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Properties of function algebras in terms of their orthogonal measures

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Abstract. In the present note, we characterize the pervasive, analytic, integrity domain and the antisymmetric function algebras respectively, defined on a compact Hausdorff space X , in terms of their orthogonal measures on X .

Keywords: compact Hausdorff space X , the sup-norm algebra $C(X)$ of all complex-valued continuous functions on X , its closed subalgebras called function algebras, pervasive (analytic, integrity domain, antisymmetric) function algebra, measure orthogonal to a function algebra

Classification: 46J10

Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra, consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a function algebra on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X .

A function algebra A on X is said to be:

pervasive whenever for any nonvoid proper closed subset F of X , the algebra A/F of all restrictions of functions in A to the set F is dense in $C(F)$ with respect to $|\cdot|_F$, the sup-norm on F ;

analytic whenever any function f in A which vanishes on a nonvoid open subset of X vanishes identically;

integrity domain whenever A has no nontrivial divisors of zero, i.e. whenever $fg = 0$ for f, g in A implies either $f = 0$ or $g = 0$;

antisymmetric provided it satisfies the following condition: any function in A , which is real-valued, is a constant one.

The mentioned notions are due to Helson, Quigley [1] and Hoffman, Singer [2].

Denote by $M(X)$ the space of all complex Borel regular measures on X , i.e. by the Riesz Representation Theorem, the dual space of $C(X)$.

Whenever A is a closed subspace of $C(X)$, let A^\perp be the annihilator of A , or the set of all measures m in $M(X)$ such that $\int f dm = 0$ for any f in A . The dual space A' of A is then canonically isomorphic to the quotient space $M(X)/A^\perp$.

Now endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

Our aim here is to characterize the above-mentioned properties of a function algebra A by means of the properties of the measures in A^\perp .

We have dealt with a part of these problems in [3].

Theorem 1 (cf. [3]). *A function algebra A on X is pervasive if and only if the closed support of any nontrivial measure in A^\perp is all X .*

PROOF: Let A be pervasive. Fix an arbitrary m in $M(X)$ such that $\emptyset \neq \text{spt } m = F \neq X$, and prove that m does not annihilate A . Without loss of generality, we may assume that $|m|$, the total variation of m on X , is equal to one. Take an f in $C(F)$ such that $\int_F f dm = 1$ and choose a g in A such that $|f - g|_F < 1$. Then

$$|1 - \int g dm| = | \int (f - g) dm | \leq |f - g|_F |m| < 1.$$

It follows that m does not annihilate g .

Let, on the contrary, A be not pervasive. Then there exists a closed nonvoid proper subset F of X and a function f in $C(F)$ which is not in the closure of A/F in the sup-norm on F . By the Hahn-Banach Separating Theorem there is an m in $M(F)$ such that $\int_F f dm \neq 0$ while $\int_F g dm = 0$ for any g in A . If \mathbf{m} is the trivial extension of m from F to X , then \mathbf{m} is in A^\perp . Indeed, $\int g d\mathbf{m} = \int_F g dm = 0$ for an arbitrary g in A . But $\text{spt } \mathbf{m} = \text{spt } m \subset F \neq X$ and, m being nontrivial, \mathbf{m} is not trivial, too. \square

Remark. Let us mention that the proof above does also work in case of A being merely a function space, not necessarily an algebra.

For a Borel set $G \subset X$, denote by $M(G)$ the set $M(X)/G$ of all restrictions of measures on X to the set G ; for m in $M(G)$ let us not distinguish between m and its trivial extension to X .

For a closed set $F \subset X$, denote by I_F the ideal of all functions in $C(X)$ which vanish on F .

Now, let us search for the dual space I'_F of I_F .

Any measure on F annihilates I_F , this implies $I^\perp_F \subset M(F)$. F is closed in X and then $M(F)$ is weak-star closed in $M(X)$. Suppose that there is an m in I^\perp_F which is not supported by F and take, by the Mazur Separation Theorem for locally convex spaces, an f in $C(X)$ such that $\int f dm = 1$ while $\int f dn = 0$ for any n in $M(F)$. But then f is in I_F , a contradiction.

So $I^\perp_F = M(F)$ and $m \rightarrow m + M(F)$ is a canonical isomorphism of $M(X - F)$ onto the quotient space $M(X)/M(F) = I'_F$. If we endow $M(X - F)$ with the w^\perp_F -topology, i.e. the weak-star topology generated by all functions in I_F , it becomes a locally convex space, and the ideal I_F becomes its dual space.

Proposition. *Let A be a function algebra on X and let F be a closed nonvoid subset of X . Then the following two conditions are equivalent.*

- (i) *The intersection $I_F \cap A$ consists of the zero function only;*
- (ii) *The set $A^\perp/X - F$ is dense in $M(X - F)$ with respect to the w^\perp_F -topology.*

PROOF: Let (ii) be not valid. Take m in $M(X - F)$ but off $\overline{A^\perp/X - F}$, where the bar denotes the w_F^\perp -closure. Then there is an f in I_F such that $\int f dm = 1$ while $\int f dn = 0$ for all n in $A^\perp/X - F$.

Now take an arbitrary p in A^\perp . Then, f being equal to zero identically on F ,

$$\int f dp = \int f d(p/X - F) = 0$$

and f is in A . But m does not annihilate f , hence $f \neq 0$. The condition (i) fails.

Let (i) be not valid. Take f in $I_F \cap A$ which does not vanish identically on X . Let m in $M(X/F)$ be such that $\int f dm = 1$ (e.g. a one-point mass). For any n in A^\perp , we have

$$0 = \int f dn = \int f d(n/X - F),$$

so that f annihilates $A^\perp/X - F$, and, by the continuity, annihilates its closure, too. But f does not annihilate m , m is not in the closure of $A^\perp/X - F$, and the condition (ii) fails. \square

Now, let us formulate two simple consequences of the Proposition.

Theorem 2. *A function algebra A on X is analytic if and only if the following condition is fulfilled:*

For an arbitrary open G , which is not dense in X , the set A^\perp/G is dense in $M(G)$ with respect to the w_{X-G}^\perp -topology.

Theorem 3. *A function algebra A on X is an integrity domain if and only if the following condition is fulfilled:*

Whenever G_1, G_2 , is a disjoint couple of open non-dense subsets of X , then the space A^\perp/G_i is $w_{X-G_i}^\perp$ -dense in $M(G_i)$ either for $i = 1$ or for $i = 2$.

Now, denote by $\text{Re } C(X)$, $\text{Re } M(X)$, the space of all real parts of the elements in $C(X)$, $M(X)$, respectively. Then $\text{Re } M(X)$ is the real dual space of $\text{Re } C(X)$. Endow $\text{Re } M(X)$ with the weak-star topology. $\text{Re } M(X)$ becomes a real locally convex space with the real dual space $\text{Re } C(X)$.

Finally, denote by M_0 the set of all measures m in $M(X)$ such that $\int dm = 0$; M_0 is then the annihilator of constant functions on X and $\text{Re } M_0$ is the real annihilator of real (or all, the same) constant functions on X .

Theorem 4. *Let A be a function algebra on X . Then A is antisymmetric if and only if the set $\text{Re } A^\perp = \{\text{Re } m, m \text{ in } A^\perp\}$ is dense in $\text{Re } M_0$ with respect to the weak-star topology.*

PROOF: Let m be in A^\perp . Then

$$0 = \int 1 dm = \int d\text{Re } m + i \int d\text{Im } m$$

and both $\text{Re } m$, $\text{Im } m$, are in $\text{Re } M_0$, so $\text{Re } A^\perp \subset \text{Re } M_0$.

Suppose $\text{Re } A^\perp$ is not dense in $\text{Re } M_0$ and fix an m in $\text{Re } M_0$ but off the weak-star closure of $\text{Re } A^\perp$. Take, by the Hahn-Banach Separating Theorem (or Mazur Theorem, more precisely) an f in $\text{Re } C(X)$ such that

$$\int f dm \neq 0 \text{ and } \int f d\text{Re } n = 0 \text{ for any } n \text{ in } A^\perp.$$

If n is in A^\perp , then both $\text{Re } n$, $\text{Im } n$, are in $\text{Re } A^\perp$, so that

$$\int f dn = \int f d\text{Re } n + i \int f d\text{Im } n = 0$$

and f annihilates A^\perp and so f lies in A . But m annihilates constants and does not annihilate f , this implies that f is not constant, so A is not antisymmetric.

Conversely, let f be a non-constant real function in A . There is a measure m in $\text{Re } M(X)$ such that

$$\int f dm \neq 0 \text{ and } \int dm = 0.$$

Then m is in M_0 . For an arbitrary n in A^\perp ,

$$\int f d\text{Re } n = \text{Re } \int f dn = 0,$$

so f annihilates $\text{Re } A^\perp$ and then, by a continuity reason, it annihilates the whole $\text{Re } A^\perp$ as well. But f does not annihilate m , thus m is not in $\text{Re } A^\perp$, consequently, $\text{Re } A^\perp$ is not dense in $\text{Re } M_0$, and Theorem 4 is proved. \square

Remark Two. Both proofs, of Theorems 2 and 4, are valid also in the case of mere function spaces. Of course, the notion of “integrity domain” in case of a function space which is not an algebra does not make sense.

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