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Weil uniformities for frames

Jorge Picado*

Abstract. In pointfree topology, the notion of uniformity in the form of a system of covers was introduced by J. Isbell in [11], and later developed by A. Pultr in [14] and [15]. Another equivalent notion of locale uniformity was given by P. Fletcher and W. Hunsaker in [6], which they called "entourage uniformity".

The purpose of this paper is to formulate and investigate an alternative definition of entourage uniformity which is more likely to the Weil pointed entourage uniformity, since it is expressed in terms of products of locales. We show that our definition is equivalent to the previous ones by proving that our category of Weil uniform frames is isomorphic to the one defined in [6].

Keywords: uniform space, frame, uniform frame, uniform frame homomorphism, C-ideal, frame coproduct, entourage, Weil uniform frame, Weil homomorphism

Classification: 06D20, 54E15, 54E55

0. Introduction

Uniformities on a set X were introduced, in the thirties, by A. Weil [17] in terms of subsets of $X \times X$ containing the diagonal $\Delta_X = \{(x, x) : x \in X\}$, called "entourages" or "surroundings" (the classical account of this subject is in Chapter II of Bourbaki [4]):

Definition 0.1 (Weil [17]). A *uniformity* on a set X is a subset $\mathcal{E}$ of $\mathcal{P}(X \times X)$ such that:

(i) $\mathcal{E}$ is a filter with respect to $\subseteq$.

(ii) Every element of $\mathcal{E}$ contains the set $\Delta_X = \{(x, x) \mid x \in X\}$.

(iii) If $E \in \mathcal{E}$ then there is a $D \in \mathcal{E}$ such that

$$D \circ D := \{(x, y) \in X \times X \mid \text{ there is a } z \in X \text{ such that } (x, z) \text{ and } (z, y) \in D\}$$

is contained in E.

(iv) If $E \in \mathcal{E}$ then the set $E^{-1} := \{(x, y) \in X \times X \mid (y, x) \in E\}$ is also in $\mathcal{E}$.

The members of the filter $\mathcal{E}$ are called *entourages*. A *uniform space* is a set X together with a uniformity on X.

A map $h : X \longrightarrow Y$, where X and Y are uniform spaces, is *uniformly continuous* if $(h \times h)^{-1}(E)$ is an entourage of X whenever E is an entourage of Y.

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Let $\mathcal{E}$ be a uniformity on X. For each $A \subseteq X$ and each $E \in \mathcal{E}$, let $E[A] = \bigcup_{x\in A} E[x]$, where $E[x] = \{y \in X \mid (x,y) \in E\}$. For $A, B \subseteq X$, we write $A \overset{\mathcal{E}}{\lhd} B$ if $E[A] \subseteq B$ for some $E \in \mathcal{E}$. The following result is well known:

Proposition 0.2. *Let $(X,\mathcal{E})$ be a uniform space and let $\mathcal{T}_{\mathcal{E}}$ be the associated topology on X. Then, for every $U \in \mathcal{T}_{\mathcal{E}}$, $U = \bigcup\{V \in \mathcal{T}_{\mathcal{E}} \mid V \overset{\mathcal{E}}{\lhd} U\}$.* □

It was the approach to uniform spaces via covers of J.W. Tukey [16] that was first studied in the pointless context of locales: in [11] J. Isbell introduced a theory of locale (frame) uniformities in terms of covers (a *cover* of a locale L is a subset C of L such that $\bigvee C = 1$), later developed in detail by A. Pultr ([14], [15]). In [11] the author also suggested a theory of locale uniformities by entourages but, intentionally, put it aside: "Entourages ought to work, but not in the present state of knowledge of product locales".

Subsequently, J.L. Frith [10] studied uniform-type structures from a more categorical point of view, also making use of frame covers, and P. Fletcher and W. Hunsaker [6] introduced an entourage-like theory of uniformities, which they proved to be equivalent to the covering one. As in the spatial setting, entourage-like theories have shown to be more manageable than the covering ones in the study of quasi-uniformities (see [7], [8], [9] and [13] and compare them with Frith's theory of quasi-uniformities via conjugate covers [10]).

Here, following the hint of J. Isbell, we present an alternative approach to entourage uniformities, expressed in terms of the coproduct $L \oplus L$, showing this way that entourages in the style of Weil do work in the pointless context.

The paper is organized as follows. After Section 1, which reviews all the required background, namely the theory of entourage frame uniformities of P. Fletcher and W. Hunsaker and the construction of binary coproducts of frames in terms of C-ideals, we present, in Section 2, an alternative notion of frame uniformity. We do so by introducing uniform structures on a frame L in terms of subsets of the coproduct $L \oplus L$. In Sections 3, 4 and 5 the nexus between this type of uniformity and the one of P. Fletcher and W. Hunsaker is spelled out in detail, leading to Theorem 5.4 which states that the two corresponding categories are isomorphic:

We show, in Section 3, that a uniformity $\mathcal{U}$ in the sense of P. Fletcher and W. Hunsaker yields a Weil uniformity $\psi(\mathcal{U})$. Likewise, a Weil uniformity $\mathcal{E}$ yields a uniformity, in the sense of P. Fletcher and W. Hunsaker, $\phi(\mathcal{E})$. Since we work with $L \oplus L$ as the frame of C-ideals of $L \times L$, the knowledge of some facts about C-ideals is essential for our approach. Namely, we need to know more about the C-ideal of $L \times L$ generated by a down-set U of $L \times L$: the point appears to be Lemma 3.3, which enables us to control, in terms of smallness, all the pairs (x,y) that go into the C-ideal generated by the diagonal $\{(x,x) \in L \times L \mid x \text{ is } f\text{-small}\}$, and Lemma 3.1. The correspondences $(L,\mathcal{U}) \longmapsto (L,\psi(\mathcal{U}))$ and $(L,\mathcal{E}) \longmapsto (L,\phi(\mathcal{E}))$ give rise, respectively, to the functors Ψ and Φ. In the last section, we conclude that $\Phi\Psi = id$ and $\Psi\Phi = id$.

1. Background

1.1 The category UFrm of (entourage) uniform frames.

Recall that L is a *frame* if it is a complete lattice satisfying the (infinite) distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge t \mid t \in S\} \qquad (a \in L, S \subseteq L)$$

and that a *frame homomorphism* is a map preserving finitary meets, including the unit 1, and arbitrary joins, including the zero 0.

For general facts about frames we refer to Johnstone [12].

The following definitions and notations are taken from [6].

Let L be a frame and let $\mathcal{F}$ be the collection of all order-preserving maps from L to L. If $f \in \mathcal{F}$ and $x \in L$, then x is *f-small* if $x \leq f(y)$ whenever $x \wedge y \neq 0$. If $\mathcal{B} \subseteq \mathcal{F}$ and $x, y \in L$, the relation $y \overset{\mathcal{B}}{\lhd} x$ means that there is an $f \in \mathcal{B}$ such that $f(y) \leq x$.

Definition 1.1.1 (Fletcher and Hunsaker [6])**.** A set $\mathcal{B}$ of sup-preserving maps from L to L is an (*entourage*) *uniformity base* on L provided that for $f \in \mathcal{B}$ and $x, y \in L$:

(i) $\mathcal{B}$ is a filter base with respect to $\leq$.
(ii) The collection of all f-small elements of L is a cover of L.
(iii) There exists a $g \in \mathcal{B}$ such that $g \circ g \leq f$.
(iv) $x \wedge f(y) = 0$ if and only if $f(x) \wedge y = 0$.
(v) $x = \bigvee \{y \in L \mid y \overset{\mathcal{B}}{\lhd} x\}$.

A subset $\mathcal{U}$ of $\mathcal{F}$ is called an (*entourage*) *uniformity* on L if it is generated by an (entourage) uniformity base $\mathcal{B}$ on L, i.e.

$$\mathcal{U} = \{f \in \mathcal{F} \mid \text{ there is a } g \in \mathcal{B} \text{ such that } g \leq f\}.$$

An (*entourage*) *uniform frame* is a pair $(L,\mathcal{U})$ where L is a frame and $\mathcal{U}$ is an (entourage) uniformity on L. Let $(L,\mathcal{U})$ and $(M, \mathcal{V})$ be (entourage) uniform frames. An (*entourage*) *uniform frame homomorphism* $h : (L,\mathcal{U}) \longrightarrow (M, \mathcal{V})$ is a frame map $h : L \longrightarrow M$ such that for every $f \in \mathcal{U}$ there exists a $g \in \mathcal{V}$ such that $g \circ h \leq h \circ f$.

In the sequel, when referring to these entourage uniformities and entourage uniform frames, we shall forget the word "entourage" and we shall call them just uniformities and uniform frames, respectively. The category of uniform frames and uniform frame homomorphisms will be denoted by UFrm.

Remark 1.1.2. Note that condition (ii) implies that $x \leq f(x)$ for every x (and, consequently, that $f^n \leq f^{n+1}$, for every natural n) . Indeed, we have

$$x = x \wedge \bigvee \{y \in L \mid y \text{ is } f\text{-small}\} = \bigvee \{x \wedge y \mid y \text{ is } f\text{-small and } x \wedge y \neq 0\} \leq f(x).$$

From this fact we may also conclude that, if $y\overset{\mathcal{B}}{\lhd}x$, then $y \leq x$.

We say that $f \in \mathcal{F}$ is *symmetric* if it satisfies condition (iv) of Definition 1.1.1.

In [6] the authors established an isomorphism between UFrm and the category of (covering) uniform frames defined by J. Isbell [11].

1.2 Coproducts of frames.

For a subset A of a poset $(X, \leq)$, let $\downarrow A = \{x \in X \mid x \leq a \text{ for some } a \in A\}$. The set A is said to be a *down-set* if $\downarrow A = A$. We shall denote by $\mathcal{D}(X)$ the frame of all down-sets in X.

Let L be a frame. Recall (cf., e.g., [5], [12]) that the coproduct of the frame L by itself

$$L \xrightarrow{u_1} L \oplus L \xleftarrow{u_2} L$$

can be constructed as follows: Take $L \times L$ with the obvious order. A down-set $A \subseteq L \times L$ is a *C-ideal* if

$$\{x\} \times S \subseteq A \;\Rightarrow\; (x, \bigvee S) \in A$$

and

$$S \times \{y\} \subseteq A \;\Rightarrow\; (\bigvee S, y) \in A.$$

Put $L \oplus L$ as the frame of all C-ideals of $L \times L$. Observe that the case $S = \emptyset$ implies that every C-ideal contains the set $\{(x, 0), (0, x) \mid x \in L\}$ which we shall denote by $\mathcal{N}$. Obviously, each $\downarrow (x, y) \cup \mathcal{N}$ is a C-ideal. It is denoted by $x \oplus y$. Finally put $u_1(x) = x \oplus 1$ and $u_2(y) = 1 \oplus y$. Thus $x \oplus y = u_1(x) \wedge u_2(y)$. Every element of $L \oplus L$ is of the form $\bigvee_{\gamma \in \Gamma}(x_\gamma \oplus y_\gamma)$, for some subset $\{(x_\gamma, y_\gamma) \mid \gamma \in \Gamma\}$ of $L \times L$. For any morphism $h : L \longrightarrow M$, we write $h \oplus h : L \oplus L \longrightarrow M \oplus M$ for the morphism given by $(h \oplus h) \circ u_i = v_i \circ h$ $(i = 1, 2)$. Obviously,

$$(h \oplus h)\left(\bigvee_{\gamma \in \Gamma}(x_\gamma \oplus y_\gamma)\right) = \bigvee_{\gamma \in \Gamma}(h(x_\gamma) \oplus h(y_\gamma)).$$

2. Uniformities in terms of the frame coproduct $L \oplus L$: the category WUFrm of Weil uniform frames

In the sequel L will always denote a frame. If $U, V \in \mathcal{D}(L \times L)$ we denote by $U \cdot V$ the set

$$\{(x, y) \in L \times L \mid \text{ there is a } z \in L \setminus \{0\} \text{ such that } (x, z) \in U \text{ and } (z, y) \in V\}$$

and by $U \circ V$ the C-ideal generated by $U \cdot V$.

Note that the operation $\circ$ is associative (but generally not commutative) and so bracketing is unnecessary for repeated compositions such as

$$U^n = U \circ U \circ \cdots \circ U \qquad (n \text{ factors}).$$

Further we have:

For any $U \in \mathcal{D}(L \times L)$, $U^{-1} := \{(x,y) \in L \times L \mid (y,x) \in U\}$ and $C_U := \{x \in L \mid (x,x) \in U\}$.

For any $U \in \mathcal{D}(L \times L)$ and $x \in L$, $st(x, C_U) := \bigvee\{y \in C_U \mid y \wedge x \neq 0\}$ (the "star" of x in C_U).

For any $\mathcal{E} \subseteq L \oplus L$ and $x, y \in L$, $y \overset{\mathcal{E}}{\lhd} x$ means that $E \circ y \oplus y \subseteq x \oplus x$, for some $E \in \mathcal{E}$.

Definition 0.1 and Proposition 0.2 motivate our definition of Weil uniformity:

Definition 2.1. A *Weil uniformity base* on L is a set $\mathcal{E}$ of C-ideals such that:

(i) $\mathcal{E}$ is a filter base with respect to $\subseteq$.
(ii) For each $E \in \mathcal{E}$, C_E is a cover of L.
(iii) For any $E \in \mathcal{E}$ there is a $D \in \mathcal{E}$ such that $D \circ D \subseteq E$.
(iv) For each $E \in \mathcal{E}$, E^{-1} is also in $\mathcal{E}$.
(v) For each $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$.

A *Weil uniformity* is a set $\mathcal{E}$ of C-ideals, called *entourages*, for which there exists a Weil uniformity base $\mathcal{E}' \subseteq \mathcal{E}$ such that each member of $\mathcal{E}$ contains some member of $\mathcal{E}'$. Clearly $\mathcal{E}$ is a Weil uniformity if and only if it is a filter satisfying conditions (ii)–(v) above.

Usually it is more convenient to work with uniformities bases rather than to work with uniformities.

A *Weil uniform frame* is just a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a Weil uniformity on L. These are the objects of the category WUFrm whose morphisms — the *Weil homomorphisms* — are those frame maps $h : (L, \mathcal{E}) \longrightarrow (M, \mathcal{G})$ such that $(h \oplus h)(E) \in \mathcal{G}$ for every $E \in \mathcal{E}$.

Remarks 2.2. Let $\mathcal{E}$ be a Weil uniformity.

(i) It is useful to note that the *symmetric* entourages E of $\mathcal{E}$, i.e. those for which $E = E^{-1}$, form a base for $\mathcal{E}$. In fact, if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$ so $E \cap E^{-1}$ is a symmetric entourage contained in E.
(ii) Note that condition (ii) of Definition 2.1 implies that $y \leq st(y, C_E)$, for every $y \in L$ and $E \in \mathcal{E}$. Therefore $y \overset{\mathcal{E}}{\lhd} x$ implies that $y \leq x$ since, for any $z \in C_E$ satisfying $z \wedge y \neq 0$, the pair (z, y) belongs to $E \circ y \oplus y$.
Condition (ii) also implies that every entourage E is contained in E^2: Consider $(x, y) \in E$. We have

$$y = y \wedge \bigvee\{z \mid (z,z) \in E\} \leq \bigvee\{z \mid (z,z) \in E, y \wedge z \neq 0\}.$$

For any z such that $(z,z) \in E$ and $y \wedge z \neq 0$, $(x,z) \in E^2$, since $(x, y \wedge z), (y \wedge z, z) \in E$. Therefore $(x, \bigvee\{z \mid (z,z) \in E, y \wedge z \neq 0\}) \in E^2$ and, consequently, $(x,y) \in E^2$.

Proposition 2.3. *Let $\mathcal{E}$ be a Weil uniformity on L and let $x, y \in L$. The following assertions are equivalent:*

(i) $y \overset{\mathcal{E}}{\lhd} x$.
(ii) $y \oplus y \circ E \subseteq x \oplus x$, *for some* $E \in \mathcal{E}$.
(iii) $E \circ y \oplus 1 \subseteq x \oplus 1$, *for some* $E \in \mathcal{E}$.
(iv) $1 \oplus y \circ E \subseteq 1 \oplus x$, *for some* $E \in \mathcal{E}$.
(v) $st(y, C_E) \leq x$, *for some* $E \in \mathcal{E}$.

Proof: We only prove that statements (i) and (v) are equivalent because the proofs that each one of (ii), (iii) and (iv) is equivalent to (v) are similar.

(i) $\Rightarrow$ (v): Cf. Remark 2.2 (ii).

(v) $\Rightarrow$ (i): In order to show that $y \overset{\mathcal{E}}{\lhd} x$ it suffices to prove that $D \circ y \oplus y \subseteq x \oplus x$ for any symmetric $D \in \mathcal{E}$ such that $D^2 \subseteq E$. So, consider $a, b, c \in L$ such that $(a, b) \in D$, $(b, c) \leq (y, y)$ and $a, b \neq 0$. Then $(a, b) \in D^2$ and, by the symmetry of D, $(b, a) \in D^2$, which forces $(a \vee b, a \vee b) \in D^2 \subseteq E$, as (a, a) and (b, b) also belong to D^2. Thus $a \vee b \in C_E$ and, therefore, $a \leq st(y, C_E)$, since $(a \vee b) \wedge y \geq b \neq 0$. Hence $a \leq x$ and $c \leq y \leq st(y, C_E) \leq x$ which implies that $(a, c) \in x \oplus x$. $\square$

Furthermore, $y \overset{\mathcal{E}}{\lhd} x$ implies, trivially, that $E \circ 1 \oplus y \subseteq 1 \oplus x$, for some $E \in \mathcal{E}$.

So, condition (v) of 2.1 could be formulated in the following equivalent way:

For each $U \in L \oplus L$, $U = \bigvee\{V \in L \oplus L \mid V \overset{\mathcal{E}}{\lhd} U\}$, where $V \overset{\mathcal{E}}{\lhd} U$ means that $E \circ V \subseteq U$ for some $E \in \mathcal{E}$.

Indeed, for every $U = \bigvee_{\gamma \in \Gamma}(a_\gamma \oplus b_\gamma) \in L \oplus L$, we have

$$a_\gamma \oplus b_\gamma = \left(\bigvee\{x \in L \mid x \overset{\mathcal{E}}{\lhd} a_\gamma\}\right) \oplus \left(\bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} b_\gamma\}\right) = \bigvee\{x \oplus y \mid x \overset{\mathcal{E}}{\lhd} a_\gamma, y \overset{\mathcal{E}}{\lhd} b_\gamma\},$$

since, for every $\gamma \in \Gamma$, $a_\gamma = \bigvee\{x \in L \mid x \overset{\mathcal{E}}{\lhd} a_\gamma\}$ and $b_\gamma = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} b_\gamma\}$. But $x \overset{\mathcal{E}}{\lhd} a_\gamma$ and $y \overset{\mathcal{E}}{\lhd} b_\gamma$ imply, respectively, that $E_1 \circ x \oplus 1 \subseteq a_\gamma \oplus 1$ and $E_2 \circ 1 \oplus y \subseteq 1 \oplus b_\gamma$, for some $E_1, E_2 \in \mathcal{E}$, thus $E \circ x \oplus y \subseteq (E \circ x \oplus 1) \cap (E \circ 1 \oplus y) \subseteq a_\gamma \oplus b_\gamma$, for $E = E_1 \cap E_2 \in \mathcal{E}$. Consequently, $\bigvee\{x \oplus y \mid x \overset{\mathcal{E}}{\lhd} a_\gamma, y \overset{\mathcal{E}}{\lhd} b_\gamma\} \subseteq \bigvee\{V \in L \oplus L \mid V \overset{\mathcal{E}}{\lhd} a_\gamma \oplus b_\gamma\}$.

Conversely, for every $x \in L$, $x \oplus 1 = \{V \in L \oplus L \mid V \overset{\mathcal{E}}{\lhd} x \oplus 1\} \leq (\bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}) \oplus 1$, because $V \overset{\mathcal{E}}{\lhd} x \oplus 1$ implies that, for every $(a, b) \in V$, $a \overset{\mathcal{E}}{\lhd} x$, i.e. that $V \subseteq (\bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}) \oplus 1$. Hence $x \leq \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$.

3. The functor $\Psi : \mathsf{UFrm} \longrightarrow \mathsf{WUFrm}$

Before stating a way of obtaining a Weil uniformity base from a uniformity base we need some technical lemmas.

The map

$$k_0 : \mathcal{D}(L \times L) \longrightarrow \mathcal{D}(L \times L)$$
$$U \longmapsto \{(x, \bigvee S) \mid \{x\} \times S \subseteq U\} \bigcup \{(\bigvee S, y) \mid S \times \{y\} \subseteq U\}$$

is a preclosure operator (it is, even, a prenucleus [1]) and, consequently,

$$Fix(k_0) := \{U \in \mathcal{D}(L \times L) \mid k_0(U) = U\} = L \oplus L$$

is a closure system, and the associated closure operator is then given by $k(U) = \bigcap\{V \in L \oplus L \mid U \subseteq V\}$, i.e. $k(U)$ is the C-ideal generated by U. Moreover we have:

Lemma 3.1. *Let $U, V \in \mathcal{D}(L \times L)$. Then:*

(i) $k_0(U) \cdot V \subseteq k_0(U \cdot V)$ *and* $U \cdot k_0(V) \subseteq k_0(U \cdot V)$.
(ii) $k(U) \circ k(V) = U \circ V$.
(iii) *If U is symmetric then $k(U)$ is symmetric.*

Proof:

(i) Consider $(x, y) \in k_0(U) \cdot V$ and $z \neq 0$ such that $(x, z) \in k_0(U)$ and $(z, y) \in V$. If $(x, z) = (x, \bigvee S)$ for some S with $\{x\} \times S \subseteq U$, there is a non-zero $s \in S$ such that $(x, s) \in U$ and $(s, y) \in V$ and, therefore, $(x, y) \in U \cdot V \subseteq k_0(U \cdot V)$. On the other hand, if $(x, z) = (\bigvee S, z)$ for some S with $S \times \{z\} \subseteq U$, $(s, y) \in U \cdot V$ for each $s \in S$ and, therefore, $(x, y) \in k_0(U \cdot V)$.
By symmetry, we also have that $U \cdot k_0(V) \subseteq k_0(U \cdot V)$.

(ii) It suffices to show that $k(U) \cdot k(V) \subseteq k(U \cdot V)$. For this, consider the non-empty set

$$\mathbb{E} = \{A \in \mathcal{D}(L \times L) \mid U \subseteq A \subseteq k(U), A \cdot V \subseteq k(U \cdot V)\}.$$

By (i), if $A \in \mathbb{E}$ then also $k_0(A) \in \mathbb{E}$. On the other hand, for any non-void $X \subseteq \mathbb{E}$, $\bigcup X \in \mathbb{E}$, since $(\bigcup X) \cdot V \subseteq \bigcup_{A \in X}(A \cdot V)$. Therefore $S = \bigcup_{A \in \mathbb{E}} A$ belongs to $\mathbb{E}$, i.e. $\mathbb{E}$ has a largest element S. But $k_0(S) \in \mathbb{E}$ so $S = k_0(S)$, i.e. S is a C-ideal. Hence $k(U) = S \in \mathbb{E}$ and, consequently, $k(U) \cdot V \subseteq k(U \cdot V)$. By symmetry, $U \cdot k(V) \subseteq k(U \cdot V)$.
In conclusion, we have $k(U) \cdot k(V) \subseteq k(U \cdot k(V)) \subseteq k^2(U \cdot V) = k(U \cdot V)$, as desired.

(iii) Consider a symmetric $U \in \mathcal{D}(L \times L)$ and let

$$\mathbb{E} = \{A \in \mathcal{D}(L \times L) \mid U \subseteq A \subseteq k(U), A^{-1} = A\}.$$

The set $\mathbb{E}$ is non-empty and $k_0(A) \in \mathbb{E}$ whenever $A \in \mathbb{E}$. Moreover, for any non-void $X \subseteq \mathbb{E}$, $\bigcup X \in \mathbb{E}$. Therefore $\mathbb{E}$ has a largest element S which must be $k(U)$ since $S = k_0(S)$ and $U \subseteq S \subseteq k(U)$. This says that $k(U)$ is symmetric. ∎

The map k can also be constructed from k_0, by transfinite induction over the class Ord of ordinals:

if one defines, for any $U \in \mathcal{D}(L \times L)$ and any ordinal β,

- $k_0^0(U) = U$
- $k_0^\beta(U) = k_0(k_0^\alpha(U))$, if $\beta = \alpha + 1$
- $k_0^\beta(U) = \bigvee\{k_0^\alpha(U) \mid \alpha < \beta\}$, if β is a limit ordinal,

then $k = \bigvee_{\gamma \in Ord} k_0^\gamma$. So, the lemma can be proved by transfinite induction. The approach we followed (cf. [1] and [2]) in the above lemma avoids the use of ordinals.

Let $\mathcal{U} \subseteq \mathcal{F}$. For each $f \in \mathcal{U}$ define $E_f = k(\{(z,z) \in L \times L \mid z \text{ is } f\text{-small}\})$ and denote the set $\{E_f \mid f \in \mathcal{U}\}$ by $\mathcal{E}_{\mathcal{U}}$.

Lemma 3.2. *Consider $f \in \mathcal{F}$.*

(i) *If $X = \{x_\gamma \mid \gamma \in \Gamma\}$ is a set of f-small elements such that $x_\gamma \wedge x_\delta \neq 0$ for every $\gamma, \delta \in \Gamma$, then $\bigvee X$ is f^2-small.*

(ii) *If f is symmetric and x is f-small, then $f(x)$ is f^3-small.*

PROOF:

(i) Cf. proof of Proposition 2.2 of [6].

(ii) Trivial. □

We say that a pair $(x,y) \in L \times L$ is *f-small* if $x \vee y \leq f(z)$ whenever $x \wedge z \neq 0$ and $y \wedge z \neq 0$. Note that this does not imply that x and y are f-small. However, (x,x) is f-small if and only if x is f-small.

Lemma 3.3. *Let $f \in \mathcal{F}$ satisfy $f(x) \geq x$ for all $x \in L$. If $(x,y) \in E_f$ then (x,y) is f-small.*

PROOF: Since E_f is symmetric it suffices to show that $x \leq f(z)$ whenever $(x,y) \in E_f$ and $y \wedge z \neq 0$. But this is consequence of the following result:

$$\text{if } f(x_1) \leq y_1 \text{ and } f(x_2) \leq y_2 \text{ then } E_f \circ x_1 \oplus x_2 \subseteq y_1 \oplus y_2. \tag{1}$$

In fact, by (1), $E_f \circ z \oplus z \subseteq f(z) \oplus f(z)$ thus $(x,z) \in f(z) \oplus f(z)$ because $(x, y \wedge z) \in E_f$ and $(y \wedge z, z) \in z \oplus z$.

So, let us show (1):

Denote the set $\{(z,z) \mid z \text{ is } f\text{-small }\}$ by F. Then $E_f \circ x_1 \oplus x_2 = k(\downarrow F) \circ k(\downarrow \{(x_1,x_2)\})$. Hence, according to Lemma 3.1 (ii), $E_f \circ x_1 \oplus x_2 = \downarrow F \circ \downarrow \{(x_1,x_2)\}$. Consider $(a,b) \in \downarrow F$ and $(b,c) \in \downarrow \{(x_1,x_2)\}$ with $b \neq 0$. We have $(a,b) \leq (z,z)$ for some f-small element z and $z \wedge x_1 \geq b \neq 0$. Then $a \leq z \leq f(x_1) \leq y_1$ and, on the other hand, $c \leq x_2 \leq f(x_2) \leq y_2$, thus $(a,c) \in y_1 \oplus y_2$. □

Remark 3.4. In the sequel, we only need the following particular case of Lemma 3.3:

Let $f \in \mathcal{F}$ satisfy $f(x) \geq x$ for all $x \in L$. If $(x,x) \in E_f$ then x is f-small.

We are now able to prove that $\mathcal{E}_{\mathcal{U}}$ is a Weil uniformity base whenever $\mathcal{U}$ is a uniformity base:

Proposition 3.5. *Let $\mathcal{U}$ be a uniformity base on a frame L. Then $\mathcal{E}_{\mathcal{U}}$ is a Weil uniformity base on L.*

Proof: We just need to check conditions (i)–(v) of Definition 2.1:

(i) Let $E_f, E_g \in \mathcal{E}_{\mathcal{U}}$. Take h in $\mathcal{U}$ such that $h \leq f \wedge g$. If x is h-small, then it is also f-small and g-small, so $E_h \subseteq E_f \cap E_g$. Thus $\mathcal{E}_{\mathcal{U}}$ is a filter base.

(ii) It is obvious.

(iii) We first prove that $E_f^2 \subseteq E_{f^2}$, for every $f \in \mathcal{U}$. According to Lemma 3.1(ii) we have $E_f^2 = (\downarrow F)^2$, where $F = \{(z,z) \mid z \text{ is } f\text{-small}\}$. So, consider $(a,b) \leq (x,x)$ and $(b,c) \leq (y,y)$, x and y f-small elements and $b \neq 0$. Then $x \wedge y \geq b \neq 0$ and thus, by Lemma 3.2 (i), $x \vee y$ is f^2-small. Hence $(a,c) \in E_{f^2}$.
Now, let $E_g \in \mathcal{E}_{\mathcal{U}}$ and take $f \in \mathcal{U}$ such that $f^2 \leq g$. Then $E_f^2 \subseteq E_{f^2} \subseteq E_g$.

(iv) It is obvious, by Lemma 3.1 (iii).

(v) This is an immediate consequence of the fact that

$$x \overset{\mathcal{U}}{\lhd} y \text{ if and only if } x \overset{\mathcal{E}_{\mathcal{U}}}{\lhd} y$$

that we prove next.
If $x \overset{\mathcal{U}}{\lhd} y$ then there is an $f \in \mathcal{U}$ with $f(x) \leq y$. We claim that $st(x, C_{E_f}) \leq y$. Consider $(z,z) \in E_f$ such that $z \wedge x \neq 0$. By Remark 3.4, z is f-small thus $z \leq f(x) \leq y$. Hence $st(x, C_{E_f}) \leq y$.
Conversely, assume that $st(x, C_{E_f}) \leq y$ and consider $g \in \mathcal{U}$, symmetric, such that $g^3 \leq f$. We require $g(x) \leq y$. Certainly, since g is sup-preserving,

$$\begin{aligned} g(x) &= g\left(x \wedge \bigvee\{z \mid z \text{ is } g\text{-small}\}\right) \\ &= \bigvee\{g(x \wedge z) \mid z \text{ is } g\text{-small and } x \wedge z \neq 0\} \\ &\leq \bigvee\{g(z) \mid z \text{ is } g\text{-small and } x \wedge z \neq 0\}. \end{aligned}$$

But the symmetry of g implies that $g(z)$ is g^3-small (recall part (ii) of Lemma 3.2) and then f-small so $(g(z), g(z)) \in E_f$. In case $x \wedge z \neq 0$, $g(z) \wedge x$ is also non-zero so $g(z) \leq y$. Therefore, $g(x) \leq y$. □

In what follows, if $\mathcal{U}$ is a uniformity on L, then $\psi(\mathcal{U})$ denotes the Weil uniformity for which $\mathcal{E}_{\mathcal{U}}$ is a base. The correspondence $(L,\mathcal{U}) \longmapsto (L,\psi(\mathcal{U}))$ is functorial. Indeed, it is the function on objects of a functor $\Psi : \mathsf{UFrm} \longrightarrow \mathsf{WUFrm}$ whose function on morphisms is described in the following proposition:

Proposition 3.6. *Let $(L,\mathcal{U})$ and $(M,\mathcal{V})$ be uniform frames and let $h : (L,\mathcal{U}) \longrightarrow (M,\mathcal{V})$ be a uniform frame homomorphism. Then $h : (L,\psi(\mathcal{U})) \longrightarrow (M,\psi(\mathcal{V}))$ is a Weil homomorphism.*

Proof: Consider $E_f \in \mathcal{E}_{\mathcal{U}}$ $(f \in \mathcal{U})$ and take $f' \in \mathcal{U}$, symmetric, such that ${f'}^3 \leq f$. Since h is a uniform frame homomorphism there is a $g \in \mathcal{V}$ such that $g \circ h \leq h \circ f'$. We only have to show that $E_g \subseteq (h \oplus h)(E_{{f'}^3})$ because $(h \oplus h)(E_{{f'}^3}) \subseteq (h \oplus h)(E_f)$. So, consider a non-zero g-small element x of M. Since

$$x = \bigvee\{x \wedge h(z) \mid z \in L, z \text{ is } f'\text{-small}\},$$

there is a $z \in L$, f'-small, such that $x \wedge h(z) \neq 0$, which implies that $x \leq g \circ h(z) \leq h \circ f'(z)$. Consequently, $(h \oplus h)(f'(z) \oplus f'(z)) \subseteq (h \oplus h)(E_{{f'}^3})$, that is, $h \circ f'(z) \oplus h \circ f'(z) \subseteq (h \oplus h)(E_{{f'}^3})$, and so $(x,x) \in (h \oplus h)(E_{{f'}^3})$. Thus $E_g \subseteq (h \oplus h)(E_{{f'}^3})$, as we claimed. $\square$

4. The functor $\Phi : \mathsf{WUFrm} \longrightarrow \mathsf{UFrm}$

Let $\mathcal{E} \subseteq L \oplus L$. For each $E \in \mathcal{E}$ define $f_E : L \longrightarrow L$ by $f_E(x) = st(x, C_E)$ and denote the set $\{f_E \mid E \in \mathcal{E}\}$ by $\mathcal{U}_{\mathcal{E}}$.

Proposition 4.1. *Let $\mathcal{E}$ be a Weil uniformity base on a frame L. Then $\mathcal{U}_{\mathcal{E}}$ is a uniformity base on L.*

Proof: Easily, every f_E $(E \in \mathcal{E})$ is sup-preserving. We need to check conditions (i)–(v) of Definition 1.1.1.

(i) Let $f_D, f_E \in \mathcal{U}_{\mathcal{E}}$. In order to prove that $\mathcal{U}_{\mathcal{E}}$ is a filter base just take f_F for some entourage F such that $F \subseteq D \cap E$.

(ii) It is an immediate consequence of the fact that x is f_E-small whenever $x \in C_E$.

(iii) For $f_E \in \mathcal{U}_{\mathcal{E}}$ consider $D \in \mathcal{E}$ such that $D^2 \subseteq E$. We claim that ${f_D}^2 \leq f_E$. In fact, ${f_D}^2(x) = \bigvee\{y \in L \mid (y,y) \in D, y \wedge st(x, C_D) \neq 0\}$. Consider $y \in L$ with $(y,y) \in D$ and $y \wedge st(x, C_D) \neq 0$. Then there is a $z \in L$ such that $(z,z) \in D$, $z \wedge x \neq 0$ and $z \wedge y \neq 0$. Therefore $(y, y \wedge z) \in D$ and $(y \wedge z, z) \in D$ thus $(y,z) \in E$. Similarly, $(z,y) \in E$. Also $(y,y),(z,z) \in D^2 \subseteq E$. But E is a C-ideal so $(y \vee z, y \vee z) \in E$. In conclusion, $(y \vee z, y \vee z) \in E$ and $(y \vee z) \wedge x \geq z \wedge x \neq 0$, hence $y \leq f_E(x)$.

(iv) Let $E \in \mathcal{E}$ and $x, y \in L$. Then we have that

$$x \wedge f_E(y) = 0 \Leftrightarrow \bigvee\{x \wedge u \mid (u,u) \in E \text{ and } u \wedge y \neq 0\} = 0 \tag{2}$$

and, analogously,

$$f_E(x) \wedge y = 0 \Leftrightarrow \bigvee\{y \wedge u \mid (u,u) \in E \text{ and } u \wedge x \neq 0\} = 0. \tag{3}$$

Obviously (2) and (3) are equivalent.

(v) Trivial, since $y \overset{\mathcal{E}}{\lhd} x$ if and only if $y \overset{\mathcal{U}_\mathcal{E}}{\lhd} x$. □

In what follows, if $\mathcal{E}$ is a Weil uniformity on L, then $\phi(\mathcal{E})$ denotes the uniformity generated by $\mathcal{U}_\mathcal{E}$. The correspondence $(L,\mathcal{E}) \longmapsto (L,\phi(\mathcal{E}))$ is functorial:

Proposition 4.2. *Let $(L,\mathcal{E})$ and $(M,\mathcal{G})$ be Weil uniform frames and let $h : (L,\mathcal{E}) \longrightarrow (M,\mathcal{G})$ be a Weil homomorphism. Then $h : (L,\phi(\mathcal{E})) \longrightarrow (M,\phi(\mathcal{G}))$ is a uniform frame homomorphism.*

Proof: Let $f_E \in \mathcal{U}_\mathcal{E}$, where $E \in \mathcal{E}$. Take $D \in \mathcal{E}$, symmetric, such that $D^2 \subseteq E$. Since h is a Weil homomorphism, $(h \oplus h)(D) \in \mathcal{G}$. In order to show that $h : (L,\phi(\mathcal{E})) \longrightarrow (M,\phi(\mathcal{G}))$ is uniform it suffices to show that $f_{(h\oplus h)(D)} \circ h \leq h \circ f_E$.

So, fix $x \in L$ and take $y \in M$ such that $(y,y) \in (h \oplus h)(D)$ and $y \wedge h(x) \neq 0$. Then $(y, y \wedge h(x)) \in (h \oplus h)(D)$ and $(y \wedge h(x), h(x)) \in h(x) \oplus h(x)$ and, consequently, $(y, h(x)) \in (h \oplus h)(D) \circ h(x) \oplus h(x)$. Further, since D is of the form $\bigvee_{\gamma\in\Gamma}(a_\gamma \oplus b_\gamma)$, for some subset $\{(a_\gamma, b_\gamma) \mid \gamma \in \Gamma\}$ of $L \times L$, we have that

$$\begin{aligned}(h \oplus h)(D) \circ h(x) \oplus h(x) &= (h \oplus h)\left(\bigvee_{\gamma\in\Gamma}(a_\gamma \oplus b_\gamma)\right) \circ h(x) \oplus h(x) \\ &= \langle \bigcup_{\gamma\in\Gamma} \big(h(a_\gamma) \oplus h(b_\gamma)\big)\rangle \circ \langle {\downarrow}(h(x), h(x))\rangle,\end{aligned}$$

so, by Lemma 3.1 (ii), $(h\oplus h)(D) \circ h(x) \oplus h(x) = \bigcup_{\gamma\in\Gamma}((a_\gamma) \oplus h(b_\gamma)) \circ {\downarrow}(h(x), h(x))$. But $\bigcup_{\gamma\in\Gamma}((a_\gamma) \oplus h(b_\gamma)) \circ {\downarrow}(h(x), h(x))$ is contained in $h \circ f_E(x) \oplus h(x)$: For any $(y,z) \in \bigcup_{\gamma\in\Gamma}((a_\gamma) \oplus h(b_\gamma)) \circ {\downarrow}(h(x), h(x)) \setminus \mathcal{N}$, there is a $w \in L \setminus \{0\}$ and a $\gamma \in \Gamma$ such that $(y,w) \leq (h(a_\gamma), h(b_\gamma))$ and $(w,z) \leq (h(x), h(x))$. It follows that $y \leq h(a_\gamma \vee b_\gamma)$ and, therefore, that $y \leq h \circ f_E(x)$. Indeed, $(a_\gamma \vee b_\gamma) \wedge x \neq 0$ because $h(b_\gamma \wedge x) \geq w \neq 0$ and, by the symmetry of D, $(a_\gamma \vee b_\gamma, a_\gamma \vee b_\gamma) \in D^2 \subseteq E$.

In conclusion, we have that $(y, h(x)) \in (h \oplus h)(D) \circ h(x) \oplus h(x) \subseteq h \circ f_E(x) \oplus h(x)$. Hence $y \leq h \circ f_E(x)$ which implies that $f_{(h\oplus h)(D)}(h(x)) \leq h(f_E(x))$, as required. □

We shall denote the functor defined above by Φ.

5. The isomorphism between WUFrm and UFrm

Finally, let us show that functors Φ and Ψ define an isomorphism between the categories WUFrm and UFrm.

Lemma 5.1. *If $g \in \mathcal{F}$ is symmetric and the set of g-small elements is a cover, then $g \leq f_{E_{g^3}}$.*

Proof: By definition $f_{E_{g^3}}(x) = st(x, C_{E_{g^3}})$. On the other hand

$$g(x) = g\left(\bigvee\{x \wedge y \mid y \text{ is } g\text{-small}\}\right) = \bigvee\{g(x \wedge y) \mid y \text{ is } g\text{-small and } x \wedge y \neq 0\}.$$

Consider any g-small element y such that $x \wedge y \neq 0$. By Lemma 3.2 (ii), $g(x \wedge y)$ is g^3-small so $(g(x \wedge y), g(x \wedge y)) \in E_{g^3}$. Since $g(x \wedge y) \wedge (x \wedge y) = x \wedge y \neq 0$, it follows that

$$g(x \wedge y) \leq f_{E_{g^3}}(x \wedge y) \leq f_{E_{g^3}}(x).$$

Hence $g(x) \leq f_{E_{g^3}}(x)$. □

The corresponding property for C-ideals is the following:

Lemma 5.2. *If D is a symmetric C-ideal such that C_D is a cover of L, then $D \subseteq E_{f_{D^2}}$.*

Proof: Consider $(x, y) \in D \setminus \mathcal{N}$. Then $(x \vee y, x \vee y) \in D^2$ because (x, y), (y, x), (x, x) and (y, y) belong to D^2. Since every member of C_{D^2} is f_{D^2}-small, $(x \vee y, x \vee y) \in E_{f_{D^2}}$ and, consequently, $(x, y) \in E_{f_{D^2}}$. □

Proposition 5.3. *Let $\mathcal{E}$ be a Weil uniformity on L and let $\mathcal{U}$ be a uniformity on L. Then $\psi\phi(\mathcal{E}) = \mathcal{E}$ and $\phi\psi(\mathcal{U}) = \mathcal{U}$.*

Proof: We first show that $\psi\phi(\mathcal{E}) = \mathcal{E}$. Consider $\mathcal{E}_{\phi(\mathcal{E})} = \{E_g \mid g \in \phi(\mathcal{E})\}$. By Proposition 3.5, $\mathcal{E}_{\phi(\mathcal{E})}$ is a Weil uniformity base. It suffices to show that it is a base for $\mathcal{E}$. To prove this, consider $E_g \in \mathcal{E}_{\phi(\mathcal{E})}$ and $E \in \mathcal{E}$ such that $g = f_E$. One can take $D \in \mathcal{E}$, symmetric, such that $D^2 \subseteq E$. By Lemma 5.2, $D \subseteq E_g$ so $\mathcal{E}_{\phi(\mathcal{E})} \subseteq \mathcal{E}$. Finally assume $D \in \mathcal{E}$ and choose $D' \in \mathcal{E}$ such that $D'^2 \subseteq D$. We show that $E_{f_{D'}} \subseteq D$:

Consider $x \neq 0$, $f_{D'}$-small. We have that $x \leq \bigvee\{z \mid (z, z) \in D', z \wedge x \neq 0\}$. Since x is $f_{D'}$-small, we have $x \leq f_{D'}(z) = \bigvee\{y \mid (y, y) \in D, y \wedge z \neq 0\}$, for any z such that $(z, z) \in D'$ and $x \wedge z \neq 0$. For each y in this set we have that $(z, z \wedge y)$, $(z \wedge y, y) \in D'$, which implies that $(z, y) \in D'^2$. Therefore $(z, x) \in D'^2$ and, consequently, $(x, x) \in D'^2 \subseteq D$.

Now let us prove the second equality. Consider $\mathcal{U}_{\psi(\mathcal{U})} = \{f_E \mid E \in \psi(\mathcal{U})\}$. By Proposition 4.1, $\mathcal{U}_{\psi(\mathcal{U})}$ is a uniformity base. It suffices to show that it is a base for $\mathcal{U}$. For any $f_E \in \mathcal{U}_{\psi(\mathcal{U})}$ take $g \in \mathcal{U}$ with $E = E_g$ and consider $h \in \mathcal{U}$, symmetric, such that $h^3 \leq g$. We know, by Lemma 5.1, that $h \leq f_E$, so $f_E \in \mathcal{U}$, i.e. $\mathcal{U}_{\psi(\mathcal{U})} \subseteq \mathcal{U}$. Finally, let $g \in \mathcal{U}$. By Remark 3.4, y is g-small whenever $(y, y) \in E_g$. Therefore $f_{E_g}(x) = \bigvee\{y \in L \mid (y, y) \in E_g, y \wedge x \neq 0\} \leq g(x)$. □

In conclusion, the functors Φ and Ψ are mutually inverse. Thus:

Theorem 5.4. *The categories* WUFrm *and* UFrm *are isomorphic.* □

Remark 5.5. In the spatial setting, by dropping the symmetry condition one gets the notion of quasi-uniformity. Here, if we drop the symmetry (condition (iv) in Definition 2.1), we must observe the following: the equivalence between conditions (i) and (ii) of Proposition 2.3 is no longer valid. In fact, we have two distinct order relations, $\overset{\mathcal{E}}{\triangleleft}_1$ and $\overset{\mathcal{E}}{\triangleleft}_2$, induced by the family $\mathcal{E}$ of C-ideals:

$$x\overset{\mathcal{E}}{\triangleleft}_1 y \;\equiv\; \exists\, E\in\mathcal{E}: E\circ x\oplus x\subseteq y\oplus y \quad\text{and}\quad x\overset{\mathcal{E}}{\triangleleft}_2 y \;\equiv\; \exists\, E\in\mathcal{E}: x\oplus x\circ E\subseteq y\oplus y.$$

These order relations define the subframes

$$L_i := \left\{x\in L : x=\bigvee\{y\in L \mid y\overset{\mathcal{E}}{\triangleleft}_i x\}\right\} \quad (i=1,2)$$

of L. Then, in order to get the appropriate definition of *Weil quasi-uniform frame*, we have to replace the admissibility condition (v) in Definition 2.1 by the condition (equivalent, under symmetry) that the triple (L, L_1, L_2) is a biframe [3].

Indeed, the corresponding category of Weil quasi-uniform frames and Weil homomorphisms is isomorphic to the category of quasi-uniform frames of J.L. Frith [10] (cf. [13] for the details), defined in terms of the so called conjugate cover pairs of biframes, and, therefore, it is also isomorphic to the category of quasi-uniform frames of P. Fletcher, W. Hunsaker and W. Lindgren [9]. Thus one can rephrase in our context all the results of [7] and [8].

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