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Abstract. The author studies some characteristic properties of semiprime ideals. The semiprimeness is also used to characterize distributive and modular lattices. Prime ideals are described as the meet-irreducible semiprime ideals. In relatively complemented lattices they are characterized as the maximal semiprime ideals. D-radicals of ideals are introduced and investigated. In particular, the prime radicals are determined by means of $\hat{C}$-radicals. In addition, a necessary and sufficient condition for the equality of prime radicals is obtained.

Keywords: semiprime ideal, prime ideal, congruence of a lattice, allele, lattice polynomial, meet-irreducible element, kernel, forbidden exterior quotients, D-radical, prime radical

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1. Introduction

The notion of a semiprime ideal was introduced by Rav in [8] in the following way: An ideal I of a lattice L is said to be *semiprime* if the implication

$$(a \wedge b \in I \ \ \& \ \ a \wedge c \in I) \Rightarrow a \wedge (b \vee c) \in I$$

is true for every $a, b, c \in L$.

In a recent paper, a new method was used to characterize the semiprime ideals by means of lattice quotients. For a detailed description of the method see [3], whereas for a comparative study of this technique against a classical background see [1]. The semiprime ideals in lattices have been studied in [6], [2] and [4].

For completeness we include some definitions here.

Let a, b be elements of a lattice L. If $a \leq b$, we say that these elements form a *quotient* b/a of L. We write $b/a \sim_w d/c$ if either

$$b = a \vee d \ \ \& \ \ a \wedge d \geq c$$

or

$$a = b \wedge c \ \ \& \ \ b \vee c \leq d.$$

If there exist quotients y_i/x_i such that

$$b/a = y_0/x_0 \sim_w y_1/x_1 \sim_w \cdots \sim_w y_n/x_n = d/c,$$

we write $b/a \approx_w d/c$.

A quotient b/a is called an *allele* if there exists a quotient d/c satisfying $b/a \approx_w \approx_w d/c$ and such that either $b \le c$ or $d \le a$. The set of all the alleles of L will be denoted by $\mathbf{A}(L)$.

Let $\hat{C}(L)$ denote the smallest congruence θ of L for which the quotient lattice L/θ is distributive. It can be shown [1] that $(a,b) \in \hat{C}(L)$ if and only if there exist $a_i \in L$ satisfying

$$a_0 = a \wedge b \le a_1 \le a_2 \cdots \le a_m = a \vee b \tag{1}$$

and such that $a_{i+1}/a_i \in \mathbf{A}(L)$ for every $i = 0, 1, \ldots, m-1$.

Proposition 1. *Let I be an ideal of a lattice L. Then the following conditions are equivalent:*

(i) *the ideal I is semiprime;*
(ii) *for any $a, \tilde{a}, b$ of L,*

$$(b \wedge a \in I \ \ \& \ \ b \wedge \tilde{a} \in I \ \ \& \ \ a \vee \tilde{a} \ge b) \Rightarrow b \in I;$$

(iii) *there is no allele b/a of L with $a \in I$ and $b \notin I$;*
(iv) *for any x, y of L,*

$$(x \in I \ \ \& \ \ x \le y \ \ \& \ \ (x,y) \in \hat{C}(L)) \Rightarrow y \in I;$$

(v) *for any x, y of L,*

$$(x \in I \ \ \& \ \ (x,y) \in \hat{C}(L)) \Rightarrow y \in I;$$

(vi) *the ideal $(I]_{Id(L)}$ generated by I in the ideal lattice $Id(L)$ is semiprime.*

Proof: (i) $\Leftrightarrow$ (ii). Clearly, any semiprime ideal satisfies (ii).

Suppose now that $x \wedge y \in I$ and $x \wedge z \in I$. Put $a = y$, $\tilde{a} = z$ and $b = x \wedge (y \vee z)$. From (ii) it follows that $x \wedge (y \vee z) \in I$.

(i) $\Leftrightarrow$ (iii). This is Main Theorem of [3].

(iii) $\Leftrightarrow$ (iv) and (iv) $\Leftrightarrow$ (v). Immediate.

(i) $\Leftrightarrow$ (vi). This has been proved by Rav [8]. □

Corollary 2. (i) *Let $x \in L$. Then the principal ideal $(x]$ is semiprime if and only if there is no allele y/x with $y > x$.*

(ii) *An ideal X of L is semiprime if and only if there is no ideal Y satisfying $X \subsetneqq Y$ and $Y/X \in \mathbf{A}(Id(L))$.*

Proof: (i) Suppose that $(x]$ satisfies the condition and let q/i be an allele with $i \in (x]$. Since $(i,q) \in \hat{C}(L)$, $(x, x \vee q) \in \hat{C}(L)$. By the assumption and (1), $x \vee q \in (x]$ and so $q \in (x]$. Thus $(x]$ is semiprime.

The remainder follows from Proposition 1 (i).

(ii) Use (i) and Proposition 1 (v). □

2. Properties characterizing semiprime ideals

First we need some notation.

Let I be an ideal of L and let $M \subset L$. By M^*_I we mean the set of all $a \in L$ such that $a \wedge m \in I$ for every $m \in M$. We write m^*_I (or simply m^*) instead of $\{m\}^*_I$.

Note that the ideal I is semiprime if and only if m^*_I is an ideal of L for every $m \in L$.

Given an ideal I of L, let ψ and θ be relations defined on L in the following way:

$$(a,b) \in \psi \Leftrightarrow a^*_I = b^*_I; \ (a,b) \in \theta \Leftrightarrow (a \wedge b)^*_I = (a \vee b)^*_I.$$

The relation ψ was used by Rav in the proof of his Main Theorem in [8]. Note that $\theta \subset \psi$. However, the converse inclusion need not be true.

Theorem 3. *The following conditions are equivalent for any ideal I of a lattice L:*

(i) *The ideal I is semiprime.*
(ii) *The relation ψ satisfies $\psi \supset \hat{C}(L)$.*
(iii) *The relation θ satisfies $\theta \supset \hat{C}(L)$.*
(iv) *The relations θ and ψ satisfy $\theta = \psi \supset \hat{C}(L)$.*

Proof: (i) $\Rightarrow$ (iv). Let $a^* = b^*$ and let $z \in (a \wedge b)^*$. Then $z \wedge a \wedge b \in I$, which gives $z \wedge a \in b^* = a^*$. Hence $z \wedge a \in I$ and, similarly, $z \wedge b \in I$. Since I is semiprime, it follows that $z \wedge (a \vee b) \in I$. Consequently, $z \in (a \vee b)^*$ and this implies $(a \wedge b)^* = (a \vee b)^*$. Thus $\theta = \psi$. By [8, p. 109], L/θ is distributive and so $\theta \supset \hat{C}(L)$.

(iv) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (ii). Use $\theta \subset \psi$.

(ii) $\Rightarrow$ (i). Let $q/i \in \mathbf{A}(L)$ be such that $i \in I$. Then $(i,q) \in \hat{C}(L) \subset \psi$, and, therefore, $q^* = i^* = L$. This yields $q \in I$. □

Theorem 4. *An ideal I of a lattice L is semiprime if and only if*

$$[(a \vee b) \wedge c]^*_I \supset [a \vee (b \wedge c)]^*_I \tag{2}$$

for every $a, b, c \in I$.

Proof: Suppose I is semiprime and let $x \in [a \vee (b \wedge c)]^*$. Then $x \wedge [a \vee (b \wedge c)] \in I$, and, a fortiori,

$$x \wedge c \wedge a \in I \ \ \& \ \ x \wedge c \wedge b \in I.$$

Since I is semiprime, $x \wedge c \wedge (a \vee b) \in I$. Therefore, $x \in [(a \vee b) \wedge c]^*$.

Suppose that (2) is valid and let $a \wedge c \in I$ and $b \wedge c \in I$. Replace a in (2) by $a \wedge c$. Then

$$\{[(a \wedge c) \vee b] \wedge c\}^* \supset [(a \wedge c) \vee (b \wedge c)]^*. \tag{3}$$

Since $(a \wedge c) \vee (b \wedge c) \in I$, it is readily seen that $\{[(a \wedge c) \vee b] \wedge c\}^* = L$. Accordingly, $[(a \wedge c) \vee b] \wedge c \in I$, and, by (2), $c \in [b \vee (a \wedge c)]^* \subset [(b \vee a) \wedge c]^*$. Hence $(a \vee b) \wedge c \in I$. □

Theorem 5. *An ideal I of a lattice L is semiprime if and only if the following implication holds for every $a, b, c \in L$:*

$$[(c \wedge a)^*_I \supset (c \wedge b)^*_I \ \ \& \ \ (c \vee a)^*_I \supset (c \vee b)^*_I] \Rightarrow a^*_I \supset b^*_I. \tag{4}$$

Proof: First we shall suppose that I is semiprime. Then we can consider the quotient lattice L/ψ where ψ was defined above. If x/ψ, $y/\psi \in L/\psi$, then $x/\psi \leq y/\psi$ if and only if $x^*_I \supset y^*_I$. Hence the antecedent of (4) can be rewritten as

$$c/\psi \wedge a/\psi \leq c/\psi \wedge b/\psi \ \ \& \ \ c/\psi \vee a/\psi \leq c/\psi \vee b/\psi.$$

This, together with a result of M. Molinaro [7, p. 75], implies that $a/\psi \leq b/\psi$. Thus $a^* \supset b^*$.

Finally, let (4) be valid and let x, y and z be arbitrary elements of L. Let $a = (x \vee y) \wedge z$, $b = x \vee (y \wedge z)$ and $c = y$. Then

$$c \wedge a = y \wedge z \leq c \wedge b = y \wedge [x \vee (y \wedge z)]$$

and

$$c \vee a = y \vee [(x \vee y) \wedge z] \leq c \vee b = x \vee y.$$

Consequently we have

$$(c \wedge a)^* \supset (c \wedge b)^* \ \ \& \ \ (c \vee a)^* \supset (c \vee b)^*.$$

By assumption, $a^* \supset b^*$. From Theorem 4 we see that I is semiprime. □

Theorem 6. *An ideal I of a lattice L is semiprime if and only if for any lattice polynomial $p(x_1, x_2, \dots, x_n)$ and any choice of elements $a_1, a_2, \dots, a_n \in L$ the relations*

$$p(a_1, a_2, \dots, a_n) \in I \ \ \& \ \ a_1 \hat{C}(L) a_2 \hat{C}(L) \dots \hat{C}(L) a_n$$

imply $a_1, a_2, \dots, a_n \in I$.

Proof: Let I be semiprime and let $p(a_1, a_2, \dots, a_n) \in I$. Then

$$\begin{aligned} I &= p(a_1, a_2, \dots, a_n)/\psi = p(a_1/\psi, a_2/\psi, \dots, a_n/\psi) \\ &= p(a_1/\psi, a_1/\psi, \dots, a_1/\psi) = a_1/\psi. \end{aligned}$$

Thus $a_1 \in I$ and the same is true for the other a_i.

Now suppose that the stated implication is true and let $p(x_1, x_2) = x_1 \wedge x_2$. If $a \leq b$ are such that $a \in I$ and $(a, b) \in \hat{C}(L)$, then $p(a, b) = a \in I$. We therefore have from Proposition 1 (iv) that I is semiprime. □

3. Semiprimeness as a descriptive tool

Theorem 7. *A lattice L is distributive if and only if every principal ideal $(a]$ $(a \in L)$ is semiprime.*

Proof: Let $I = ((a \wedge b) \vee (a \wedge c)]$ be semiprime. Since $a \wedge b$ and $a \wedge c$ belong to I, we get $a \wedge (b \vee c) \in I$. Thus $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$ and we conclude that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Evidently, every ideal of a distributive lattice is semiprime. □

Theorem 8. *A lattice L is modular if and only if for any $a, b, c \in L$, the ideal $(a \vee [b \wedge (a \vee c)]]$ is a semiprime ideal of the sublattice generated by a, b, c in L.*

Proof: Suppose that L is modular and let M denote the sublattice generated by a, b, c. Then, by modularity, $I = (a \vee [b \wedge (a \vee c)]] = ((a \vee b) \wedge (a \vee c)]$. Now M is isomorphic to a quotient lattice of the free modular lattice M_{28} (see [5, p. 64]) with three generators x, y, z. However, a closer inspection of the quotient lattices of M_{28} shows that in any of these quotient lattices the ideal corresponding to $((x \vee y) \wedge (x \vee z)]$ is semiprime. Hence also our ideal I is semiprime.

Conversely, suppose the ideal $I = (a \vee [b \wedge (a \vee c)]]$ is semiprime. Note that $a \wedge (a \vee c) \in I$ and $b \wedge (a \vee c) \in I$. Consequently, $(a \vee b) \wedge (a \vee c) \in I$. Thus $(a \vee b) \wedge (a \vee c) = a \vee [b \wedge (a \vee c)]$ and L is modular. □

Theorem 9. *Let I be a semiprime ideal of a lattice L. Then I is prime if and only if I is a meet-irreducible element of the ideal lattice $Id(L)$.*

Proof: One easily shows that each prime ideal is a meet-irreducible element in $Id(L)$.

It remains to show that every semiprime ideal I which is meet-irreducible in $Id(L)$ is also prime. To do this, consider $b, c \in L$ satisfying $b \wedge c \in I$.

We first note that the inclusion in $I \subset (I \vee (b]) \cap (I \vee (c])$ can be replaced by the equality sign. Indeed, let $x \in (I \vee (b]) \cap (I \vee (c])$. Then there exist $i, j \in I$ and $b_1 \leq b$, $c_1 \leq c$ such that $x \leq (i \vee b_1) \wedge (j \vee c_1)$. Hence $x \leq (h \vee b_1) \wedge (h \vee c_1)$ where $h = i \vee j \in I$. But $b_1 \wedge c_1 \leq b \wedge c \in I$. Therefore, $h \vee (b_1 \wedge c_1) \in I$.

Now $L/\hat{C}(L)$ is distributive, and so $(h \vee (b_1 \wedge c_1), (h \vee b_1) \wedge (h \vee c_1)) \in \hat{C}(L)$. Since I is semiprime, we have, by Proposition 1 (iv), $(h \vee b_1) \wedge (h \vee c_1) \in I$. Consequently, $x \in I$. Combining this with the meet-irreducibility of I we can derive easily that either $b \in I \vee (b] = I$ or $c \in I \vee (c] = I$. □

Corollary 10. *Let $(a]$ be a semiprime ideal of a lattice L. Then $(a]$ is prime if and only if a is a meet-irreducible element of the lattice L.*

Proof: Use the fact that a is a meet-irreducible element of L if and only if $(a]$ is a meet-irreducible element of $Id(L)$. □

By [8, p. 108], any semiprime ideal of L is the kernel of a congruence of L. Hence the following lemma can be applied to semiprime ideals.

Lemma 11. *Let I be an ideal of a lattice L which is the kernel of a congruence θ of L.*

Then

$$(I \wedge J \supset K \wedge J \ \ \& \ \ I \vee J \supset K \vee J) \Rightarrow I \supset K$$

for any ideals J, K of L.

PROOF: Let $k \in K$. Since $K \subset I \vee J$, there exist $i \in I$ and $j \in J$ such that $k \leq i \vee j$. At the same time, $j \wedge k \in J \wedge K \subset I$. Hence $(i, j \wedge k) \in \theta$ and, consequently, $(j, i \vee j) \in \theta$. From $j \leq j \vee k \leq i \vee j$ it follows that $(j, j \vee k) \in \theta$. But then $(j \wedge k, k) \in \theta$. Since I is the kernel of θ and $j \wedge k \in I$, we get $k \in I$. □

Lemma 12. *Let I be a semiprime ideal of a lattice L and let $a, b \in L$ be such that $a \wedge b \in I$.*

Then either $(a] \vee I \neq L$ or

$$(a] \vee I = L \ \ \& \ \ b \in I.$$

PROOF: Suppose that $(a] \vee I = L$. Put $J = (a]$, $K = (b]$ and use Lemma 11. It follows that $b \in K \subset I$. □

The following theorem generalizes a result of Chevalier [6, p. 383] stated for orthomodular lattices.

Theorem 13. *Let L be a relatively complemented lattice. Then a proper ideal I of L is prime if and only if it is a maximal semiprime ideal of L.*

PROOF: It is well-known that in a relatively complemented lattice every proper prime ideal is maximal.

What remains to be shown is that any maximal semiprime ideal $I \neq L$ is prime. Let I be an ideal having these properties and let $a \wedge b \in I$ for some $a, b \in I$.

Suppose first that

$$(a] \vee I \neq L \ \ \& \ \ a \notin I. \tag{5}$$

Then $(a] \vee I$ is not semiprime and, by Proposition 1 (iv), there exist $p \in (a] \vee I$ and $q \notin (a] \vee I$ such that $(p, q) \in \hat{C}(L)$ with $p \leq q$. But $p \in (a] \vee I$ means that $p \leq a \vee i$ for a suitable $i \in I$. Now

$$p \leq q \wedge (a \vee i) \leq q \ \ \& \ \ (p, q) \in \hat{C}(L).$$

Hence $(q \wedge (a \vee i), q) \in \hat{C}(L)$ and, therefore,

$$(a \vee i, q \vee a \vee i) \in \hat{C}(L). \tag{6}$$

Let r^+ be a relative complement of $a \vee i$ in the interval $[i, a \vee i \vee q]$. From (6) we can see that $(i, r^+) \in \hat{C}(L)$. If r^+ belonged to I, then $r^+ \vee a \vee i$ would belong to $(a] \vee I$. But then

$$q \leq a \vee i \vee q = r^+ \vee a \vee i \in (a] \vee I,$$

a contradiction.

Thus $r^+ \notin I$, $i \in I$ and, moreover, $(i, r^+) \in \hat{C}(L)$. But this contradicts Proposition 1 (iv).

We may therefore assume that (5) and a similar statement for b are not true.

However, if $(a] \vee I = L$ or $(b] \vee I = L$, then we can use Lemma 12. Thus either $a \in I$ or $b \in I$ and we are done. □

We now turn our attention to the prime radicals. Recall [8, p. 111] that the *prime radical* rad(I) of an ideal I in a lattice L is the intersection of all the semiprime ideals of L which contain I.

There is a simple way how to generalize this notion [4]: Given any lattice L, let $D(L)$ denote a congruence of L and let D be the class of all these congruences. We shall say that an ideal I of L is an *ideal with forbidden exterior quotients* in D, if the implication

$$(a \leq b \ \ \& \ \ (a, b) \in D(L) \ \ \& \ \ a \in I) \Rightarrow b \in I$$

is true for any choice of a and b in L.

From Proposition 1 (iv) we conclude that an ideal I is semiprime if and only if it is an ideal with forbidden exterior quotients in $\hat{C}$ where $\hat{C}$ denotes the class of all congruences $\hat{C}(L)$.

If I is an ideal of L, we put

$$\Gamma_D(I) = \{x \in L; (\exists i) i \in I \ \ \& \ \ (i, x) \in D(L)\}$$

calling it the *D-radical* of I.

Proposition 14. *The D-radical of an ideal I is equal to the intersection of all the ideals with forbidden exterior quotients in D containing I.*

Proof: Straightforward. □

Corollary 15. *The $\hat{C}$-radical of any ideal I in a lattice L is equal to the prime radical of I.* □

Let I and J be ideals of a lattice L. If $\Gamma_D(I) \subset \Gamma_D(J)$, then it is clear that for any $i \in I$ there exists $j \in J$ such that $(i, j) \in D(L)$. From this remark we could deduce directly a simple characterization of the case where $\Gamma_D(I) = \Gamma_D(J)$. However, there is another approach which seems to be more fruitful:

Theorem 16. *The following two conditions on ideals I, J of a lattice L are equivalent:*

(i) $\Gamma_D(I) = \Gamma_D(J)$.

(ii) *For any $i \in I$ and any $j \in J$ there exist $i_1 \in I$ and $j_1 \in J$ such that*

$$i \leq i_1 \ \ \& \ \ j \leq j_1 \ \ \& \ \ (i_1, j_1) \in D(L).$$

Proof: Suppose first that $\Gamma_D(I) = \Gamma_D(J)$ and let $i \in I$, $j \in J$.

Since $i \in \Gamma_D(I) \subset \Gamma_D(J)$, there exists $j_2 \in J$ such that (i, j_2) belongs to $D(L)$. Then $(i \vee j, j_2 \vee j) \in D(L)$. It follows from $j_2 \vee j \in \Gamma_D(J) \subset \Gamma_D(I)$ that there exists $i_2 \in I$ such that $(i_2, j \vee j_2) \in D(L)$. Hence

$$(i \vee i_2, i \vee j \vee j_2) \in D(L) \ \ \& \ \ (i \vee j \vee i_2, i \vee j \vee j_2) \in D(L). \tag{7}$$

Now $i \vee i_2 \in \Gamma_D(I) \subset \Gamma_D(J)$ and so there is $j_3 \in J$ with $(i \vee i_2, j_3) \in D(L)$. Therefore,

$$(i \vee i_2 \vee j, j_3 \vee j) \in D(L). \tag{8}$$

Put $i_1 = i \vee i_2$, $j_1 = j \vee j_3$. Then using (7) and (8), we get $(i_1, j_1) \in D(L)$ and it is evident that $i \leq i_1$ and $j \leq j_1$.

Next suppose conversely that I and J satisfy the condition (ii). By symmetry, it is sufficient to prove that $\Gamma_D(I) \subset \Gamma_D(J)$.

Let $x \in \Gamma_D(I)$. Then there exists $i \in I$ with $(x, i) \in D(L)$. Let j be an element of J. By the assumption, there are $i_1 \geq i$, $j_1 \geq j$ such that $(i_1, j_1) \in D(L)$. However, from $(x, i) \in D(L)$ we obtain $(x \vee i_1 \vee j_1, i_1 \vee j_1) \in D(L)$. Similarly, $(i_1, j_1) \in D(L)$ implies that $(i_1 \vee j_1, j_1) \in D(L)$. Therefore, $(x \vee i_1 \vee j_1, j_1) \in D(L)$ and, consequently, $x \vee i_1 \vee j_1 \in \Gamma_D(J)$. Since $\Gamma_D(J)$ is an ideal, we have $x \in \Gamma_D(J)$. □

Corollary 17. *Let a, b be elements of a lattice L. Then*

(i) *the D-radical $\Gamma_D((a])$ is equal to the D-radical $\Gamma_D((b])$ if and only if $(a, b) \in D(L)$;*
(ii) *the prime radical $\operatorname{rad}((a])$ is equal to the prime radical $\operatorname{rad}((b])$ if and only if $(a, b) \in \hat{C}(L)$.*

Proof: (i) Suppose $\Gamma_D((a]) = \Gamma_D((b])$. By Theorem 16, there are a_1, b_1 such that

$$a \leq a_1 \ \ \& \ \ b \leq b_1 \ \ \& \ \ a_1 \in (a] \ \ \& \ \ b_1 \in (b] \ \ \& \ \ (a_1, b_1) \in D(L).$$

Hence $(a, b) \in D(L)$.

Conversely, suppose $(a, b) \in D(L)$. For any $i \in (a]$ and $j \in (b]$ we then can put $i_1 = a$, $j_1 = b$ and use Theorem 16.

(ii) Now immediate. □

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