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Strong subdifferentiability of norms and geometry of Banach spaces

G. Godefroy, V. Montesinos, V. Zizler

Dedicated to the memory of Josef Kolomý

Abstract. The strong subdifferentiability of norms (i.e. one-sided differentiability uniform in directions) is studied in connection with some structural properties of Banach spaces. It is shown that every separable Banach space with nonseparable dual admits a norm that is nowhere strongly subdifferentiable except at the origin. On the other hand, every Banach space with a strongly subdifferentiable norm is Asplund.

Keywords: strong subdifferentiability of norms, Asplund spaces, renormings, weak compact generating

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1. Introduction

Let X be a real Banach space and $x \in X$. A norm $\|\cdot\|$ on X is said to be *strongly subdifferentiable* (SSD) *at* x if the one-sided limit

$$\lim_{t\to 0^+} \frac{1}{t}\left(\|x+th\| - \|x\|\right)$$

exists uniformly on h in the unit sphere S_X of X ([8], [4], [5], [1], [6]). If a norm $\|\cdot\|$ on X is strongly subdifferentiable at every point of X we say that $\|\cdot\|$ is an SSD norm on X. Similarly, if $\|\cdot\|$ is Gâteaux or Fréchet differentiable on $X\backslash\{0\}$ we say that $\|\cdot\|$ is a *Gâteaux* or *Fréchet differentiable norm* on X. Note that a norm $\|\cdot\|$ on a Banach space X is Fréchet differentiable at $x \in X$ if and only if $\|\cdot\|$ is both Gâteaux differentiable and SSD at x. From the monotonicity of the differentiation quotient for convex functions and from the classical Dini theorem it follows that every norm on a finite dimensional space is SSD at every point of the space. It can be shown (see e.g. [4]) that the canonical sup-norm $\|\cdot\|_\infty$ on ℓ_∞ is SSD at $x = \{x_n\}$ if and only if $\|x\| \notin \{|x_n|; |x_n| \neq \|x\|_\infty\}'$, where $\{\cdot\}'$ denotes the derived set of $\{\cdot\}$. Therefore the sup-norm on c_0 is SSD at every point of c_0. On the other hand, we show in Proposition 5 below that

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the set S of all SSD points of the sup-norm of the space ℓ_∞ is not a G_δ set in ℓ_∞ (although S contains a dense G_δ set of points of Fréchet differentiability (see e.g. [2, p. 26, 29])). It is known that various classes of Banach spaces can be characterized by various differentiability properties of norms (see e.g. [2]). The class of separable Banach spaces with nonseparable dual is strictly larger than the class of separable Banach spaces with an equivalent Gâteaux differentiable norm which is nowhere Fréchet differentiable (see e.g. [2, p. 101, 104]). In the main result in this paper, Theorem 1, the separable spaces with nonseparable dual are characterized as those separable Banach spaces that admit norms which are nowhere strongly subdifferentiable except at the origin. In the third part of this paper some applications of the strong subdifferentiability of norms in the geometry of Banach spaces are shown. For example, it is proved that a Banach space with an SSD norm is Asplund.

We use a standard notation in this note. We denote by B_X the closed unit ball of X and by S_X its unit sphere. For $x \in S_X$ we let

$$J(x) = \{x^* \in S_{X^*};\ x^*(x) = 1\}.$$

We will use the following analogue of the Šmulyan lemma: The norm $\|\cdot\|$ is SSD at $x \in S_X$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{dist}\big(x^*, J(x)\big) < \varepsilon$ whenever $x^* \in B_{X^*}$ is such that $x^*(x) > 1-\delta$ (see [5]). We recall that a subspace Y of the dual Banach space $(X, \|\cdot\|)^*$ is called 1-norming if for every $x \in X$ we have

$$\|x\| = \sup\,\{x^*(x);\ x^* \in B_{X^*} \cap Y\}.$$

The *ball topology* b_X on a Banach space X is the weakest topology on X in which the balls $B(x,\delta) = \{y \in X;\ \|y-x\| \le \delta\}$, $x \in X$ and $\delta > 0$, are closed. We refer to [7] for more on this subject. Furthermore, recall that a biorthogonal system $\{x_\alpha, x_\alpha^*\}_{\alpha\in\Lambda} \subset X \times X^*$ is a shrinking Markuševič basis for a Banach space X if $\overline{\operatorname{span}}\{x_\alpha\} = X$ and $\overline{\operatorname{span}}^{\|\cdot\|^*}\{x_\alpha^*\} = X^*$. Note that the set $\{x_\alpha\} \cup \{0\}$ is then a weak compact set that generates X. We refer to [14] and [17] for more on Markuševič bases and to [2] for the definition and basic properties of the projectional resolutions of the identity on nonseparable Banach spaces as well as for some unexplained notions and results used in this paper.

2. SSD characterization of separable Asplund spaces

The following theorem is a main result in this paper.

Theorem 1. *Any separable Banach space with nonseparable dual admits an equivalent norm that is nowhere strongly subdifferentiable except at the origin.*

Proof: If a separable Banach space X contains an isomorphic copy of ℓ_1, then X admits an equivalent Gâteaux differentiable norm that is nowhere Fréchet differentiable ([3], see also [2, p. 101, 104]). Such a norm is nowhere SSD except at the origin. We will therefore assume that X is a separable Banach space that does not contain an isomorphic copy of ℓ_1 and such that X^* is nonseparable. Without loss of generality we assume that the original norm $|\cdot|$ of X is Gâteaux differentiable (see e.g. [2, Theorem II.3.1]).

There is a subset $K^* := \{x^*_\varepsilon, \varepsilon \in \{0,1\}^{\mathbb{N}}\}$ of S_{X^*} that is w^*-homeomorphic to the Cantor set in $[0,1]$ and a subset $K^{**} := \{x^{**}_\varepsilon;\ \varepsilon \in \{0,1\}^{\mathbb{N}}\}$ of $\frac{5}{4}B_{X^{**}}$ such that $x^{**}_\varepsilon(x^*_\gamma) = \delta_{\varepsilon\gamma}$ (the Kronecker delta) for $\varepsilon,\ \gamma \in \{0,1\}^{\mathbb{N}}$ ([18]). Denote by $L := K^* \cup (-K^*)$, $\quad C := \overline{\operatorname{conv}}^{w^*}(L)$ and $B^* := B_{X^*} + C$, where B_{X^*} is the dual unit ball for the norm $|\cdot|$ on X. Clearly, B^* is the dual unit ball for an equivalent norm $\|\cdot\|$ on X. Choose an arbitrary element $x^* \in K^*$ and let $H := \{x^*\}_\perp \subset X$. Using the fact that X^* contains a countable separating set for X we let $\{H_n\}$ be a sequence of closed hyperplanes in X with $H_1 = H$ and such that $\cap H_n = \{0\}$. Furthermore, we let T_1 be the identity map on X and for $n = 2, 3, \ldots$ we let T_n be an isomorphism of X onto itself such that $T_n(H_n) = H$. We assume without loss of generality that the norms of T_n are uniformly bounded. We let $\{M_n\}$ be the collection of all elementary clopen (i.e. closed and open) subsets of K^* or $-K^*$.

For $n \in \mathbb{N}$ we let $\|\cdot\|_n$ be a seminorm defined on X by

$$\|x\|_n = \sup\,\{f_n(t^*)\langle x, t^*\rangle :\ t^* \in M_n\}, \quad x \in X,$$

where $f_n \in C(M_n)$ is such that $0 \le f_n$ and $\sup_{M_n} f_n = 1$ is attained at a unique point $a^*_n \in M_n$.

Finally, we define an equivalent norm $|||\cdot|||$ on X by

$$|||x||| = \sum_{n=1}^{\infty} 2^{-n}\|T_n x\| + \sum_{n,m=1}^{\infty} 2^{-(n+m)}\|T_n x\|_m.$$

We will show that $|||\cdot|||$ is SSD only at the origin. In doing so, we will use the following fact that directly follows from the monotonicity of the differentiation quotient for convex functions: If φ_1 and φ_2 are two convex functions and φ_1 is not SSD at x, then $\varphi_1 + \varphi_2$ is not SSD at x.

Hence to finish the proof we only need to show that given $x \in X\backslash\{0\}$, at least one of the component norms or seminorms in the definition of $|||\cdot|||$ is not SSD at x.

Assume first that $x \notin H$ and let the face of C determined by x be denoted by $F(C,x)$, i.e.

$$F(C,x) = \{x^* \in C;\ x^*(x) = \sup\,\{y^*(x);\ y^* \in C\}\}.$$

x is not identically zero on K, hence $\sup\,\{y^*(x);\ y^* \in C\} > 0$. Note that $\operatorname{Ext} F(C,x) \subset \operatorname{Ext} C \subset L$ by Milman's theorem, where $\operatorname{Ext} C$ denotes the set of all extreme points of C. Thus, $L \cap F(C,x)$ is a nonempty w^*-closed subset of L.

We consider two cases: first assume that x is constant on no set M_n that intersects $L \cap F(C,x)$. Then, using the fact that x is a positive constant on

$F(C,x)$, we can find a sequence $\{x_n^*\}\subset L\backslash F(C,x)$ that is w^*-convergent to an $x^*\in L\cap F(C,x)$. We may and do assume that the sequence $\{x_n^*\}$ and x^* are in K^*. We show that

$$|\cdot|^* - \text{ dist }\big(x_n^*, F(C,x)\big) \geq \Big(\|\cdot\|^* - \text{ dist }\big(x_n^*, F(C,x)\big)\Big) \geq \tfrac{2}{5}\,.$$

To see this, denoting the corresponding biorthogonal functionals to $\{x_n^*\}$ by $\{x_n^{**}\}$ we have

$$x_n^{**}(x_n^* - e^*) \geq 1$$

for every $e^*\in$ Ext $F(C,x)$.

Since X does not contain an isomorphic copy of ℓ_1, every $x^{**}\in X^{**}$ is the w^*-limit of a sequence in X ([15]) hence it satisfies the barycentric calculus and thus, by the Lebesgue Dominated Convergence Theorem, we have

$$x_n^{**}(x_n^* - x^*) \geq 1$$

for every $x^*\in F(C,x)$.

Hence for every $x^*\in F(C,x)$ and every $n\in\mathbb{N}$ we have

$$\|x_n^* - x^*\|^* \geq 2^{-1}|x_n^* - x^*|^* \geq 2^{-1}\cdot \tfrac{4}{5}\, x_n^{**}(x_n^* - x^*) \geq \tfrac{2}{5}\,.$$

Note that the Gâteaux differentiability of the norm $|\cdot|$ implies that $F(B_{X^*},x)$ is a singleton, say u_0^*. Therefore

$$F(B^*,x) = F(B_{X^*},x) + F(C,x) = u_0^* + F(C,x).$$

Hence for all $n\in\mathbb{N}$ we have

$$\begin{aligned}
&\|\cdot\|^* - \text{dist }\big(u_0^* + x_n^*, F(B^*,x)\big)\\
&\qquad = \|\cdot\|^* - \text{dist }\big(u_0^* + x_n^*, u_0^* + F(C,x)\big)\\
&\qquad = \|\cdot\|^* - \text{dist }\big(x_n^*, F(C,x)\big) \geq \tfrac{2}{5}\,.
\end{aligned}$$

We have $\|u_0^* + x_n^*\|^* \leq 1$ for all $n\in\mathbb{N}$ and

$$\begin{aligned}
\lim\,(u_0^* + x_n^*)(x) &= (u_0^* + x^*)(x)\\
&= \sup\ \{x^*(x);\ x^*\in B_{X^*}\} + \sup\,\{x^*(x);\ x^*\in C\}\\
&= \|x\| = 1.
\end{aligned}$$

By the variant of Šmulyan's lemma stated in Introduction, we have that $\|\cdot\|$ is not SSD at x.

Second, assume $x\notin H$ and there is an $n\in\mathbb{N}$ such that M_n intersects $L\cap F(C,x)$ and x is a (positive) constant on M_n. We shall show that $\|\cdot\|_n$ is not SSD at x. We assume without loss of generality that $\|x\|_n = 1$. We observe that the seminorm

$\|\cdot\|_n$ is Gâteaux differentiable at x (see e.g. [2, p. 5]). Therefore to show that $\|\cdot\|_n$ is not SSD at x is equivalent to show that $\|\cdot\|_n$ is not Fréchet differentiable at x. To see this we let t_k^* and s_k^* be points of M_n different from a_n^* such that (as elements in $[0,1]$) $t_k^* < s_k^*$ and

$$\min\{f_n(t_k^*), f_n(s_k^*)\} > 1 - k^{-2}$$

for all $k \in \mathbb{N}$.

Furthermore, for $k \in \mathbb{N}$, we let $x_k^{**} \in X^{**}$ be defined by

$$x_k^{**} = x_{t_k^*}^{**} - x_{s_k^*}^{**},$$

where $x_{t_k^*}^{**}$ and $x_{s_k^*}^{**}$ denote now the corresponding biorthogonal functionals in the set K^{**}. From Goldstine's theorem, for $k \in \mathbb{N}$ we find $y_k \in X$ such that $\|y_k\| \le \|x_k^{**}\|$ and

$$\langle y_k, t_k^* - s_k^*\rangle > \langle x_k^{**}, t_k^* - s_k^*\rangle - \frac{1}{2} \ \Big(= \frac{3}{2}\Big).$$

We thus have

$$\Big\|x + \frac{1}{k}y_k\Big\|_n \ge f_n(t_k^*)\Big\langle x + \frac{1}{k}y_k, t_k^*\Big\rangle > \Big(1 - \frac{1}{k^2}\Big)\Big(1 + \frac{1}{k}\langle y_k, t_k^*\rangle\Big),$$

$$\Big\|x - \frac{1}{k}y_k\Big\|_n \ge f_n(s_k^*)\Big\langle x - \frac{1}{k}y_k, s_k^*\Big\rangle > \Big(1 - \frac{1}{k^2}\Big)\Big(1 - \frac{1}{k}\langle y_k, s_k^*\rangle\Big).$$

So

$$\begin{aligned} k\Big(\Big\|x + \frac{1}{k}y_k\Big\|_n + \Big\|x - \frac{1}{k}y_k\Big\|_n - 2\|x\|_n\Big) > \\ > k\Big(1 - \frac{1}{k^2}\Big)\Big(2 + \frac{1}{k}\Big\langle t_k^* - s_k^*, y_k\Big\rangle - 2k > \\ > \Big(k - \frac{1}{k}\Big)\Big(2 + \frac{3}{2k}\Big) - 2k = \frac{3}{2} - \frac{2}{k} - \frac{3}{2k^2} \ge \frac{1}{8} \end{aligned}$$

for all $k = 2, 3, \ldots$. Hence $\|\cdot\|_n$ is not Fréchet differentiable at x and this finishes the proof if $x \notin H$.

In general, given $x \in X\backslash\{0\}$, there is $n \in \mathbb{N}$ such that $x \notin H_n$ and thus $T_n x \notin H$. By the argument above, one of the norms or seminorms $\|\cdot\|$ or $\|\cdot\|_m$ is not SSD at $T_n x$. Therefore the norm $|||\cdot|||$ is not SSD at x. This concludes the proof of Theorem 1. □

Note that the statement in Theorem 1 actually provides for a characterization of separable spaces with nonseparable dual (cf. e.g. [2, Theorem I.5.7]).

3. Applications of the strong subdifferentiability

The following result shows a few applications of the strong subdifferentiability in the geometry of Banach spaces.

Theorem 2. *Let the norm $\|\cdot\|$ of a Banach space X be strongly subdifferentiable. Then*

(i) *The space X is an Asplund space* (*see* [5], [1], [6]).

(ii) *If X has $\|\cdot\|$-norm 1 projectional resolution of the identity $\{P_\alpha\}_{\alpha\le\mu}$, then $\{P^*_\alpha\}_{\alpha\le\mu}$ is a projectional resolution of the identity on X^*. If moreover X as well as every complemented subspace Y of X with dens$Y$ less than dens$X$ has a $\|\cdot\|$-norm 1 projectional resolution of the identity, then X has a shrinking Markuševič basis; in particular, this applies to X with dens$X=\aleph_1$.*

(iii) *Any weakly closed bounded subset of X is an intersection of finite unions of balls in X* ([6]).

In the proof of Theorem 2 as well as in many other results on SSD norms, the following result plays a crucial rôle.

Lemma 3 ([6]). *Let the norm of a Banach space X be strongly subdifferentiable. Then X^* contains no proper norm closed 1-norming subspace.*

Proof: We give a proof that is slightly different from that in [6]. Assume first that X is separable and $\|\cdot\|$ is an SSD norm on X. Suppose that $N\subset X^*$ is a norm closed 1-norming proper subspace of X^* with respect to $\|\cdot\|$. Let $\{x^*_n\}$ be a w^*-dense sequence in $B_{X^*}\cap N$ (which is then w^*-dense in B_{X^*} as N is 1-norming). Since the norm $\|\cdot\|$ is SSD, from the variant of Šmulyan's lemma discussed in Introduction it follows that

$$\operatorname{dist}\big(\{x^*_n\},J(x)\big)=0$$

for every $x\in S_X$. Let

$$B=S_{X^*}\cap\big(\cup_n B(x^*_n,\tfrac12)\big),$$

where $B(x^*_n,\,\frac12)$ denotes the ball of radius 1/2 centered at x^*_n.

Then B is a subset of S_{X^*} on which any $x\in X$ attains its norm. Since N is a proper subspace of X^*, we can choose $x^{**}\in N^\perp\subset X^{**}$ with $\|x^{**}\|^{**}=1$ and $x^*_0\in B_{X^*}$ such that $x^{**}(x^*_0)>4/5$. From Goldstine's theorem we find a sequence $\{x_k\}\subset B_X$ such that for all $n\in\{0,1,2,\dots\}$ we have

$$\lim_k x^*_n(x_k)=x^{**}(x^*_n).$$

We may and do assume that $x^*_0(x_k)>4/5$ for all $k\in\mathbb{N}$. By Simons' inequality ([16], see e.g. [2, Lemma I.3.7]) we have

$$\inf\big\{\|y\|;\ y\in\ \operatorname{conv}\{x_k\}\big\}\le\sup_{x^*\in B}\{\limsup x^*(x_k)\}.$$

Since $x^{**}(x_n^*) = 0$ for all $n \in \mathbb{N}$ and $\|x_k\| \le 1$ for all $k \in \mathbb{N}$, from the definition of the set B it follows that

$$\limsup_k |x^*(x_k)| \le \tfrac{1}{2}$$

for all $x^* \in B$. Hence from Simons' inequality it follows that there is $y \in \operatorname{conv}\{x_k\}$ such that $\|y\| < 3/4$. However, since $x_0^*(x_k) > 4/5$ for every $k \in \mathbb{N}$, we have $x_0^*(y) \ge 4/5$. Thus $\|y\| \ge x_0^*(y) \ge 4/5$. This contradiction completes the proof if X is separable. □

The proof for the general case follows from the first part of this proof and from the following variant of the Mazur separable exhaustion argument.

Lemma 4. *Let X be a Banach space. Assume that X^* contains a proper norm-closed 1-norming subspace. Then there is a separable subspace Y of X and a proper norm-closed 1-norming subspace of Y^*.*

PROOF: Let N be a proper norm-closed 1-norming subspace of X^* and let $x_0^* \in X^*$ be such that $\operatorname{dist}(x_0^*, N) = 1$. For a subspace $G \subset N$ we say that a subspace $F \subset X$ is *G-good* if for every $g^* \in \operatorname{span}\{G, x_0^*\}$ we have

$$\|g^*\| = \sup\{g^*(f);\ f \in S_X \cap F\}.$$

Given a subspace $F \subset X$ we say that a subspace $H \subset X^*$ is *F-norming* if for every $f \in F$ we have

$$\|f\| = \sup\{h^*(f);\ h^* \in H \cap S_{X^*}\}.$$

We construct a sequence $\{G_n\}$ of separable subspaces of N and a sequence $\{F_n\}$ of separable subspaces of X as follows: Choose $g_1^* \in N$ arbitrarily, put $G_1 = \operatorname{span}\{g_1^*\}$ and let F_1 be a separable subspace of X that is G_1-good. If $G_1, G_2, \dots, G_n$ and $F_1, F_2, \dots, F_n$ have been chosen, pick a separable subspace $G_{n+1} \subset N$ such that $G_{n+1} \supset G_n$ and that G_{n+1} is F_n-norming and then choose a separable subspace F_{n+1} of X such that $F_{n+1} \supset F_n$ and F_{n+1} is G_{n+1}-good.

Put $Y := \overline{\cup F_n} \subset X$ and $G := \overline{\cup G_n}^{\|\cdot\|^*} \subset N$. Denote by Re the restriction map of X^* to Y^*. Then Re is an isometry of a subspace G of N onto a norm closed subspace $N_1 :=$ Re G of Y^* which is clearly 1-norming in Y^*. If $g^* \in G_n$, then

$$\begin{aligned}\|\mathrm{Re}\, x_0^* - \mathrm{Re}\, g^*\|_{Y^*} &= \sup\{|x_0^*(y) - g^*(y)|;\ y \in S_X \cap Y\}\\ &\ge \sup\{|x_0^*(f) - g^*(f)|;\ f \in S_X \cap F_n\}\\ &= \|x_0^* - g^*\|_{X^*}^* \ge 1.\end{aligned}$$

From this it follows that $\operatorname{dist}(N_1, \mathrm{Re}\, x_0^*) \ge 1$ in Y^*. Hence N_1 is a proper subspace of Y^*. This completes the proof of Lemma 4 and thus finishes the proof of Lemma 3. □

Proof of Theorem 2: (i) Let Y be a separable subspace of X and let $\{x_n^*\}$ be a w^*-dense sequence in B_{Y^*}. Then the subspace $N = \overline{\text{span}}^{\|\cdot\|^*}\{x_n^*\}$ is 1-norming and the restriction of the norm $\|\cdot\|$ to Y is SSD. Therefore, by Lemma 3 we have $N = Y^*$. This shows that Y^* is separable. Therefore X is an Asplund space.

(ii) The only thing we need to prove is that for every limit ordinal $\gamma \le \mu$ we have $\overline{\bigcup_{\alpha<\gamma} P_\alpha^*(X^*)}^{\|\cdot\|^*} = P_\gamma^*(X^*)$ (see e.g. [12] for details).

To show the latter we first notice that since $P_\gamma^* x^* = \lim_{\alpha\to\gamma} P_\alpha^* x^*$ in the w^*-topology, for all $x^* \in X^*$, and that $\|P_\alpha^*\| = 1$ for all $\alpha \le \mu$, the restriction $\overline{\bigcup_{\alpha<\gamma} P_\alpha^* X^*}^{\|\cdot\|^*}_{\restriction P_\gamma X}$ is a 1-norming subspace in $(P_\gamma X)^*$. By Lemma 3 we thus have $\overline{\bigcup_{\alpha<\gamma} P_\alpha^* X^*}^{\|\cdot\|^*}_{\restriction P_\gamma X} = (P_\gamma X)^*$. Let $x^* \in P_\gamma^* X^*$. There exists $y^* \in \overline{\bigcup_{\alpha<\gamma} P_\alpha^* X^*}^{\|\cdot\|^*}$ such that $y^*_{\restriction P_\gamma X} = x^*_{\restriction P_\gamma X}$. Then $x^* = P_\gamma^* x^* = P_\gamma^* y^* = y^*$. We obtain that $\overline{\bigcup_{\alpha<\gamma} P_\alpha^*(X^*)}^{\|\cdot\|^*} = P_\gamma^*(X^*)$.

(iii) We only need to show that the ball topology b_X coincides on bounded sets with the weak topology ([7]). For this, it is enough to show that if Y is a separable subspace of X and $x^* \in X^*$, then the restriction $x^*_{\restriction Y}$ is b_Y-continuous on B_Y ([7, Proposition 2.5]). For this, in turn is sufficient to prove that $x^*_{\restriction Y}$ belongs to all 1-norming norm closed subspaces of Y^* ([7, Theorem 2.4]). This is clearly so by Lemma 3. We refer to [6] for a selfcontained proof of (iii). □

The following proposition shows that the set of all SSD points of a norm on an Asplund space may in general be a non-G_δ set in the space.

Proposition 5. *The set of all* SSD *points of any given norm on any Banach space X is an $F_{\sigma\delta}$ set in the space X. The set of all* SSD *points of the sup-norm on ℓ_∞ is not a G_δ set in ℓ_∞.*

Proof: Let $\|\cdot\|$ be the norm of a Banach space X. For $k,\ n \in \mathbb{N}$ we denote by $F_{k,n}$ the set of all $x \in X$ such that

$$\sup_{h\in S_X} \{|\tfrac{1}{t_1}\,(\|x+t_1h\| - \|x\|) - \tfrac{1}{t_2}\,(\|x+t_2h\| - \|x\|)|\} \le \tfrac{1}{k}$$

whenever $0 < t_1 \le t_2 < 1/n$.

For $k,\ n \in \mathbb{N}$, the sets $F_{k,n}$ are closed and for the set S of all SSD points of the norm $\|\cdot\|$ we have

$$S = \bigcap_k \big(\bigcup_n F_{k,n}\big).$$

Let X be ℓ_∞ endowed with the sup-norm $|\cdot|_\infty$. Define the map $\Psi:\ \{0,1\}^{\mathbb{N}} \to X$ by

$$\Psi(\varepsilon) = \sum_{i=1}^{\infty} 2^{-i}\varepsilon(i)\chi_{[i,\infty[},\ \varepsilon \in \{0,1\}^{\mathbb{N}},$$

where $\chi_{[i,\infty[}$ denotes the characteristic function of $[i,\infty[$ in $\mathbb{N}$. It follows that $\Psi(\varepsilon)$ is an SSD point for the norm $|\cdot|_\infty$ if and only if $\varepsilon \in Q$, where

$$Q = \{\varepsilon \in \{0,1\}^{\mathbb{N}};\ \mathrm{card}\,\{i;\ \varepsilon(i) = 1\} < \infty\}$$

(cf. e.g. [4]).

Since Q is a countable dense subset of a perfect compact metric space $\{0,1\}^{\mathbb{N}}$, Q is not a G_δ set in $\{0,1\}^{\mathbb{N}}$. Moreover, denoting by S the set of SSD points, we have $\Psi^{-1}(S) = Q$ and Ψ is a continuous map of $\{0,1\}^{\mathbb{N}}$ into X. Therefore S is not a G_δ set in X. $\square$

We finish this paper with a list of a few open problems in this area.

Problems 6.

(i) Does every Asplund space admit an equivalent SSD norm? In particular, does the space $C(K)$ admit an equivalent SSD norm whenever K is a tree space?
Recall that R. Haydon found an example of a tree such that the space $C(K)$ does not admit any equivalent Gâteaux differentiable norm. He also found many other connections between properties of trees and renormings ([9], see also [2, Chapter VII]).

(ii) Assume that a Banach space X admits an equivalent Gâteaux differentiable norm and that X admits also an equivalent SSD norm. Does X admit an equivalent Fréchet differentiable norm?
Recall that S. Troyanski showed that X admits an equivalent locally uniformly rotund norm provided that X admits both strictly convex norms and norms with the Kadec-Klee property ([19], see also [2, Corollary IV.3.6]).

(iii) Assume that the norm of a separable Banach space X has the property that its restriction to every infinite dimensional closed subspace $Y \subset X$ has a point of Fréchet differentiability on Y. Is then X^* necessarily separable? The statement might be considered as a sort of linearization in the argument in the Baire Great Theorem (cf. the Jayne-Rogers Baire 1 selectors [11], see also [2, Chapter I.4]).

(iv) Assume that X is separable and X^* is nonseparable. Does there exist on X an equivalent norm that is "uniformly non SSD" in the sense analogous to the roughness of norms?
Note that the answer to this question is positive if, for instance, X contains an isomorphic copy of ℓ_1 ([3], see e.g. [2, p. 101 and 104]).

(v) Assume that X is (in general nonseparable) non Asplund space. Does X admit an equivalent norm that is nowhere SSD except at the origin?

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