On isometric embeddings of Hilbert's cube into c

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Abstract. In our note, we prove the result that the Hilbert's cube equipped with l_p-metrics, $p \geq 1$, cannot be isometrically embedded into c.

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1. Introduction

Aharoni [1] proved that every separable metric space can be Lipschitz embedded into c_0. His proof was simplified by Assouad [2] who also improved the Lipschitz constants given by Aharoni's construction. This fact was further generalized by Pelant in [3] using the theorem that metric spaces uniformly homeomorphic to subspaces of some $c_0(\kappa)$ are exactly those satisfying the A.H. Stone paracompactness theorem in a uniform way, i.e. in which for any uniform cover, one can find a uniform refinement which is locally finite. Some further improvements of Lipschitz constants were given in [3]. For Lipschitz embeddings of compact metric spaces into c_0, these improvements give the best possible estimates, i.e. for any compact metric space (X, d) and any $\varepsilon > 0$, there is $F : X \to c_0$, s.t.

$$\frac{1}{1+\varepsilon} d(x,y) \leq ||F(x) - F(y)||_{c_0} \leq d(x,y) \text{ for each } x, y \in X.$$

On the other hand, it is shown in [3] that the Hilbert's cube equipped with l_1-metrics cannot be isometrically embedded into c_0.
In our note, we prove the analogous result for the Hilbert's cube endowed by l_p-metrics, $p \geq 1$ and the space c. Moreover, we show that there exists a compact subset of c which cannot be isometrically embedded into c_0, i.e. there is a nonformal difference between c and c_0.

2. Notation and results

Let I be a closed unit interval $[0, 1]$ and as usually $I^{\aleph_0}$ be the Hilbert's cube. For $p \geq 1$, $I^{\aleph_0}$ constitutes the metric space $I_p = (I^{\aleph_0}, \rho_p)$ by the metric ρ_p, where for each $x, y \in I^{\aleph_0}$

$$\rho_p(x,y) = (\sum_{i=1}^{\infty} \frac{|x_i - y_i|^p}{2^i})^{\frac{1}{p}}.$$

Let c be the set of all real sequences $x = \{\xi_n\}$ such that finite $\lim_{n\to\infty} \xi_n = \xi_\infty$ exists, endowed by the norm $||x|| = \sup_n |\xi_n|$. On a normed linear space $(c, ||\quad||)$ we consider the induced metric $\sigma(x, y) = ||x - y||$. A subspace c_0 consists of all sequences $x = \{\xi_n\}$ such that $\lim_{n\to\infty} \xi_n = 0$. In the metric space X, $B_X(r, s)$ denotes the closed ball of the center $r \in X$ and the radius s, $S_X(r, s)$ denotes its sphere. By $x = \{\xi\}$ we mean a constant sequence. Recall that using Ascoli-Arzela Theorem, we have a characterization of a relatively compact infinite subset of c.

Proposition. *A set $\{x_\alpha\}_{\alpha\in\mathcal{A}} = \{\{\xi_{\alpha,m}\}\}_{\alpha\in\mathcal{A}}$ is a relatively compact subset of c, if the following two conditions are satisfied:*

$$\{x_\alpha\}_{\alpha\in\mathcal{A}} \text{ is equi-bounded, i.e., } \sup_{\alpha\in\mathcal{A}} ||x_\alpha|| < \infty,$$

$$\{x_\alpha\}_{\alpha\in\mathcal{A}} \text{ is uniformly convergent, i.e., } \lim_{n\to\infty} \sup_{\substack{\alpha\in\mathcal{A}\\ m\geq n}} |\xi_{\alpha,m} - \xi_{\alpha,\infty}| = 0.$$

□

Theorem 1. *There exists a compact set K in c which cannot be isometrically embedded into c_0.*

Proof: Let K contain $\{0\}$ and the sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ of elements of c defined by the equalities

(i) $a_k = \{\alpha_l\}$, $\alpha_k = 1 + \frac{1}{2^k}$, $\alpha_l = 1$ for $l \neq k$,

(ii) $b_k = \{\beta_l\}$, $\beta_l = -\alpha_l$ for each l.

By Proposition, the reader can easily verify that K is a compact subset of c and for different positive integers k, l, we have from (i), (ii)

(iii) $\sigma(a_k, a_l) = \sigma(b_k, b_l) = \frac{1}{2^{\min(k,l)}}$, $\quad \sigma(a_k, b_l) = 2 + \frac{1}{2^{\min(k,l)}}$
$\sigma(a_k, \{0\}) = \sigma(b_k, \{0\}) = 1 + \frac{1}{2^k}$, $\quad \sigma(a_k, b_k) = 2 + \frac{1}{2^{k-1}}$.

Suppose that an isometry F from K into c_0 exists. Without loss of generality we can assume that $\{0\}$ is a fixed point of an isometry F. Denote all images in $F(K)$ by 'tilde', i.e. $F(K) = \tilde{K}$ and $F(a_k) = \tilde{a}_k$ for $a_k \in K$. Since the property of F, $\tilde{K}$ is a compact subset of $c_0 \subset c$ and an analogous equalities as (iii) can be written for elements of $\tilde{K}$. By Proposition $\tilde{K}$ is uniformly convergent and there exists a positive integer k_0 such that for each $\tilde{x} = \{\tilde{\xi}_n\} \in \tilde{K}$

(iv) $\sup_{n>k_0} |\tilde{\xi}_n| < \frac{1}{2}$.

Consider a pair $\tilde{a}_k = \{\tilde{\alpha}_l\}$, $\tilde{b}_k = \{\tilde{\beta}_l\}$ from $\tilde{K}$. Since $\sigma(\tilde{a}_k, \tilde{b}_k) = 2 + \frac{1}{2^{k-1}}$ there exists $l_1 \in \{1, 2, ..., k_0\}$ such that

(v) $|\tilde{\alpha}_{l_1} - \tilde{\beta}_{l_1}| = 2 + \frac{1}{2^{k-1}}$, $|\tilde{\alpha}_{l_1}| = |\tilde{\beta}_{l_1}| = 1 + \frac{1}{2^k}$.

Because $\tilde{K}$ is infinite and the condition (iv) holds, we have the equalities (v) with an index l_1 for infinitely many $\{k_i\}$ and pairs $\tilde{a}_{k_i}, \tilde{b}_{k_i}$. Then for $i \neq j$ either $\sigma(\tilde{a}_{k_i}, \tilde{a}_{k_j}) = 2 + \frac{1}{2^{k_i}} + \frac{1}{2^{k_j}}$ or $\sigma(\tilde{a}_{k_i}, \tilde{b}_{k_j}) = 2 + \frac{1}{2^{k_i}} + \frac{1}{2^{k_j}}$. Since F is distance-preserving and by (iii) we have a contradiction.

□

Theorem 2. *There is no isometric embedding of $I_p = (I^{\aleph_0}, \rho_p)$ to (c, σ).*

Proof: To the contrary suppose that such an isometric embedding $F : I_p \to c$ exists. Without loss of generality we can assume that $F(\{\frac{1}{2}\}) = \{0\}$. Using a notation stated above we can write

(vi) $B_{I_p}(\{\frac{1}{2}\}, \frac{1}{2}) = I_p$, $F(S_{I_p}(\{\frac{1}{2}\}, \frac{1}{2})) \subset S_c(\{0\}, \frac{1}{2})$.

It is clear from the definitions of metrics ρ_p, σ that

(vii) $S_{I_p} = S_{I_p}(\{\frac{1}{2}\}, \frac{1}{2}) = \{0,1\}^{\aleph_0}$, $S_c = S_c(\{0\}, \frac{1}{2}) \subset [-\frac{1}{2}, \frac{1}{2}]^{\aleph_0}$,

(viii) for any $x \in S_{I_p}$ there exists a single opposite $y \in S_{I_p}$ with $\rho_p(x, y) = 1$.

In what follows we shall denote this opposite element by x'. The sphere S_c can be divided to three disjoint sets K, L, M by the way $K = \{\, x \in S_c,\ \lim |x_i| < \frac{1}{2}\}$, $L = \{\, x \in S_c,\ \lim x_i = \frac{1}{2}\}$, $M = \{\, x \in S_c,\ \lim x_i = -\frac{1}{2}\}$. Note that

(ix) $\mathrm{card}\{\, i,\ |x_i| = \frac{1}{2}\} < \infty$ for each $x \in K$.

Let for $x \in S_c$, $E_+(x) = \{\, i,\ x_i = \frac{1}{2}\}$, $E_-(x) = \{\, i,\ x_i = -\frac{1}{2}\}$ and define on S_c the equivalence relation by the following : elements $x, y \in S_c$ are equivalent if and only if $E_+(x) = E_+(y)$ and $E_-(x) = E_-(y)$. According to (ix) there is a countable set of the equivalence classes τ_k which forms the decomposition $\{\tau_k\}$ of K. So, we have

(x) $S_c = (\cup \tau_k) \cup L \cup M$.

Because of (vii) the set S_{I_p} is uncountable, hence we have from (vi), (x) that one of the following cases must be realized :

I. There is a positive integer k such that $\mathrm{card}(\tau_k \cap F(S_{I_p})) \geq 2$.

Choose different $a, b \in (\tau_k \cap F(S_{I_p}))$. Since σ is a metric and a, b are equivalent ($\in K$) the relations

(xi) $0 < \sigma(a, b) < 1$

hold. By (viii) for $x \in S_{I_p}$, $x = F^{-1}(a)$, there exists $x' \in S_{I_p}$ with the property $\rho_p(x, x') = 1$. If we put $d = F(x')$, then

$\sigma(a, d) = \rho_p(x, x') = 1$,

hence by (xi) $b \neq d$. Now, the reader can easily verify that because of $E_+(a) = E_+(b)$ and $E_-(a) = E_-(b)$ we even have

$\sigma(b, d) = \rho_p(F^{-1}(b), x') = 1$.

This implies $F^{-1}(b) = F^{-1}(a) = x$, hence we have $a = b$. But this is a contradiction.

II. The set $L \cap F(S_{I_p})$ is uncountable and $\operatorname{card}(\tau_k \cap F(S_{I_p})) \leq 1$ for every positive integer k.
Then first of all by (viii), $M \cap F(S_{I_p}) = \emptyset$ and for all but countably many $a \in (L \cap F(S_{I_p}))$ an opposite element $a'' = F((F^{-1}(a))')$ which is guaranteed by (viii) belongs again to $L \cap F(S_{I_p})$. Thus, if we define for $n \in \mathbb{N}$ the sets G_n by

$$G_n = \{\ a \in (L \cap F(S_{I_p})),\ \inf_{i \geq n+1} a_i > -\frac{1}{2}\ \&\ \inf_{i \geq n+1} a_i'' > -\frac{1}{2}\},$$

there exists $m \in \mathbb{N}$ for which G_m is infinite (uncountable). Similarly as above the reader can easily see that there exist two different elements a, b in G_m such that $E_+(a) = E_+(b)$ and $E_-(a) = E_-(b)$, i.e.

$$\sigma(a, a'') = \sigma(b, a'') = 1.$$

Hence we have a contradiction.

III. The set $M \cap F(S_{I_p})$ is uncountable and $\operatorname{card}(\tau_k \cap F(S_{I_p})) \leq 1$ for every positive integer k.
This case is analogous to the previous one.
The proof of Theorem 2 is finished. □

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