \left(\overline{L_1^o \cap C}\right) \cup \left(\overline{L_2^o \cap C}\right) = \\ = \overline{\left(L_1^o \cup L_2^o\right) \cap C} \subseteq \overline{(L_1 \cup L_2)^o \cap C},$$

therefore $(L_1 \cup L_2) \cap C = \overline{(L_1 \cup L_2)^o \cap C}$.

(ii) If $L$ has property $(*)$ with respect to $C$ and $L^o \subset C'$ then

$$\overline{L^o \cap (C \cap C')} = \overline{L^o \cap C} \supset L \cap C \supset L \cap \left(C \cap C'\right),$$

so $\overline{L^o \cap (C \cap C')} = L \cap \left(C \cap C'\right)$.
Conversely, if $L$ has property $(*)$ with respect to $C \cap C'$ and $L \subset C'$ then

$$L \cap C = L \cap \left(C \cap C'\right) = \overline{L^o \cap (C \cap C')} = \overline{(L^o \cap C') \cap C} = \overline{L^o \cap C},$$

so $L$ has property $(*)$ with respect to $C$.

At this point, to get stronger results we need to assume more properties of the space $X$. □

In the following lemma, local connectedness plays a crucial role, as the counter-example at the end of the proof shows.

The idea of the proof is a topological version of the *mean value theorem*, which says that if a connected set $V$ intersects both $A$ and $A^c$, then $V$ contains at least one point in $\partial A$.

**Lemma 2.3.** *Suppose $X$ locally connected.*
*If $\{ L_k \}_{k \in \mathbb{N}}$ is a countable collection of closed set and $L$ is the closure of the set $\bigcup_{k=1}^{\infty} L_k$, then every point in $\partial L$ is a limit of points in $\bigcup_{k=1}^{\infty} \partial L_k$.*

*In particular, if all the $L_k$'s have property $(*)$ with respect to a closed set $C$, then also $L$ has property $(*)$ with respect to $C$.*

Proof: Take $x_0 \in \partial L$. If $x_0$ belongs to some $L_k$, then $x_0$ is in $\partial L_k$ and there is nothing to prove. Suppose $x_0 \in \partial L \setminus \bigcup_k L_k$ and choose any connected neighborhood $V$ of $x_0$.

Since $x_0$ is limit of points in $\bigcup_k L_k$, then there exists $m \in \mathbb{N}$ such that $V \cap L_m \neq \emptyset$.

$V$ is a connected set that contains points in $L_m$ and $x_0 \in L_m^c$.

It follows that $V$ has to contain points in $\partial L_m$, otherwise $V$ would be the union of the non empty open sets $V \cap L_m^o$ and $V \cap L_m^c$.

The conclusion comes from the fact that the family of connected neighborhoods of $x_0$ forms a neighborhood system around $x_0$. □

If the local connectedness is not assumed, then the thesis of Lemma 2.3 does not hold, even if $X$ is connected. An easy counter-example comes from a variation of the classical *ladder*, which is the subspace $X$ of the euclidean plane formed by the union of segments of the form $I_n = \{\frac{1}{n}\} \times [0,1]$, $n \in \mathbb{N} \setminus \{0\}$, together with the segments $[0,1] \times \{0\}$, $[0,1] \times \{1\}$ and the square $[-1,0] \times [0,1]$.

If we take the sequence of the $I_n$'s, which are closed, then the closure of their union $L$ contains the segment $I_0 = \{0\} \times [0,1]$. Every point in $I_0$ is in the boundary of $L$, whereas the only boundary points of the $I_n$'s are the extremes.

We come now to the main result of this section, which is about the existence of *enough* compact sets satisfying property $(*)$ with respect to a given closed set. The proof presented is rather technical and the extra assumption of local compactness is made.

**Proposition 2.4.** *Suppose $X$ is locally connected and locally compact and $C$ a closed subset of $X$. Then*

(i) *Every point $x_0 \in X$ has a compact neighborhood $V_{x_0}$ that has property $(*)$ with respect to $C$; the family of all such neighborhoods forms a neighborhood system of $x_0$.*

(ii) *If $L$ is any compact set, then there exists a compact set $L' \supseteq L$ that has property $(*)$ with respect to $C$. If $L$ is connected and the space $X$ is locally connected, then also $L'$ can be found connected.*

(iii) *$C$ is covered by the family of compact sets satisfying property $(*)$ with respect to $C$.*

PROOF: (i) If $x_0 \in C^o$ we can choose $\delta > 0$ such that $V_{x_0} = B[x_0, \delta]$ is compact and contained in $C^o$. Then $\overline{V^o_{x_0} \cap C} = \overline{V^o_{x_0}} \supseteq V_{x_0} = V_{x_0} \cap C$, therefore $V_{x_0}$ has property $(*)$ with respect to $C$. If $x_0 \notin C$ we choose $V_{x_0}$ a compact ball around $x_0$ that has empty intersection with $C$; then $V_{x_0} \cap C = \emptyset = V^o_{x_0} \cap C$ and $V_{x_0}$ has property $(*)$ with respect to $C$.

For the general case, we are going to define *recursively* an increasing sequence $V_n$ of compact neighborhoods of $x_0$ in the following way.

Let $l > 0$ be such that the closed ball $B[x_0, 2l]$ is compact and set $V_0 = B[x_0, l]$, which is a compact neighborhood of $x_0$.

Let now $n > 0$. We describe how to build $V_n$ once $V_{n-1}$ is given.

Consider the set $W_n = (V_{n-1} \cap C) \setminus \left(\overline{V^o_{n-1} \cap C}\right) \subset \partial V_{n-1}$, i.e. $W_n$ is the set of points in $V_{n-1} \cap C$ which are not limit points for $V^o_{n-1} \cap C$.

$W_n$ is relatively compact because it is a subset of $V_{n-1}$, therefore it is covered by a finite number of open balls $U^{(n)}_1, U^{(n)}_2, \dots, U^{(n)}_{k_n}$ of radius $l \cdot 2^{-n}$ centered in its points.

Define $V_n$ as the union of $V_{n-1}$ and the closure of the $U^{(n)}_j$'s.

Evidently $V_n \supseteq V_{n-1}$, moreover $V_n$ is compact, since it is a closed subset of $B[x_0, 2l]$. Namely, if $y \in V_n$, then by the triangle inequality $\mathrm{d}\,(x_0, y) \leq l \cdot \sum_{m=0}^{n} 2^{-m} < 2l$.

Let us look at the points in $\partial V_n$.

If $y \in \partial V_n$ then either $y \in \partial V_{n-1}$, or there exists a point $z \in V_n^o \cap C$ with $\mathrm{d}\,(z, y) \leq l \cdot 2^{-n}$. In fact, if $y \notin \partial V_{n-1}$, then $y$ belongs to one of the balls $U_j^{(n)}$, whose center is in $V_n^o$ and has distance less than or equal to $l \cdot 2^{-n}$ from $y$.

Plus, if $y \in \partial V_n \cap C$, then, by construction, either $y$ is a limit point of $V_{n-1}^0 \cap C$ or $y$ is in one of the $U_j^{(n)}$'s, in which case there exists $z \in V_n^o \cap C$ with $\mathrm{d}(z, y) \leq l \cdot 2^{-n}$.

Define $V_{x_0}$ to be the closure of $\bigcup_{n=0}^{\infty} V_n$, which is a compact neighborhood of $x_0$.

We claim that $V_{x_0}$ has property $(*)$ with respect to $C$. To prove this, we need to show that if $x \in \partial V_{x_0} \cap C$ then there exists a sequence in $V_{x_0}^o \cap C$ converging to $x$.

From Lemma 2.3 we have that every point $x \in \partial V_{x_0}$ is limit of points in $\partial V_n$.

Now, if $x \in \partial V_{x_0} \cap C$ and $(y_m)_{m \in \mathbb{N}} \subseteq \bigcup_{n \in \mathbb{N}} \partial V_n$ is a sequence converging to $x$, then either for some $\overline{n} \in \mathbb{N}$, $y_m \in \partial V_{\overline{n}}$ for $m \geq \overline{n}$, or there exists a subsequence $(y_{m_k})_{k \in \mathbb{N}}$ of $(y_m)_{m \in \mathbb{N}}$ and a subsequence $(V_{n_k})_{k \in \mathbb{N}}$ of $(V_n)_{n \in \mathbb{N}}$ such that $y_{m_k} \in \partial V_{n_k} \setminus V_{n_k - 1}$ for $k \in \mathbb{N}$.

Suppose $y_m \in \partial V_{\overline{n}}$ for all $m \geq \overline{n}$.

Then $x \in \partial V_{\overline{n}} \cap \partial V_{x_0}$, therefore $x \in \partial V_n$ for every $n \geq \overline{n}$. Indeed $x \in V_{\overline{n}} \subseteq V_n$ for all $n \geq \overline{n}$, and $x \notin V_{x_0}^o \supseteq V_n^o$. Thus, either $x$ is a limit point for one of the $V_n^o \cap C \subseteq V_{x_0}^o \cap C$, or, for $n \geq \overline{n}$, there exists a point $z_n \in V_n^o \cap C \subseteq V_{x_0}^o \cap C$ with $\mathrm{d}(z_n, x) \leq l \cdot 2^{-n}$. In both cases, $x$ is a limit point of $V_{x_0}^o \cap C$.

Suppose now that $(y_{m_k})_{k \in \mathbb{N}}$ and $(V_{n_k})_{k \in \mathbb{N}}$ are subsequences, respectively, of $(y_m)_{m \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$, such that $y_{m_k} \in \partial V_{n_k} \setminus V_{n_k - 1}$ for every $k \in \mathbb{N}$.

Then there is a sequence $(z_k)_{k \in \mathbb{N}}$ such that $z_k \in V_{n_k}^o \cap C \subseteq V_{x_0}^o \cap C$ and $\mathrm{d}(z_k, y_{m_k}) \leq l \cdot 2^{-n_k}$ for every $k \in \mathbb{N}$; therefore $\lim_k z_k = \lim_k y_{m_k} = \lim_m y_m = x$ and $x$ is a limit point for $V_{x_0}^o \cap C$.

The second statement of (i) follows easily from the fact that the number $l$ can be arbitrarily small.

(ii) For $x \in L$, take $V_x$ as in (i).

Then the family $\{V_x^o\}_{x \in L}$ covers $L$, and since $L$ is compact, $L$ is covered by a finite number of them, say $V_{x_1}, V_{x_2}, \cdots, V_{x_n}$.

Take $L' = \bigcup_{i=1}^{n} V_{x_i}$. $L'$ is a compact superset of $L$ that has property $(*)$ with respect to $C$ from part (i) of Lemma 2.2.

Moreover, if $X$ is locally connected, the $V_x$'s can be chosen to be connected. If $L$ is connected, since each of the $V_{x_i}$'s has non empty intersection with $L$ and $L \subset \bigcup_{i=1}^{n} V_{x_i}$, then also $\bigcup_{i=1}^{n} V_{x_i}$ is connected.

(iii) Obvious from (i). □

**Remark 2.5.** As it was pointed out by our referee, in Proposition 2.4 (i), local compactness is not essential, since every point has a local basis of closed neighborhoods with property $(*)$ with respect to $C$. It is also true that each bounded and separable subset $L$ of $X$ admits a bounded superset $L'$ which has property $(*)$ with respect to $C$.

## 3. Equivalence of $\tau_{aw}$ and $\tau_K^c$ on $\mathbb{G}$

Let $X$ and $Y$ be two locally connected, locally compact metric spaces.

We will show the equivalence of $\tau_{aw}$ and $\tau_K^c$ in $\mathbb{G}$ by proving that the two topologies have the same converging generalized sequences.

Let us consider a directed set $A$ and a net $\Gamma(f_\alpha, \Omega_\alpha)$ in $\mathbb{G}$ indexed by elements of $A$.

**Lemma 3.1.** *If $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_{aw}$-convergent to $\Gamma(f_0, \Omega_0)$, then for every closed and bounded set $B \subset X \times Y$*

$$K \limsup_\alpha [\Gamma(f_\alpha, \Omega_\alpha) \cap B] \subseteq \Gamma(f_0, \Omega_0) \cap B\,.$$

Proof: It follows easily from the well known fact that the $\tau_{aw}$-convergence implies the Kuratowski convergence. See [7], [8] for reference. □

To revert the inclusion in Lemma 3.1 we need property $(*)$, discussed in the previous section.

**Lemma 3.2.** *If $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_{aw}$-convergent to $\Gamma(f_0, \Omega_0)$, then for every closed set $B$ satisfying property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$ it follows*

$$K \liminf_\alpha [\Gamma(f_\alpha, \Omega_\alpha) \cap B] \supset \Gamma(f_0, \Omega_0) \cap B\,.$$

Proof: Let us consider the closed sets $C_\alpha = \Gamma(f_\alpha, \Omega_\alpha) \cap B$.

Since $K \liminf_\alpha C_\alpha$ is a closed set, we will need to show that every point $\overline{x} \in \Gamma(f_0, \Omega_0) \cap B$ is a cluster point for $K \liminf_\alpha C_\alpha$. Due to the property $(*)$, it will suffice to consider only points $\overline{x} \in \Gamma(f_0, \Omega_0) \cap B^o$.

Let $\overline{x} \in \Gamma(f_0, \Omega_0) \cap B^o$. Since $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_{aw}$-convergent to $\Gamma(f_0, \Omega_0)$, for every $\varepsilon > 0$, the point $\overline{x}$ lies in $B[\Gamma(f_\alpha, \Omega_\alpha), \varepsilon]$ eventually. $B^o$ is open, therefore we can find $\overline{\varepsilon}$ small enough so that that ball $B(\overline{x}, \overline{\varepsilon})$ is entirely contained in $B^o$. It follows that $B(\overline{x}, \overline{\varepsilon})$ has non empty intersection with $\Gamma(f_\alpha, \Omega_\alpha) \cap B$ eventually and $\overline{x} \in K \liminf_\alpha C_\alpha$. □

Observe that one can easily produce a counter-example to Lemma 3.2 if the assumption of property $(*)$ is dropped.

Namely, if $B = [0,1] \times [0,1] \subset \mathbb{R}^2$, $f_n(x) \equiv 1 + \dfrac{1}{n}$ and $f_0(x) \equiv 1$ on $[0,1]$, then $\Gamma(f_n, [0,1])$ is $\tau_{aw}$-convergent to $\Gamma(f_0, [0,1])$, $\Gamma(f_0, \Omega_0) \cap B = \Gamma(f_0, \Omega_0)$, but $\Gamma(f_n, [0,1]) \cap B = \emptyset$ for every $n$.

If we put together the results of Lemma 3.1 and Lemma 3.2 we get the following:

**Corollary 3.3.** *If $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_{aw}$-convergent to $\Gamma(f_0, \Omega_0)$, then for every closed and bounded set $B \subset X \times Y$ satisfying the property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$,* *the sequence $C_\alpha = \Gamma(f_\alpha, \Omega_\alpha) \cap B$ is convergent to $\Gamma(f_0, \Omega_0) \cap B$ in the sense of Kuratowski.*

PROOF: It follows directly from the inclusion $K \liminf_\alpha C_\alpha \subseteq K \limsup_\alpha C_\alpha$. $\square$

The following step is to relate global and local Kuratowski convergence of closed sets.

**Lemma 3.4.** *If $D_\alpha$ is a net of closed sets in $X \times Y$ converging in the sense of Kuratowski to the closed set $D_0$ and $B \subset X \times Y$ is a closed set satisfying property $(*)$ with respect to $D_0$,*
*then sequence $D_\alpha \cap B$ converges in the sense of Kuratowski to $D_0 \cap B$.*

PROOF: If $D_\alpha \cap B = \emptyset$ frequently, then $K \liminf_\alpha (D_\alpha \cap B) = \emptyset$. Moreover, since $D_\alpha$ converges, then $(K \lim_\alpha D_\alpha) \cap B^o = D_0 \cap B^o = \emptyset$, thus $D_0 \cap B = \overline{D_0 \cap B^o} = \emptyset$.

Then $K \limsup_\alpha (D_\alpha \cap B) \subseteq (K \limsup_\alpha D_\alpha) \cap B = D_0 \cap B = \emptyset$ and $K \lim_\alpha (D_\alpha \cap B) = D_0 \cap B$.

If $D_\alpha \cap B \neq \emptyset$ eventually, then we can assume without loss of generality that $D_\alpha \cap B \neq \emptyset$ for all $\alpha$. If that was not the case, we could work with the net $D'_\alpha$ defined by:

$$D'_\alpha = \begin{cases} D_\alpha & \text{if } D_\alpha \cap B \neq \emptyset; \\ B & \text{otherwise,} \end{cases}$$

which has the same asymptotic properties of $D_\alpha$.

We need to show the following facts:

(a) if $y_{\alpha_\beta} \in D_{\alpha_\beta} \cap B$ is a net converging to $y_0$, then $y_0$ is in $D_0 \cap B$;
(b) if $y_0$ is in $D_0 \cap B$ then there exists a net $y_\alpha \in D_\alpha \cap B$ converging to $y_0$.

For (a), consider that since $D_\alpha$ converges in the sense of Kuratowski to $D_0$, then $y_0 \in D_0$. Moreover, since $y_{\alpha_\beta} \in B$ and $B$ is closed, then $y_0 \in B$, so $y_0 \in D_0 \cap B$.

For (b), it will be enough to consider a point $y_0$ in $D_0 \cap B^o$, since this is a *dense* subset of $D_0 \cap B$ by hypothesis and since $K \limsup_\alpha (D_\alpha \cap B)$ is closed.

Let $y_0 \in D_0 \cap B^o$. Since $D_\alpha$ converges to $D_0$ and $y_0 \in D_0$, then there exists a net $y'_\alpha \in D_\alpha$ converging to $y_0$. $B$ is a neighborhood of $y_0$, so that the sequence $y'_\alpha$ is *eventually* in $B$. If we choose arbitrarily a net $z_\alpha \in D_\alpha$ and we define

$$y_\alpha = \begin{cases} y'_\alpha & \text{if } y'_\alpha \in B, \\ z_\alpha & \text{otherwise,} \end{cases}$$

then $y_\alpha \in D_\alpha \cap B$ and $\lim_\alpha y_\alpha = y_0$, so that (b) holds and the lemma is proved. $\square$

Notice that Lemma 3.4 holds in any metric space, without any compactness or connectedness assumption, although the request of property $(*)$ may trivialize the conclusion.

We now use this result to prove that, for a sequence of functions, Kuratowski convergence on compact subsets of their domains is the same as Kuratowski convergence on compact subsets of their graphs.

**Proposition 3.5.** *The sequence* $\Gamma(f_\alpha, \Omega_\alpha)$ *is* $\tau_K^c$*-convergent to* $\Gamma(f_0, \Omega_0)$ *in* $\mathbb{G}$ *if and only if the sequence*

$$C_\alpha = \Gamma(f_\alpha, \Omega_\alpha) \cap B$$

*converges in the sense of Kuratowski to* $\Gamma(f_0, \Omega_0) \cap B$ *for every compact set* $B \subset X \times Y$ *satisfying the property* $(*)$ *with respect to* $\Gamma(f_0, \Omega_0)$.

Proof: Suppose $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_K^c$-convergent to $\Gamma(f_0, \Omega_0)$ and $B \subset X \times Y$ is a compact set satisfying the property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$.

Let $\pi : X \times Y \longmapsto X$ be the projection onto the first factor and $\Delta = \pi(B) \subset X$.

We can assume that $\Delta$ satisfies property $(*)$ with respect to $\Omega_0$. If not, we can change $\Delta$ in the rest of the proof with a suitable compact superset $\Delta' \supset \Delta$ that satisfies $(*)$ with respect to $\Omega_0$.

Since $B$ has property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$ and $B \subset \Delta \times Y$, then from part (ii) of Proposition 2.1 it follows that $B$ has property $(*)$ with respect to $\Gamma(f_0, \Omega_0) \cap (\Delta \times Y) = \Gamma(f_0, \Omega_0 \cap \Delta)$.

The sequence $\Gamma(f_\alpha, \Omega_\alpha \cap \Delta)$ converges in the sense of Kuratowski to $\Gamma(f_0, \Omega_0 \cap \Delta)$, and by Lemma 3.4 also $\Gamma(f_\alpha, \Omega_\alpha \cap \Delta) \cap B = \Gamma(f_\alpha, \Omega_\alpha) \cap B$ converges in the sense of Kuratowski to $\Gamma(f_0, \Omega_0 \cap \Delta) \cap B = \Gamma(f_0, \Omega_0) \cap B$.

Conversely, suppose that for every compact $B \subset X \times Y$ satisfying property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$ the sequence $\Gamma(f_\alpha, \Omega_\alpha) \cap B$ converges in the sense of Kuratowski to $\Gamma(f_0, \Omega_0) \cap B$.

Let $\Delta \subset X$ be a compact set such that $\Delta \cap \Omega_0 = \overline{\Delta^o \cap \Omega_0} \neq \emptyset$.

Consider the compact set $K = f_0(\Delta \cap \Omega_0)$, let $L \subset Y$ be a compact set such that $L^o \supset K$ and define $B = \Delta \times L \subset X \times Y$.

Then $B^o \cap \Gamma(f_0, \Omega_0) = \Gamma(f_0, \Omega_0 \cap \Delta^o)$, therefore $\overline{B^o \cap \Gamma(f_0, \Omega_0)} = \Gamma(f_0, \Omega_0 \cap \Delta) = B \cap \Gamma(f_0, \Omega_0)$ and $B$ has property $(*)$. It follows that $\Gamma(f_\alpha, \Omega_\alpha) \cap B$ converges in the sense of Kuratowski to $\Gamma(f_0, \Omega_0) \cap B = \Gamma(f_0, \Omega_0 \cap \Delta)$.

Since $\Gamma(f_\alpha, \Omega_\alpha \cap \Delta) \supseteq \Gamma(f_\alpha, \Omega_\alpha) \cap B$, then

$$K \liminf_\alpha \Gamma(f_\alpha, \Omega_\alpha \cap \Delta) \supseteq K \lim_\alpha (\Gamma(f_\alpha, \Omega_\alpha) \cap B) = \Gamma(f_0, \Omega_0 \cap \Delta).$$

To prove that $K \limsup_\alpha \Gamma(f_\alpha, \Omega_\alpha \cap \Delta) \subseteq \Gamma(f_0, \Omega_0 \cap \Delta)$ suppose that $(x_{\alpha_\beta}, f_{\alpha_\beta}(x_{\alpha_\beta}))$ is a net in $\in \Gamma\left(f_{\alpha_\beta}, \Omega_{\alpha_\beta} \cap \Delta\right)$ converging to a point $(x_0, y_0) \in X \times Y$.

Then $x_0 \in \Delta$ since $x_{\alpha_\beta} \in \Delta$. Moreover, since the net $(x_{\alpha_\beta}, f_{\alpha_\beta}(x_{\alpha_\beta}))$ is convergent and $X \times Y$ is locally compact, we can find a compact subset $C$ of $X \times Y$ that satisfies property $(*)$ with respect to $\Omega_0$ and such that $(x_{\alpha_\beta}, f_{\alpha_\beta}(x_{\alpha_\beta})) \in C$ eventually.

Since $\Gamma(f_\alpha, \Omega_\alpha) \cap C$ converges in the sense of Kuratowski to $\Gamma(f_0, \Omega_0) \cap C$, it follows that $(x_0, y_0) \in \Gamma(f_0, \Omega_0)$, so $(x_0, y_0) \in \Gamma(f_\alpha, \Omega_\alpha \cap \Delta)$.

This says that

$$K \limsup_\alpha \Gamma(f_\alpha, \Omega_\alpha \cap \Delta) \subseteq \Gamma(f_0, \Omega_0 \cap \Delta),$$

so that $K \lim_\alpha \Gamma(f_\alpha, \Omega_\alpha \cap \Delta) = \Gamma(f_0, \Omega_0)$ and the proposition is proved. $\square$

Putting together the results of 3.3 and 3.5, we have the following

**Corollary 3.6.** *If $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_{aw}$-convergent to $\Gamma(f_0, \Omega_0)$, then $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_K^c$-convergent to $\Gamma(f_0, \Omega_0)$. In other words, $\tau_{aw}$ is finer than $\tau_K^c$.*

We now prove the converse to Corollary 3.6. A little technical problem arises from the asssumption of property $(*)$ in the definition of $\tau_K^c$, where the same assumption is not requested in the definition of $\tau_{aw}$.

We make now a stronger assumption on the metric spaces $X$, $Y$, that from now on will be assumed to be $b$-compact spaces (in such spaces, closed bounded sets are compact).

**Proposition 3.7.** *If $\Gamma(f_\alpha, \Omega_\alpha) \cap B$ is convergent in the sense of Kuratowski to $\Gamma(f_0, \Omega_0) \cap B$ for every compact $B \subset X \times Y$ satisfying property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$, then $\Gamma(f_\alpha, \Omega_\alpha)$ is $\tau_{aw}$-convergent to $\Gamma(f_0, \Omega_0)$.*

Proof: Suppose $\Gamma(f_\alpha, \Omega_\alpha) \cap B$ Kuratowski convergent to $\Gamma(f_0, \Omega_0) \cap B$ for every $B$ compact satisfying property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$ and let $C \subset X \times Y$ be a closed and bounded set, therefore compact.

We need to show that, for every $l \in \mathbb{N}$, $\Gamma(f_0, \Omega_0) \cap C \subset B\left[\Gamma(f_\alpha, \Omega_\alpha), \frac{1}{l}\right]$ and $\Gamma(f_\alpha, \Omega_\alpha) \cap C \subset B\left[\Gamma(f_0, \Omega_0), \frac{1}{l}\right]$ eventually.

We can assume that $C$ satisfies property $(*)$ with respect to $\Gamma(f_0, \Omega_0)$, otherwise we could consider, instead of $C$, a suitable compact superset $C' \supset C$ that does.

Since $\Gamma(f_\alpha, \Omega_\alpha) \cap C \subset K \liminf_\alpha [\Gamma(f_\alpha, \Omega_\alpha) \cap C]$, then for every $p \in \Gamma(f_\alpha, \Omega_\alpha) \cap C$ and $l \in \mathbb{N}$ the intersection $B\left[p, \frac{1}{l}\right] \cap [\Gamma(f_\alpha, \Omega_\alpha) \cap C]$ is not empty. It follows that $p \in B\left[\Gamma(f_\alpha, \Omega_\alpha) \cap C, \frac{1}{l}\right] \subset B\left[\Gamma(f_\alpha, \Omega_\alpha), \frac{1}{l}\right]$, thus $\Gamma(f_0, \Omega_0) \cap C \subset B\left[\Gamma(f_\alpha, \Omega_\alpha), \frac{1}{l}\right]$.

For the other inclusion, suppose by absurd that $\Gamma(f_\alpha, \Omega_\alpha) \cap C \not\subset B\left[\Gamma(f_0, \Omega_0), \frac{1}{l}\right]$ infinitely often. Then we could find an integer $l$ and a net $p_{\alpha_\beta} \in \Gamma\left[f_{\alpha_\beta}, \Omega_{\alpha_\beta}\right] \cap C$ such that $d_{\Gamma(f_0,\Omega_0)}(p_{\alpha_\beta}) \geq \frac{1}{l}$.

The sequence $p_{\alpha_\beta}$ has at least one cluster point $p$, since it is contained in the compact set $C$. The point $p$ is therefore in $K \limsup_\alpha [\Gamma(f_\alpha, \Omega_\alpha) \cap C]$, but not in $\Gamma(f_0, \Omega_0) \cap C$, since $d_{\Gamma(f_0,\Omega_0)}(p) \geq \frac{1}{l}$, which is an absurd, since $\Gamma(f_0, \Omega_0) \cap C \supset K \limsup_\alpha [\Gamma(f_\alpha, \Omega_\alpha) \cap C]$.

This shows that $\Gamma(f_\alpha, \Omega_\alpha) \cap C \subset B\left[\Gamma(f_0, \Omega_0), \frac{1}{l}\right]$ and the proposition is proved. □

**Corollary 3.8.** *Let $X$ and $Y$ be locally connected $b$-compact metric spaces. Then the topologies $\tau_{aw}$ and $\tau_K^c$ are equivalent on $\mathbb{G}$.*

## 4. Conclusions

In [7], the authors have introduced on $\mathbb{G}$ the topology $\tau$ of Hausdorff convergence on compact sets when $Y = \mathbb{R}^n$ and $X$ is a closed connected subset of $\mathbb{R}$. It has been proved that the net $\Gamma(f_\alpha, \Omega_\alpha)$ in $\mathbb{G}$ is $\tau$ convergent if and only if it is $\tau_K^c$ convergent and equibounded (see [14] for reference). This fact, after the result of

Corollary 3.8, allows us to insert the $\tau_K^c$ topology in the framework of the most usual topologies of graph spaces ([7]).

In particular, from Theorem 2 and Corollaries 2 and 3 in [8], we have:

**Theorem 4.1.** *If $\Omega$ is a locally connected subspace of the b-compact metric space $X$ and $Y$ is a locally connected b-compact metric space, then $\tau_K^c$ is equivalent to the compact-open topology on $\mathbb{G}_\Omega$.*

**Theorem 4.2.** *Under the hypothesis of Theorem* 4.1, *if $X$ is compact, then $\tau_K^c$ is equivalent to the Hausdorff metric topology on $\mathbb{G}_\Omega$.*

We refer the reader to [1], [2], [7], [8], [14], for a more general view about the relationships among the different topologies in function spaces.

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