A note on linear mappings between function spaces

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Abstract. Arhangel'skiĭ proved that if X and Y are completely regular spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then X is pseudocompact if and only if Y is pseudocompact. In addition he proved the same result for compactness, σ-compactness and realcompactness. In this paper we prove that if $\phi : C_p(X) \to C_p(Y)$ is a continuous linear surjection, then Y is pseudocompact provided X is and if ϕ is a continuous linear injection, then X is pseudocompact provided Y is. We also give examples that both statements do not hold for compactness, σ-compactness and realcompactness.

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1. Introduction.

Let X be a completely regular space. By $C(X)$ we denote the set of all real-valued continuous functions on X. We endow $C(X)$ with the topology of pointwise convergence and denote that by $C_p(X)$. The function space $C_p(X)$ is a topological vector space which is a dense subspace of $\mathbb{R}^X$ with the product topology. Two function spaces $C_p(X)$ and $C_p(Y)$ are *linearly homeomorphic* if there exists a homeomorphism between $C_p(X)$ and $C_p(Y)$ which is also linear.

In [1], Arhangel'skiĭ proved the following

Theorem 1.1. *Let X and Y be completely regular spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then*

(a) *X is pseudocompact if and only if Y is pseudocompact.*
(b) *X is compact if and only if Y is compact.*
(c) *X is σ-compact if and only if Y is σ-compact.*
(d) *X is realcompact if and only if Y is realcompact.*

Instead of considering linear homeomorphisms one can also look at continuous linear surjections or continuous linear injections. In this paper we show that if $\phi : C_p(X) \to C_p(Y)$ is a continuous linear surjection then Y is pseudocompact provided X is and if ϕ is injective instead of surjective, X is pseudocompact provided Y is. Easy examples show that both statements are false for compactness, σ-compactness or realcompactness. Before we can prove our results we need some auxiliary results.

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Let X and Y be completely regular spaces, let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function and let $y \in Y$ be fixed. The function $\psi_y : C_p(X) \to \mathbb{R}$ defined by $\psi_y(f) = \phi(f)(y)$ is continuous and linear, hence an element of $L(X)$ the dual of $C_p(X)$. The evaluation mappings ξ_x ($x \in X$) defined by $\xi_x(f) = f(x)$ for $f \in C_p(X)$ form a Hamel basis for $L(X)$ (for a proof of this well-known fact we refer to [2]), so for $\psi_y \neq 0$ there are $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R} \setminus \{0\}$ such that $\psi_y = \sum_{i=1}^n \lambda_i \xi_{x_i}$. We define supp (y), the *support* of y in X, to be the finite set $\{x_1, \dots, x_n\} \subset X$. Note that for all $f \in C_p(X)$ we have $\phi(f)(y) = \sum_{i=1}^n \lambda_i f(x_i)$. We usually write $\phi(f)(y) = \sum_{z \in \text{supp } (y)} \lambda_z f(z)$, where λ_z has its obvious meaning. If $\psi_y = 0$, the support of y is defined to be the empty set. For $A \subset Y$ we denote $\bigcup\{\text{supp } (y) : y \in A\}$ by supp A. The notion of support here is a special case of a more general definition given by Arhangel'skiĭ in [1]. If $A \subset X$, we say that A is *bounded* in X if for every $f \in C_p(X)$ we have $f(A)$ is a bounded subset of $\mathbb{R}$. In [1], Arhangel'skiĭ proved the following

Proposition 1.2. *Let X and Y be completely regular spaces and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. If A is a bounded subset of Y, then* supp A *is a bounded subset of X.*

2. The results.

Lemma 2.1. *Let X and Y be completely regular spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear surjection. Then for each closed and bounded subset K of X, the set $L = \{y \in Y : \text{supp } (y) \subset K\}$ is a closed and bounded subset of Y.*

Proof: We first prove that L is closed. Take any $y \notin L$. There is $x \in \text{supp } (y)$ such that $x \notin K$. Find any $f \in C(X)$ such that $f(x) = 1$ and $f(K \cup (\text{supp } (y) \setminus \{x\})) = 0$. Then $\phi(f)(y) = \sum_{z \in \text{supp } (y)} \lambda_z f(z) = \lambda_x \neq 0$. Let $W = \{z \in Y : \phi(f)(z) \neq 0\}$. Obviously $y \in W$. If $z \in W \cap L$ we have on the one hand $\phi(f)(z) \neq 0$ and on the other hand supp $(z) \subset K$ which implies $f(\text{supp } (z)) = 0$, hence $\phi(f)(z) = 0$.

To prove that L is also bounded in Y we assume the contrary. Let $h : Y \to \mathbb{R}$ be continuous such that $h(L)$ is an unbounded subset of $\mathbb{R}$. Find $t_n \in h(L) \setminus \{0\}$ such that $\{t_n : n \in \mathbb{N}\}$ is a closed and discrete subset of $\mathbb{R}$. For each $n \in \mathbb{N}$ let $y_n \in L$ be such that $h(y_n) = t_n$ and $s_n = n \cdot \sum_{z \in \text{supp } (y_n)} |\lambda_z|$. Note that $s_n > 0$ since the surjectivity of ϕ implies supp $(y_n) \neq \emptyset$. There exists a continuous $g : \mathbb{R} \to \mathbb{R}$ such that $g(t_n) = s_n$. Since ϕ is surjective, there is $f \in C(X)$ such that $\phi(f) = g \circ h$. By boundedness of K, there is $c \in \mathbb{R}$ such that $f(K) \subset [-c, c]$. But now for $n > c$ we have,

$$\begin{aligned} s_n = \phi(f)(y_n) &= |\sum_{z \in \text{supp } (y_n)} \lambda_z \cdot f(z)| \\ &\leq \sum_{z \in \text{supp } (y_n)} |\lambda_z| \cdot |f(z)| \leq c \cdot \sum_{z \in \text{supp } (y_n)} |\lambda_z| < s_n \end{aligned}$$

which is a contradiction. □

Corollary 2.2. *Let X and Y be completely regular spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear surjection. Then Y is pseudocompact provided X is.*

PROOF: A space is pseudocompact if and only if it is bounded in itself. By definition we have $Y = \{y \in Y : \text{supp } (y) \subset X\}$. □

Recall that a *μ-space* is a space in which every closed and bounded set is compact.

Corollary 2.3. *Let X and Y be completely regular spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear surjection. If Y is a μ-space, then*

(a) *If X is compact, then Y is compact.*
(b) *If X is σ-compact, then Y is σ-compact.*

PROOF: Part (a) follows from Corollary 2.2 and the definition of a μ-space. For part (b) suppose $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is compact and $X_n \subset X_{n+1}$. Let $Y_n = \{y \in Y : \text{supp } (y) \subset X_n\}$. Then by Lemma 2.1 and the definition of a μ-space, Y_n is compact. Since $Y_n \subset Y_{n+1}$ and the support of a point is always finite, $Y = \bigcup_{n=1}^{\infty} Y_n$. □

Corollary 2.4. *Let X and Y be completely regular spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear surjection. If Y is normal and X is σ-pseudocompact, then Y is σ-pseudocompact.*

PROOF: The proof is essentially the same as the proof of part (b) of the previous corollary. Note that a σ-pseudocompact space can always be written as a countable union of closed and bounded subsets and note that in normal spaces the closed and bounded subspaces are pseudocompact. □

Corollary 2.2 is not only stronger than Theorem 1.1 (a) but our proof also provides a new one for Theorem 1.1 (a) than the original in [1]. It is however not possible to do this for the other properties in Theorem 1.1. The following example shows that we have to add some assumptions on the space Y as we did in Corollary 2.3.

Example 2.5. Let X be any pseudocompact space which is not σ-compact. Then X is also not compact and realcompact. Let $\phi : C_p(\beta X) \to C_p(X)$ be defined by $\phi(f) = f \mid X$. Then ϕ is obviously a continuous linear function. Since X is pseudocompact any real-valued continuous function on X is bounded hence can be extended over βX. This implies the surjectivity of ϕ. Note that ϕ is also injective which makes ϕ a continuous bijection. By Theorem 1.1 (a) ϕ cannot be a homeomorphism, a fact which can also be verified directly.

Lemma 2.6. *Let X and Y be completely regular spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear injection. Then* supp Y *is dense in X.*

PROOF: Suppose $x \notin \overline{\text{supp } Y}$. Let $f \in C(X)$ be such that $f(x) = 1$ and $f(\overline{\text{supp } Y}) = 0$. For each $y \in Y$ we have $f(\text{supp } (y)) = 0$ hence $\phi(f)(y) = \sum_{z \in \text{supp } (y)} \lambda_z f(z) = 0$. But then $\phi(f) = 0$ contradicting the fact that ϕ is injective. □

Corollary 2.7. *Let X and Y be completely regular spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear injection. Then X is pseudocompact provided Y is.*

PROOF: By Lemma 2.6, supp Y is dense in X. Since Y is pseudocompact, by Proposition 1.2 supp Y is bounded in X hence X is pseudocompact. □

It follows that this corollary, and hence Theorem 1.1 (a), is almost an immediate consequence of Arhangel'skiĭ's Proposition 1.2. The proof of Theorem 1.1 (a) is however completely different. Again our result does not hold for the other properties.

Example 2.8. The ordinal space ω_1 is pseudocompact but not σ-compact. Define $\phi : C_p(\omega_1) \to C_p(\omega_1 + 1)$ by

$$\phi(f)(\gamma) = \begin{cases} f(0) & \text{if } \gamma = 0 \\ f(\gamma) - f(\gamma - 1) & \text{if } 0 < \gamma < \omega_1 \\ 0 & \text{if } \gamma = \omega_1. \end{cases}$$

Then $\phi(f)$ is obviously continuous on ω_1. Since f is eventually constant on ω_1, $\phi(f)$ is eventually 0 on ω_1, hence continuous on all of $\omega_1 + 1$. So ϕ is a well-defined linear function. For continuity of ϕ we only need to show that ϕ is continuous at 0. Let $P \subset \omega_1 + 1$ be a finite set and $\varepsilon > 0$. Let $Q = \{\gamma, \gamma - 1 : \gamma \in P \cap \omega_1\}$. Then Q is a finite set and if for all $\alpha \in Q$ we have $|f(\alpha)| < \varepsilon/2$, then for each $\gamma \in P$ we have $|\phi(f)(\gamma)| < \varepsilon$. To check that ϕ is injective, take $f, g \in C_p(\omega_1)$ such that $f \neq g$. Let $\alpha < \omega_1$ be the first ordinal such that $f(\alpha) \neq g(\alpha)$. Then it easily follows that $\phi(f)(\alpha) \neq \phi(g)(\alpha)$, hence $\phi(f) \neq \phi(g)$.

3. Remarks.

1. For metric spaces, Lemma 2.1 was proved in [3] and for normal spaces it was proved in [2]. A modification of the proof made it possible to state it for all completely regular spaces.

2. Lemma 2.6 and Corollary 2.7 are also in [2]. For completeness' sake we included the short proofs.

3. Once we have Example 2.8 the natural question to ask would be if for a linear embedding $\phi : C_p(X) \to C_p(Y)$ we have X is compact, σ-compact or realcompact provided Y is. The map ϕ in Example 2.8 is however a linear embedding. To prove this fact, let P be a finite subset of ω_1 and let $\varepsilon > 0$. For $\gamma \in P$, let A_γ be the set of all predecessors of γ up to the first non-successor below γ. Then $A = \bigcup_{\gamma \in P} A_\gamma$ is finite and if for all $\alpha \in A$, $|\phi(f)(\alpha)| < \varepsilon/|A|$, then for each $\gamma \in P$, $|f(\gamma)| < \varepsilon$.

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