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# A full descriptive definition of the BV-integral

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*Abstract.* We present a Cauchy test for the almost derivability of additive functions of bounded BV sets. The test yields a full descriptive definition of a coordinate free Riemann type integral.

*Keywords:* Perimeter, partition, gage, absolute continuity

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The BV-integral, introduced in [5, Definition 5.1] under the name "variational integral", is a coordinate free generalization of the Lebesgue integral defined on all bounded Caccioppoli sets. Unlike the Lebesgue integral, it integrates partial derivatives of differentiable functions and provides the unrestricted Gauss-Green theorem. The purpose of this note is to present a complete characterization of those additive functions of bounded Caccioppoli sets that are indefinite BV-integrals (Theorem 3.9).

## 1. Preliminaries

The ambient space of this paper is $\mathbf{R}^m$, where $\mathbf{R}$ is the set of all real numbers and $m$ is a fixed positive integer. The metric in $\mathbf{R}^m$ is induced by the maximum norm, and $U(x,\varepsilon)$ denotes the open ball about $x \in \mathbf{R}^m$ of radius $\varepsilon > 0$. For a set $E \subset \mathbf{R}^m$, we denote by $\operatorname{cl} E$, $\operatorname{int} E$, $\partial E$, $d(E)$, and $|E|$ the closure, interior, boundary, diameter, and Lebesgue measure of $E$, respectively. The words "measure", "measurable", and "negligible" as well as the expressions "almost all" and "almost everywhere" always refer to the Lebesgue measure in $\mathbf{R}^m$. The *symmetric difference* of sets $A$ and $B$ is the set $A \triangle B = (A - B) \cup (B - A)$.

Let $E \subset \mathbf{R}^m$. We say an $x \in E$ is, respectively, a *density* or *dispersion* point of $E$ according to whether

$$\liminf_{\varepsilon \to 0+} \frac{|U(x,\varepsilon) \cap E|}{(2\varepsilon)^m} = 1 \qquad \text{or} \qquad \limsup_{\varepsilon \to 0+} \frac{|U(x,\varepsilon) \cap E|}{(2\varepsilon)^m} = 0\,.$$

The set of all nondispersion points of $E$ is called the *essential closure* of $E$, denoted by $\operatorname{cl}^* E$, and the set of all density points

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- of $E$ is called the *essential interior* of $E$, denoted by $\mathrm{int}^* E$. The *essential boundary* of $E$ is the set $\partial^* E = \mathrm{cl}^* E - \mathrm{int}^* E$. Clearly, $\mathrm{int}\, E \subset \mathrm{int}^* E \subset \mathrm{cl}^* E \subset \mathrm{cl}\, E$, and so $\partial^* E \subset \partial E$. If $E$ is measurable, the sets $E \triangle \mathrm{cl}^* E$, $E \triangle \mathrm{int}^* E$, and $\partial^* E$ are negligible [7, Chapter IV, Theorem 6.1]. The set $E$ is called *essentially* or *doubly closed* whenever $E = \mathrm{cl}^* E$ or $E = \mathrm{cl}^* E = \mathrm{cl}\, E$, respectively.

The $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ is denoted by $\mathcal{H}$, and a set $T \subset \mathbf{R}^m$ of $\sigma$-finite measure $\mathcal{H}$ is called *thin*. Each thin set is negligible but not vice versa. The *perimeter* (in De Giorgi's sense) of a set $A \subset \mathbf{R}^m$ is the number $\|A\| = \mathcal{H}(\partial^* A)$. A bounded set $A \subset \mathbf{R}^m$ with $\|A\| < +\infty$ is called a *Caccioppoli* or BV *set* (BV for *bounded variation* — cf. [2, Section 5.11, Theorem 1]). Each BV set is measurable [6, Corollary 13.2.5], and the family $BV$ of all BV sets is an algebra in $\mathbf{R}^m$. For $E \subset \mathbf{R}^m$, we let $BV_E = \{A \in BV : A \subset E\}$.

The *regularity* of a BV set $A$ is the number

$$r(A) = \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The isoperimetric inequality ([2, Section 5.6, Theorem 2,(i)]) shows that a sequence $\{A_n\}$ of BV sets is regular in the sense of [7, Chapter IV, Section 2] whenever $\inf_n r(A_n) > 0$.

Let $A$ be a BV set. The set of all $x \in \mathrm{int}^* A$ such that

$$\lim_{\varepsilon \to 0+} \frac{\mathcal{H}[U(x,\varepsilon) \cap \partial^* A]}{(2\varepsilon)^{m-1}} = 0$$

is called the *critical interior* of $A$, denoted by $\mathrm{int}^c A$. According to [8, Section 4], we have $\mathcal{H}(\mathrm{int}^* A - \mathrm{int}^c A) = 0$; in particular, the set $\mathrm{cl}^* A - \mathrm{int}^c A$ is negligible. The next lemma, proved in [4, Lemma 1.2], gives an important property of the critical interior.

**Lemma 1.1.** *Let $A \in BV$ and $x \in \mathrm{int}^c A$. Suppose $\{B_n\}$ is a sequence of BV sets such that $x \in \mathrm{cl}^* B_n$ and $r(B_n) > \varepsilon > 0$ for $n = 1, 2, \ldots$ . Then $x \in \mathrm{cl}^*(A \cap B_n)$ for $n = 1, 2, \ldots$, and if $\lim d(B_n) = 0$, then $r(A \cap B_n) > \varepsilon^{m+1}$ for all sufficiently large integers $n \geq 1$.*

## 2. The integral

Unless specified otherwise, by a function we always mean a real-valued function. An *additive function* in a BV set $A$ is a function $F$ defined on $BV_A$ such that

$$F(B \cup C) = F(B) + F(C)$$

for each pair of disjoint sets $B, C \in BV_A$. Such an $F$ is called *continuous* if given $\varepsilon > 0$, we can find an $\eta > 0$ so that $|F(B)| < \varepsilon$ for each $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

A *partition* is a collection $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are disjoint BV sets and $x_i \in \mathrm{cl}^* A_i$ for $i = 1, \dots, p$. When $E \subset \mathbf{R}^m$ and $\bigcup_{i=1}^p A_i \subset E$ or $\{x_1, \dots, x_p\} \subset E$, we say $P$ is a partition *in* $E$ or a partition *anchored in* $E$, respectively. Clearly, each partition in $E$ is anchored in $\mathrm{cl}^* E$. Given an $\varepsilon > 0$ and a nonnegative function $\delta$ on $E$, the partition $P$ is called

(i) $\varepsilon$-*regular* if $r(A_i) > \varepsilon$ for $i = 1, \dots, p$;

(ii) $\delta$-*fine* if $P$ is anchored in $E$ and $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$.

It is convenient to denote $\bigcup_{i=1}^p A_i$ by $\bigcup P$.

A *gage* in a set $E \subset \mathbf{R}^m$ is a nonnegative function $\delta$ defined on $\mathrm{cl}^* E$ whose *null set* $N_\delta = \{x \in \mathrm{cl}^* E : \delta(x) = 0\}$ is thin.

**Definition 2.1.** Let $A$ be a BV set and let $f$ be a function defined on $\mathrm{cl}^* A$. We say $f$ is BV-*integrable* in $A$ if there is a continuous additive function $F$ in $A$ satisfying the following condition: given $\varepsilon > 0$, we can find a gage $\delta$ in $A$ so that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each $\varepsilon$-regular $\delta$-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in $A$.

In view of [5, Propositions 7.7 and 7.8], BV-integrability coincides with variational integrability introduced in [5, Definition 5.1]. The function $F$, uniquely determined by $f$, is called the *indefinite* BV-*integral* of $f$ in $A$. If $f$ is Lebesgue integrable in $A$, it is BV-integrable in $A$, and the two indefinite integrals coincide [5, Proposition 5.8].

Let $A$ be a BV set and let $f$ be a function defined on $A \cup \mathrm{cl}^* A$. Since the BV-integral extends the Lebesgue integral, neither the BV-integrability nor the BV-integral of $f$ is affected by the values $f$ takes on negligible subsets of $\mathrm{cl}^* A$; in particular, they are not affected by the values of $f$ on $A \triangle \mathrm{cl}^* A$. Thus, in the obvious way, we can and will define the BV-integrability and BV-integral for the extended real-valued functions defined almost everywhere in $A$.

For additional properties of the BV-integral, including the Gauss-Green theorem, we refer to [5].

## 3. BV-ACG$_*$ functions

Let $E \subset \mathbf{R}^m$, let $F$ be a function defined on $BV_E$, and let $x \in \mathrm{cl}^* E$. Set

$$\underline{F}(x) = \inf_{\alpha > 0} \sup_{\delta > 0} \left[ \inf \frac{F(B)}{|B|} \right]$$

where the infimum in the brackets is taken over all sets $B \in BV_E$ with $x \in \mathrm{cl}^* B$, $d(B) < \delta$, and $r(B) > \alpha$; furthermore, let $\overline{F}(x) = -\underline{(-F)}(x)$. The extended real-valued functions $x \mapsto \underline{F}(x)$ and $x \mapsto \overline{F}(x)$ defined on $\mathrm{cl}^* E$ are denoted by $\underline{F}$ and $\overline{F}$, respectively. When $\underline{F}(x) = \overline{F}(x)$ is a real number, we denote it by $F'(x)$

and say $F$ is BV-*derivable* at $x$. By $F'$ we denote the function $x \mapsto F'(x)$ defined on the set of all $x \in \mathrm{cl}^* E$ at which $F$ is BV-derivable.

The following lemma, proved in [4, Lemma 2,3], facilitates applications of Vitali's covering theorem.

**Lemma 3.1.** *Let $F$ be an additive continuous function in a BV set $A$. If $x \in \mathrm{int}^c A$, then*

$$\underline{F}(x) = \inf_{\alpha>0} \sup_{\delta>0} \left[ \inf \frac{F(A \cap C)}{|A \cap C|} \right]$$

*where the infimum in the brackets is taken over all doubly closed BV sets $C$ with $x \in C$, $d(C) < \delta$, and $r(C) > \alpha$. In particular, $\underline{F}(x) \le \overline{F}(x)$ for each $x \in \mathrm{int}^c A$.*

**Definition 3.2.** Let $F$ be an additive continuous function in a BV set $A$. We say $F$ is BV-AC$_*$ on a set $E \subset \mathrm{cl}^* A$ if given $\varepsilon > 0$, there is an $\eta > 0$ and a gage $\delta$ in $A$ such that

$$\left| F\left(\bigcup P\right) - F\left(\bigcup Q\right) \right| < \varepsilon$$

for all $\varepsilon$-regular $\delta$-fine partitions $P$ and $Q$ in $A$ anchored in $E$ for which

$$\left| \left(\bigcup P\right) \triangle \left(\bigcup Q\right) \right| < \eta\,.$$

If $\mathrm{cl}^* A = \bigcup_{n=1}^\infty E_n$ and $F$ is BV-AC$_*$ on each $E_n$, we say that $F$ is BV-ACG$_*$.

In a more general setting BV-ACG$_*$ functions were introduced in [1]. Our results below parallel some of those obtained in [3], where a concept closely related to the BV-ACG$_*$ functions has been applied to Perron type integrals.

Following [4, Definition 2.5], we say an additive continuous function $F$ in a BV set $A$ is BV-*absolutely continuous* if given a negligible set $N \subset \mathrm{cl}^* A$ and an $\varepsilon > 0$, there is a gage $\delta$ in $A$ such that $|F(\bigcup P)| < \varepsilon$ for each $\varepsilon$-regular $\delta$-fine partition $P$ in $A$ anchored in $N$.

**Proposition 3.3.** *Each BV-ACG$_*$ function in a BV set $A$ is BV-absolutely continuous.*

Proof: Let $F$ be a BV-ACG$_*$ function in $A$. With no loss of generality, we may assume that there are disjoint sets $E_1, E_2, \ldots$ such that $\mathrm{cl}^* A = \bigcup_{n=1}^\infty E_n$ and $F$ is BV-AC$_*$ on each $E_n$. Choose a negligible set $N \subset \mathrm{cl}^* A$ and $\varepsilon > 0$, and fix an integer $n \ge 1$. Letting $Q = \emptyset$ in Definition 3.2, find a gage $\delta_n$ in $A$ and $\eta_n > 0$ so that $|F(\bigcup P)| < \varepsilon 2^{-n}$ for each $\varepsilon$-regular $\delta_n$-fine partition $P$ in $A$ anchored in $E_n$ with $|\bigcup P| < \eta_n$. There is an open set $U_n$ containing $N \cap E_n$ with $|U_n| < \eta_n$. Making $\delta_n$ smaller, we may assume that each $\delta_n$-fine partition $P$ anchored in $N \cap E_n$ is a partition in $U_n$; in particular, $|\bigcup P| < \eta_n$. Define a gage $\delta$ in $A$ by setting $\delta(x) = \delta_n(x)$ if $x \in E_n$ for $n = 1, 2, \ldots$ . If $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is an $\varepsilon$-regular $\delta$-fine partition in $E$ anchored in $N$, we obtain

$$\left| F\left(\bigcup P\right) \right| \le \sum_{n=1}^{\infty} \left| \sum_{x_i \in E_n} F(A_i) \right| < \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon \,,$$

which establishes the BV-absolute continuity of $F$. □

The following characterizations are called, respectively, the *partial* and *full descriptive definitions* of the BV-integral:

1. Among all continuous additive functions that are derivable almost everywhere in a BV set $A$, characterize those which are indefinite BV-integrals in $A$.
2. Among all continuous additive functions in a BV set $A$ (derivable or not), characterize those which are indefinite BV-integrals in $A$.

A partial descriptive definition was given in [4, Theorem 2.6] employing the concept of BV-absolutely continuous functions. Using the stronger concept of BV-ACG$_*$ functions we shall present a full descriptive definition in Theorem 3.7 below.

**Lemma 3.4.** *Let $F$ be a continuous additive function in a BV set $A$, and let*

$$E = \{x \in \mathrm{cl}^* A : \underline{F}(x) < r < s < \overline{F}(x)\} \,.$$

*If $F$ is BV-AC$_*$ on $E$, then $E$ is negligible.*

Proof: With no loss of generality we may assume $E \subset \mathrm{int}^c A$. Choose positive numbers $\varepsilon$ and $\eta$, and let $\varepsilon' = \varepsilon^{m+1}$. If $F$ is BV-AC$_*$ on $E$, we can find a positive number $\eta' \le \eta$ and a gage $\delta$ in $A$ so that

$$\left| F\left(\bigcup P\right) - F\left(\bigcup Q\right) \right| < \varepsilon$$

for all $\varepsilon'$-regular $\delta$-fine partitions $P$ and $Q$ in $A$ anchored in $E$ for which

$$\left| \left(\bigcup P\right) \triangle \left(\bigcup Q\right) \right| < 4\eta' \,.$$

Select an open set $U$ containing $E$ with $|U| < |E| + \eta'$, and let $\mathcal{R}$ and $\mathcal{S}$ be the families of all doubly closed sets $C \subset U$ such that $d(C) < \delta(x_C)$ for an $x_C \in E \cap C$, $r(C) > \varepsilon$, and respectively,

$$F(A \cap C) < r|A \cap C| \qquad \text{and} \qquad F(A \cap C) > s|A \cap C| \,.$$

In view of Lemma 1.1, making $\delta$ smaller, we may assume $r(A \cap C) > \varepsilon'$ for each $C \in \mathcal{R} \cup \mathcal{S}$. Clearly, $\mathcal{R}$ and $\mathcal{S}$ are Vitali covers of $E - N_\delta$. Since $N_\delta$ is a negligible set (in fact, a thin set), applying Vitali's covering theorem [7, Chapter IV, Theorem 3.1], we obtain collections $\{A_1, \dots, A_p\} \subset \mathcal{R}$ and $\{B_1, \dots, B_q\} \subset \mathcal{S}$, each consisting of disjoint sets and such that

$$\min\left\{\left|\bigcup_{i=1}^{p}(E \cap A_i)\right|, \left|\bigcup_{j=1}^{q}(E \cap B_j)\right|\right\} > |E| - \eta' .$$

From the definitions of $\mathcal{R}$ and $\mathcal{S}$, we see that

$$P = \{(A \cap A_1, x_{A_1}), \dots, (A \cap A_p, x_{A_p})\},$$
$$Q = \{(A \cap B_1, x_{B_1}), \dots, (A \cap B_q, x_{B_q})\}$$

are $\varepsilon'$-regular $\delta$-fine partitions in $A$ anchored in $E$. Since $\bigcup P \subset U$ and $|E \cap \bigcup P| > |E| - \eta'$, we have $|E \triangle \bigcup P| < 2\eta'$; by symmetry, also $|E \triangle \bigcup Q| < 2\eta'$. Thus $|(\bigcup P) \triangle (\bigcup Q)| < 4\eta'$, and we obtain

$$\begin{aligned}
\varepsilon &> \left|F\left(\bigcup P\right) - F\left(\bigcup Q\right)\right| \geq \sum_{j=1}^{q} F(A \cap B_j) - \sum_{i=1}^{p} F(A \cap A_i) \\
&> s\left|\bigcup Q\right| - r\left|\bigcup P\right| > s(|E| - \eta') - r(|E| + \eta') \\
&= (s - r)|E| - (s + r)\eta' \geq (s - r)|E| - |s + r|\eta .
\end{aligned}$$

The negligibility of $E$ follows from the arbitrariness of $\varepsilon$ and $\eta$. □

**Lemma 3.5.** *Let $F$ be a continuous additive function in a BV set $A$, and let*

$$E = \{x \in \mathrm{cl}^* A : \underline{F}(x) = +\infty\} .$$

*If $F$ is BV-AC$_*$ on $E$, then $E$ is negligible.*

Proof: We may assume $E \subset \mathrm{int}^c A$. Proceeding towards a contradiction, suppose $|E| > 0$. If $0 < \varepsilon < 1/(2m)$ and $F$ is BV-AC$_*$ on $E$, we can find a gage $\delta$ in $A$ and a positive $\eta < |E|/2$ so that

$$\left|F\left(\bigcup P\right) - F\left(\bigcup Q\right)\right| < 1$$

for all $\varepsilon$-regular $\delta$-fine partitions $P$ and $Q$ in $A$ anchored in $E$ for which

$$\left|\left(\bigcup P\right) \triangle \left(\bigcup Q\right)\right| < 4\eta .$$

Let $U$ be an open set containing $E$ with $|U| < |E| + \eta$. Using [5, Lemma 3.4] and Vitali's covering theorem, find disjoint closed cubes $C_1, \dots, C_p$ contained in $U$ such that $|E - \bigcup_{i=1}^{p} C_i| < \eta$ while $r(A \cap C_i) > \varepsilon$ and $d(C_i) < \delta(x_i)$ for an $x_i \in E \cap C_i$ and $i = 1, \dots, p$. Thus $P = \{(A \cap C_1, x_1), \dots, (A \cap C_p, x_p)\}$ is an $\varepsilon$-regular $\delta$-fine partition in $A$ anchored in $E$, and $|E \triangle (\bigcup P)| < 2\eta$.

In view of [5, Lemma 3.4] and Lemma 3.1, the family $\mathcal{B}$ of all closed cubes $B \subset U$ such that $r(A \cap B) > \varepsilon$, $d(B) < \delta(y_B)$ for a $y_B \in E \cap B$, and

$$\frac{F(A \cap B)}{|A \cap B|} > \frac{2}{|E|} \left| F\left(\bigcup P\right) + 1 \right| ,$$

is a Vitali cover of $E$. Using Vitali's covering theorem again, find a disjoint collection $\{B_1, \dots, B_q\} \subset \mathcal{B}$ so that $|E - \bigcup_{j=1}^{q} B_j| < \eta$. Then $Q = \{(A \cap B_1, y_{B_1}), \dots, (A \cap B_q, y_{B_q})\}$ is an $\varepsilon$-regular $\delta$-fine partition in $A$ anchored in $E$, and $|E \triangle \bigcup Q| < 2\eta$. Observing that $\sum_{j=1}^{q} |A \cap B_j| \geq |E|/2$, we obtain

$$\begin{aligned} F\left(\bigcup Q\right) &= \sum_{j=1}^{q} F(A \cap B_j) \\ &> \frac{2}{|E|} \left| F\left(\bigcup P\right) + 1 \right| \cdot \sum_{j=1}^{q} |A \cap B_j| \\ &\geq F\left(\bigcup P\right) + 1 . \end{aligned}$$

This is a contradiction, since $|(\bigcup P) \triangle (\bigcup Q)| < 4\eta$. □

**Proposition 3.6.** *A continuous additive function $F$ in a BV set $A$ that is BV-ACG$_*$ is derivable almost everywhere in* $\mathrm{cl}^* A$.

Proof: Let $\mathrm{cl}^* A = \bigcup_{n=1}^{\infty} E_n$, and let $F$ be BV-AC$_*$ on each $E_n$. The set of all $x \in \mathrm{cl}^* A$ at which $F$ is not derivable is the union of the sets

$$E_{n,+\infty} = \{x \in E_n : \underline{F}(x) = +\infty\} , \qquad E_{n,-\infty} = \{x \in E_n : \overline{F}(x) = -\infty\} ,$$

and

$$E_{n,r,s} = \{x \in E_n : \underline{F}(x) < r < s < \overline{F}(x)\}$$

where $r$ and $s$ are rational numbers. Lemma 3.5 applied to $F$ and $-F$ shows that the sets $E_{n,\pm\infty}$ are negligible, and the sets $E_{n,r,s}$ are negligible by Lemma 3.4. Since we have only countably many of these sets, the proposition follows.

**Corollary 3.7.** *Let $F$ be a continuous additive function in a BV set $A$. If $F$ is BV-ACG$_*$, then $F'$ is BV-integrable in $A$ and $F$ is its indefinite BV-integral.*

Proof: According to Proposition 3.6, the derivate $F'$ is defined almost everywhere in $\mathrm{cl}^* A$, and by Proposition 3.3, the function $F$ is BV-absolutely continuous. An application of [4, Theorem 2.6] completes the proof. □

**Proposition 3.8.** *Let $f$ be a BV-integrable function in a BV set $A$. If $F$ is the indefinite BV-integral of $f$ in $A$, then $F$ is BV-ACG$_*$.*

PROOF: We may assume that $f$ is a real-valued function defined on the whole of $\mathrm{cl}^*A$, and let $E_n = \{x \in \mathrm{cl}^*A : |f(x)| \leq n\}$ for $n = 1, 2, \dots$. Since $\mathrm{cl}^*A = \bigcup_{n=1}^{\infty} E_n$, it suffices to show that $F$ is BV-AC$_*$ on each $E_n$. To this end, fix a positive integer $n$ and let $E = E_n$. It follows from [5, Corollary 5.12] that $E$ is measurable and $f$ is Lebesgue integrable in $E$. Hence, if $\chi$ is the characteristic function of $E$ restricted to $\mathrm{cl}^*A$, then $f\chi$ is Lebesgue integrable in $A$. By [5, Proposition 5.8], the function $f\chi$ is also BV-integrable in $A$, and we denote by $G$ its indefinite BV-integral in $A$.

Choose an $\varepsilon > 0$. Using the absolute continuity of the indefinite Lebesgue integral, find an $\eta > 0$ so that $|G(Z)| < \varepsilon$ for each BV set $Z \subset A$ with $|Z| < \eta$. There is a gage $\delta$ in $A$ such that

$$\sum_{i=1}^{r} \Big| f(z_i)|C_i| - F(C_i) \Big| < \varepsilon \qquad \text{and} \qquad \sum_{i=1}^{r} \Big| f(z_i)\chi(z_i)|C_i| - G(C_i) \Big| < \varepsilon$$

for each $\varepsilon$-regular $\delta$-fine partition $R = \{(C_1, z_1), \dots, (C_r, z_r)\}$ in $A$. If such a partition $R$ is anchored in $E$, then $\chi(z_i) = 1$ for $i = 1, \dots, r$, and we have

$$\left| F\Big(\bigcup R\Big) - G\Big(\bigcup R\Big) \right| \leq \sum_{i=1}^{r} |F(C_i) - G(C_i)| < 2\varepsilon \,.$$

Now choose $\varepsilon$-regular $\delta$-fine partitions $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ and $Q = \{(B_1, y_1), \dots, (B_q, y_q)\}$ in $A$ anchored in $E$ for which $|(\bigcup P) \triangle (\bigcup Q)| < \eta$. Letting $X = \bigcup P$ and $Y = \bigcup Q$, observe that

$$\begin{aligned} |G(X) - G(Y)| &= |G(X - Y) - G(Y - X)| \\ &\leq |G(X - Y)| + |G(Y - X)| < 2\varepsilon \,; \end{aligned}$$

for $\max\{|X - Y|, |Y - X|\} \leq |X \triangle Y| < \eta$. Thus

$$|F(X) - F(Y)| \leq |F(X) - G(X)| + |F(Y) - G(Y)| + |G(X) - G(Y)| < 6\varepsilon \,,$$

which establishes $F$ is BV-AC$_*$ on $E$. □

Combining Corollary 3.7 and Proposition 3.8, we obtain the following full descriptive definition of the BV-integral.

**Theorem 3.9.** *A continuous additive function $F$ in a BV set $A$ is BV-ACG$_*$ if and only if $F'$ exists almost everywhere in $A$ and $F$ is its indefinite BV-integral.*

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