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Mehmet Baran

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The notion of closedness in topological categories

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Abstract. In [1], various generalizations of the separation properties, the notion of closed and strongly closed points and subobjects of an object in an arbitrary topological category are given. In this paper, the relationship between various generalized separation properties as well as relationship between our separation properties and the known ones ([4], [5], [7], [9], [10], [14], [16]) are determined. Furthermore, the relationships between the notion of closedness and strongly closedness are investigated in an arbitrary topological category and a characterization of each of these notions is given for some known topological categories.

Keywords: topological category, separation properties, (strongly) closed objects

Classification: 18B99, 18D15, 54A05, 54A20, 54B30, 54D10

Introduction.

Some basic concepts in general topology are the notions of separation properties which appear in many important theorems such as the Urysohn Metrization theorem, the Urysohn Lemma, the Tietze Extension theorem, among others. There are several well-known generalizations of the usual topological T_0-axiom to the topological categories that are given by [4], [5], [7], [9] [10], and [16]. Schwarz recently has shown [16] that these generalizations lead to two concepts: T_0 and separatedness. Furthermore, he has shown that every T_0 object of a topological category is separated and the converse is true under certain conditions. In [1], various generalizations of the separation properties are defined for an arbitrary topological category over Sets, the category of sets. These generalizations include not only two notions of T_0 but also one notion of T_1, and four notions of T_2, T_3, and T_4. It was shown [1] that each of these notions reduces to the corresponding classical notion in the case of topological space.

General results involving relationships among these generalized separation properties as well as interrelationships among their various forms are being investigated in [3]. One of the separation properties, namely PreT_2' [1], has already appeared in [8] as a generalized Hausdorff condition arising in the study of geometric realization functors that preserve finite limits. Furthermore, ST_2 has appeared in [14] under the name of "Hausdorff convergence space" in the case of local filter convergence spaces.

One of the other basic concepts in general topology is the notion of closedness. For example, this notion is being used in defining the separation axioms T_3 (regular) and T_4 (normal) topological spaces, and showing a topological space is Hausdorff

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if and only if Δ, the diagonal is closed, and a closed subspace of normal space is normal, among others. This is one of the reasons why the notion of closedness along with the notion of strongly closedness has been introduced in [1] for an arbitrary topological category over Sets in terms of initial lifts, final lifts, and discreteness. It is shown [1] that the notion of closedness implies closedness and they coincide when a topological space is T_1.

In this paper, we explore these notions in an arbitrary topological category over Sets, the category of sets. Moreover, we try to find: 1. The relationships between these notions which turn out to be independent of each other. 2. The relationships between two of T_2 structures, which are, in general, independent of each other. 3. The relationships between our T_2's and Nel's T_2 [14], and our T_0's and others' T_0 ([4], [5], [7], [9], [10], [16]) have been investigated.

In this section, we give, for convenience, the definition of various separation properties introduced in [1, p. 15 and 16] for an arbitrary topological category over sets.

Let E be a category and Sets be the category of sets. A functor $U : E \to$ Sets is said to be concrete if it is faithful (i.e. U is mono on hom sets) and amnestic (i.e. $U(f) =$ id and f is an isomorphism then $f =$ id). The functor U is said to be topological if it is concrete, has small (i.e. sets) fibers, and for which every U-source has an initial lift, or, equivalently, for which every U-sink has a final lift ([6, p. 125] or [11, p. 279]). Recall [11, p. 279] that the fibers of the topological functor U are nonempty cocomplete posets (i.e. posets for which each subset has an inf (infimum)). Each such topological functor has a left adjoint D called the discrete functor, where $D(b)$ is the discrete object of $U^{-1}(b)$, the fiber over b, and is characterized as the minimum element ([11, p. 279]) of the fibers of U or as those objects d of E for which every map $U(d) \to U(e)$ lifts to a map $d \to e$. The notion of indiscreteness is dual notion of discreteness.

Recall that an object X of a topological category E is called T_0-object of E if it does not have an indiscrete subspace with more than one point ([10] or [16]). If E is saturated, i.e. the class T_0E of T_0-objects is initially dense in E, then X is T_0-object iff every initial source with domain X is a monosource (i.e. X is separated in the sense of [4], [5], [7], [9], [16]). The T_0-objects of E form a quotient-reflective subcategory of E [10]; in particular, T_0E is monotopological for any topological category E [16].

Definition 1.1. A <u>Prebornological Space</u> is a pair (A, F) where F is a family of subsets of A that is closed under finite union and contains all finite nonempty subsets of A. See [11, p.530]. Furthermore, if $F \neq \phi$ and F is hereditary closed, then (A, F) is called a <u>Bornological Space</u> ([11, p. 530] or [14, p. 1376]). A morphism $(A, F) \to (A_1, F_1)$ of such spaces is a function $f : A \to A_1$ such that $f(C) \in F_1$ if $C \in F$. We denote by <u>*P Born*</u> and <u>*Born*</u>, respectively, the categories so formed and by <u>*P Born**</u> the full subcategory of *P Born* determined by those spaces (A, F) with $\phi \notin F$ ([11, p. 530]). The categories *P Born*, *P Born**, and *Born* are topological over sets.

Definition 1.2. The <u>Category of Pairs</u>, CP has as objects the pairs (A, B) where

B is a subset of A and has as morphism $(A, B) \to (A_1, B_1)$ those functions $f : A \to A_1$ such that $f(B) \subset B_1$. This forms a category which is also topological over Sets. To see this, let $U : CP \to$ Sets be defined by $U(A, B) = A$. It is clear that U is concrete and U has small fibers since $U^{-1}(A) = PA$ is a set. Let $\{f_i : A \to U(A_i, B_i) = A_i \ i \in I\}$ be any U-source in Sets. Define a subset B of A by $B = \bigcap_{i \in I} f_i^{-1}(B_i)$. It is readily seen that (A, B) is the initial lift of the given U-source. Hence U is topological.

The discrete structure (A, B) on A in CP is given by $B = \phi$.

Let A be a set and K be a function on A whose value $K(a)$ at each a in A is a set of nonempty filters on A.

Definition 1.3. A pair (A, K) is said to be a Filter Convergence Spaces if for each a in A:

1. $[a]$ belong to $K(a)$, where $[a] = \{B \subset A/a \text{ is in } B\}$.
2. If α and β are filters on A and $\alpha \subset \beta$, then $\beta \in K(a)$ if $\alpha \in K(a)$.

A morphism $(A, K) \to (B, L)$ is a function $f : A \to B$ such that $f\alpha \in L(f(a))$ if $\alpha \in K(a)$, where $f\alpha$ denotes the filter $\{U \mid U \subset B \text{ and } U \supset f(C) \text{ for some } C \in \alpha\}$. We denote by FCO, the category do formed. See [15, p. 354].

Definition 1.4. A Filter Convergence Space (A, K) is said to be a Local Filter Convergence Space if $\alpha \cap [a]$ belongs to $K(a)$ whenever α belongs to $K(a)$ ([14, p. 1374]). These spaces are the objects of the full subcategory, $LFCO$, of FCO.

Definition 1.5. The category of Constant Filter Convergence Spaces, $ConLFCO$ is the full subcategory of FCO determined by those (A, K) where K is a constant function. Similarly, we define $ConLFCO$ as the full subcategory of $LFCO$.

Lemma 1.6. *Suppose $f : X \to Y$ is a morphism in $P\,Born$, $P\,Born^*$, $Born$, or $ConLFCO$. If f has finite fibers, i.e. $f^{-1}(y)$ is a finite set for ally in Y, then f reflects discreteness, i.e. if Y is discrete, then so is X.*

PROOF:
Case 1. Suppose $f : X \to Y$ is in ($P\,Born$, $P\,Born^*$) or $Born$ and $Y = (A_1, F_1)$ is discrete, i.e. $F_1 = \{C \mid C \text{ is a (nonempty) finite subset of } A_1\}$ [11, p. 530]. Suppose $X = (A, F)$ and $W \in F$. Hence $f(W) \in F_1$ which implies $f(W)$ is a finite subset of A_1. Note that $f^{-1}f(W) = \bigcup f^{-1}(y)$, $y \in f(W)$ is finite by Corollary 6.8 of [13, p. 44], since $f^{-1}(y)$ and $f(W)$ are finite sets. But since $W \subset f^{-1}f(W)$, W is a finite set. Hence $X = (A, F)$ is discrete.

Case 2. Suppose $f : X \to Y$ is in $ConLFCO$ and $Y = (A_1, K_1)$ is discrete, i.e. $K + 1 = \{\alpha \mid \alpha \text{ contains a finite subset of } A_1\}$ ([11, p. 528]). Let $\sigma \in K$, where $X = (A, K)$. We must show that σ contains a finite subset. Since $f\sigma$ in K_1 and K_1 is discrete, there exists a finite subset U of A_1 in $f\sigma$. Hence $U \supset f(V)$ for some V in σ and consequently $f(V)$ is finite. This completes the proof. □

Let X be a set and p a point on X. Let $X \vee_p X$ be the wedge product of X with itself, i.e. two distinct copies of X identified at the point p. A point x in

$X \vee_p X$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $X \vee_p X$. Let $X^2 = X \times X$ be the cartesian product of X with itself. $X^2 \vee_\Delta X^2$ (two distinct copies of X^2 identified along the diagonal). A point (x, y) in $X^2 \vee_\Delta X^2$ will be denoted by $(x, y)_1$ $((x, y)_2)$ if (x, y) is in the first (resp. second) component of $X^2 \vee_\Delta X^2$. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$ ([1, p. 14]).

Definition 1.7. The principal p axis map, $A_p : X \vee_p X \to X^2$ is defined by $A_p(x_1) = (x_1, p)$ and $A_p(x_2) = (p, x_2)$.

Definition 1.8. The skewed p axis map, $S_p : X \vee_p X \to X^2$ is defined by $S_p(x_1) = (x_1, x_1)$ and $S_p(x_2) = (p, x_2)$.

Definition 1.9. The fold map at p, $\nabla : X \vee_p X \to X$ is given by $\nabla(x_i) = x$ for $i = 1, 2$.

Definitions 1.10. The principal axis map, $A : X^2 \vee_\Delta X^2 \to X^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map, $S : X^2 \vee_\Delta X^2 \to X^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map $\nabla : X^2 \vee_\Delta X^2 \to X^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$.

Definition 1.11. The infinite wedge product, $\vee_p^\infty X$ is formed by taking countably many distinct copies of X and identifying them at the point p. Let $X^\infty = X \times X \times \ldots$ be the countable cartesian product of X. We also need the infinite analogue A_p^∞ of the map A_p. Define $A_p^\infty : \vee_p^\infty X \to X^\infty$ by $A_p^\infty(x_i) = (p, p, \ldots, x_i, p, \ldots)$ where x_i is in the i-th component of infinite wedge and x_i is in the i-th place in $(p, p, \ldots, x_i, p, \ldots)$.

Let $U : E \to$ Sets be a topological functor, X an object in E, and p a point in $UX \doteq B$. Let $q : B \to B/F$ be the identification map identifying the nonempty subset F of B to a point $*$ ([1, p. 15 and 16]).

Definitions 1.12.

1. X is $\overline{T_0}$ at p iff the initial lift of the U-source $\{A_p : B \vee_p B \to U(X^2) = B^2$ and $\nabla : B \vee_p B \to UDB = B\}$ is discrete.
2. X is T_1 at p iff the initial lift of the U-source $\{S_p : B \vee_p B \to U(X^2) = B^2$ and $\nabla : B \vee_p B \to UDB = B\}$ is discrete.
3. X is $\overline{T_0}$ iff the initial lift of the U-source $\{A : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_D B^2 \to UD(B^2) = B^2\}$ is discrete.
4. X is T_0' iff the initial lift of the U-source $\{\text{id} : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$.
5. X is T_1 iff the initial lift of the U-source $\{S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete.
6. p is closed iff the initial lift of the U-source $\{A_p^\infty : \vee_p^\infty B \to UX^\infty = B^\infty$ and $\nabla : \vee_p^\infty B \to UDB = B\}$ is discrete.
7. $F \subset X$ is closed iff $*$ is closed in X/F.
8. $F \subset X$ is strongly closed iff X/F is T_1 at $*$.

9. <u>X is ΔT_2</u> iff the diagonal, Δ, is closed in X^2.
10. <u>X is ST_2</u> iff the diagonal, Δ, is strongly closed in X^2.

Remark 1.13. We define $p_1, p_2, \nabla_p, \pi_{ij}$ by $1+p, p+1, 1+1 : B \vee_p B \to B$ and $\pi_1 + \pi_j : B^2 \vee_\Delta B^2 \to B$, respectively where $1 : B \to B$ is the identity map, $p : B \to B$ is constant map at p, and $\pi_i : B^2 \to B$ is the i-th projection $i = 1, 2$. Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \nabla$, $\pi_1 A = \pi_{11} = \pi_1 S$, $\pi_2 A = \pi_{21} = \pi_2 S$, $\pi_3 A = \pi_{12}$ and $\pi_3 S = \pi_{22}$. When showing A_p and S_p are initial it is sufficient to show that (p_1 and p_2) and (p_1 and ∇) are initial lifts, respectively.

Lemma 1.14. *Let α be a filter on B.*

(1) *For $a \notin F$, $q\alpha \subset [a]$ iff $\alpha \subset [a]$.*
(2) *$q\alpha \subset [*]$ iff $\alpha \cup [F]$ is proper.*

Proof: See [2, p. 105]. □

Remark 1.15. Let α and β be filters on A. If $f : A \to B$ is a function, then $f(\alpha \cap B) = f\alpha \cap fB$.

Lemma 1.16. *Let α and σ be filters on B and $q : B \to B/F$ be identification map identifying F to $*$.*

(1) *If $\alpha \cup [F]$ is not proper, then $q\sigma \subset q\alpha$ iff $\sigma \subset \alpha$.*
(2) *If $\alpha \cup [F]$ is proper, then $q\sigma \subset q\alpha$ iff $\sigma \cup [F]$ is proper.*

Proof: See [2, p. 105]. □

2. Closed points.

In this section, we characterize the closed points (1.12) in the topological categories discussed in Section 1.

Theorem 2.1. *$X = (B, K)$ in FCO or LFCO is $\overline{T_0}$ at p iff for each $x \neq p$, $[x] \notin K(p)$ or $[p] \notin K(x)$.*

Proof: Suppose X is $\overline{T_0}$ at p, i.e. by [11, p. 528], 1.13 and Definition 1.12, for any filter σ on the wedge and any point z in the wedge $p_1\sigma \in K(p_1 z)$, $p_2\sigma \in K(p_2 z)$, and $\nabla\sigma = [\nabla z]$ or $[\phi]$ iff $\sigma = [z]$ or $[\phi]$. We will show that for any $x \neq p$ if $[x] \in K(p)$, then $[p] \notin K(x)$. Suppose $[p] \in K(x)$. Let $\sigma = [(p, x)]$. Clearly $p_1\sigma = [p] \in K(x)$, $p_2\sigma = [x] \in K(p)$, and $\nabla\sigma = [x]$. Since X is $\overline{T_0}$ at p, we get a contradiction.

Similarly, one can show that for any $x \neq p$ if $[p] \in K(x)$, then $[x] \notin K(p)$.

On the other hand if the conditions hold, we will show that X is $\overline{T_0}$ at p.

If σ satisfies $p_1\sigma \in K(p_1(x, p))$, $p_2\sigma \in K(p_2(x, p))$, and $\nabla\sigma = [x]$ or $[\phi]$, then it follows easily that $\sigma = [(x, p)]$, $[(p, x)]$, $[\phi]$ or $\sigma \supset [(x, p) \cup (p, x)]$. We wish to show that $\sigma = [(x, p)]$ or $[\phi]$. If $\sigma = [(p, x)]$, then $p_1\sigma = [p] \in K(x)$, $p_2\sigma = [x] \in K(p)$, a contradiction. Hence, $\sigma \neq [(p, x)]$. If $\sigma = [(x, p) \cup (p, x)]$, then $p_1\sigma = [x \cup p] \subset [p]$, and $p_2\sigma = [p \cup x] \subset [x]$, and consequently $[p] \in K(x)$ and $[x] \in K(p)$, a contradiction. Hence, $\sigma \neq [(x, p) \cup (p, x)]$. We next show that the case $\sigma \supset [(x, p) \cup (p, x)]$ with $\sigma \neq [(x, p) \cup (p, x)]$ and $\sigma \neq [\phi]$ cannot occur either. To this end, we show that if $[\phi] \neq \sigma \neq [(x, p) \cup (p, x)]$, then $\sigma \supset [(x, p) \cup (p, x)]$ iff $\sigma = [(x, p)]$ or $[(p, x)]$. Clearly if $\sigma = [(x,p)]$ or $[(p,x)]$, then $\sigma \supset [(x,p) \cup (p,x)]$. Conversely, if $\sigma \supset [(x,p) \cup (p,x)]$ with $[\phi] \neq \sigma \neq [(x,p) \cup (p,x)]$, then there exists $U \in \sigma$ such that $U \neq \{(x,p),(p,x)\}$ and $U \neq \phi$. Since $\{(x,p),(p,x)\} \in \sigma$, a filter, $U \cap \{(x,p),(p,x)\} = (x,p)$ or (p,x) is in σ, i.e. $\sigma = [(x,p)]$ or $[(p,x)]$. We have already shown above that $\sigma \neq [(p,x)]$. Hence $\sigma = [(x,p)]$ or $[\phi]$. Similarly, it can be shown that if σ satisfies $p_1\sigma \in K(p)$, $p_2\sigma \in K(x)$, and $\nabla\sigma = [x]$ or $[\phi]$, then $\sigma = [(p,x)]$ or $[\phi]$. If σ satisfies $p_1\sigma \in K(p)$, $p_2\sigma \in K(p)$, and $\nabla\sigma = [p]$ or $[\phi]$, then $\sigma = [(p,p)]$ or $[\phi]$ (since $\nabla^{-1}(p) = (p,p)$). Hence X is $\overline{T_0}$ at p. □

Theorem 2.2. *$X = (B,K)$ in FCO or LFCO is T_1 at p iff for each $x \neq p$, $[x] \notin K(p)$ and $[p] \notin K(x)$.*

Proof: Suppose X is T_1 at p, i.e. by [11, p. 528], 1.13 and 1.12 for any filter σ on the wedge and any point z in the wedge $p_1\sigma \in K(p_1 z)$, $\nabla\sigma \in K(\nabla z)$, $\nabla\sigma = [\nabla z]$ or $[\phi]$ iff $\sigma = [z]$ or $[\phi]$. If $[x] \in K(p)$ for $x \neq p$, then let $\sigma = [(x,p)]$. Clearly, $p_1\sigma = [x] \in K(p)$, $\nabla\sigma = [x] \in K(x)$. Since X is T_1 at p, $\sigma = [(p,x)]$, a contradiction since $x \neq p$. Hence $[x] \notin K(p)$. If $[p] \in K(x)$, $x \neq p$, then let $\sigma = [(p,x)]$. Clearly $p_1\sigma = [p] \in K(x)$ and $\nabla\sigma = [x] \in K(x)$, a contradiction ($X$ is T_1 at p). Hence $[p] \notin K(x)$. Conversely, if σ satisfies $p_1\sigma \in K(x)$, $\nabla\sigma \in K(x)$, and $\nabla\sigma = [x]$ or $[\phi]$, then it follows easily that $\sigma = [(x,p)]$, $[(p,x)]$, $[\phi]$ or $\sigma \supset [(x,p) \cup (p,x)]$. We must show that $\sigma = [(x,p)]$ or $[\phi]$. If $\sigma = [(p,x)]$, then $p_1\sigma = [p] \in K(x)$, a contradiction. If $\sigma = [(x,p) \cup (p,x)]$ then $p_1\sigma = [x \cup p] \subset [p]$ and consequently $[p] \in K(x)$, a contradiction. If $\sigma \supset [(x,p) \cup (p,x)]$ with $[\phi] \neq \sigma \neq [(x,p) \cup (p,x)]$, then by the same argument used in 2.1, we get $\sigma = [(x,p)]$ or $[(p,x)]$ and consequently $\sigma = [(x,p)]$ or $[\phi]$. Similarly, if σ satisfies $p_1 \in K(p)$, $\nabla\sigma \in K(x)$, and $\nabla\sigma = [x]$ or $[\phi]$, then $\sigma = [(p,x)]$ or $[\phi]$. If σ satisfies $p_1\sigma \in K(p)$, $\nabla\sigma \in K(p)$, and $\nabla\sigma = [p]$ or $[\phi]$, then $\sigma = [(p,p)]$ or $[\phi]$ (since $\nabla^{-1}(p) = (p,p)$). Hence X is T_1 at p. □

Theorem 2.3. *$X = (B,K)$ in ConFCO is $\overline{T_0}$ at p iff for each $x \neq p$, $[x] \cap [p] \notin K$.*

Proof: Suppose X is $\overline{T_0}$ at p, i.e. by [11, p. 528], 1.13 and 1.12, for any filter σ on the wedge, $p_1\sigma \in K$, $p_2\sigma \in K$ and $\nabla\sigma = [x]$ or $[\phi]$ for some x iff $\sigma = [z]$ or $[\phi]$ for some z in the wedge. If $[x] \cap [p] \in K$ for some $x \neq p$, then let $\sigma = [(x,p) \cup (p,x)]$. By 1.15, $p_1\sigma = [x \cup p] = p_2\sigma \in K$ and $\nabla\sigma = [x]$. But $\sigma \neq [z]$ for any point z in the wedge, a contradiction. Hence we must have $[x] \cap [p] \notin K$ for all $x \neq p$. Conversely, suppose the condition holds. If σ satisfies $p_1\sigma \in K$ and $\nabla\sigma = [x]$ or $[\phi]$ for some x (1.12 and 1.13), then it follows easily that $\sigma = [(x,p)]$, $[(p,x)]$, $[\phi]$ or $\sigma \supset [(x,p) \cup (p,x)]$. We show that the last case cannot occur. If $\sigma = [(x,p) \cup (p,x)]$, then by 1.15 $p_1\sigma = [x \cup p] = p_2\sigma \in K$, a contradiction. If $\sigma \supset [(x,p) \cup (p,x)]$ and $[\phi] \neq \sigma \neq [(x,p) \cup (p,x)]$, then by the same argument used in the proof of 2.1, it follows that $\sigma = [(x,p)]$ or $[(p,x)]$. Therefore $\sigma = [(x,p)]$, $[(p,x)]$ or $[\phi]$, i.e. X is $\overline{T_0}$ at p. □

Theorem 2.4. *$X = (B,K)$ in ConFCO is T_1 iff for each $x \neq p$, $[x] \cap [p] \notin K$.*

Proof: The proof is similar to the proof of 2.3 since X is T_1 at p means by [11, p. 528], 1.12 and 1.13, that for any filter σ, $p_1\sigma \in K$, $\nabla\sigma \in K$, and $\nabla\sigma = [x]$ or $[\phi]$ for some x iff $\sigma = [z]$ or $[\phi]$ for some z in the wedge. □

Lemma 2.5. *If* $\nabla : (B \vee_p B, K) \to (B, K_d)$ *is in anyone of* $ConLFCO$, $P\,Born$, $P\,Born^*$, *or* $Born$, *where* K_d *is discrete structure on* B, *then* K *is discrete.*

PROOF: This follows from 1.6 since the fibers of ∇ are finite. □

Lemma 2.6. *If* $f : X \to Y$ *is in* CP, *then* f *reflects discreteness, i.e. if* Y *is discrete, then so is* X.

PROOF: If $Y = (B, U)$ is discrete, i.e. $U = \phi$ (1.12) but $X = (A, V)$ is not discrete, i.e. $V \neq \phi$, then $f(V) \neq \phi$. But also since $f : X \to Y$ is in CP, it follows that $f(V) \subset \cap$, and consequently $f(V) = \phi$, a contradiction. □

Theorem 2.7. *All* X *in* $COnLFCO$ *are* $\overline{T_0}$ *at* p *and* T_1 *at* p.

PROOF: This follows from 2.5 and Definition 1.12. □

Theorem 2.8. *All* X *in* $P\,Born$, $P\,Born^*$ *or* $Born$ *are* $\overline{T_0}$ *at* p *and* T_1 *at* p.

PROOF: This follows from 2.5 and Definition 1.12. □

Theorem 2.9. *All objects in* CP, *the category of pairs, are* $\overline{T_0}$ *at* p *and* T_1 *at* p.

PROOF: This follows from 2.6 and Definition 1.12 since discreteness is reflected. □

Theorem 2.10. *Let* $X = (B, K)$ *be in one of* FCO *or* $LFCO$. *A point* p *in* B *is closed iff* X *is* $\overline{T_0}$ *at* p, *i.e. by* 2.1 *for each* $x \neq p$, $[x] \notin K(p)$ *or* $[p] \notin K(x)$.

PROOF: Suppose p is closed, i.e. by Definition 1.13, [11, p. 528], and 1.12, for any filter σ on the infinite wedge and for any point z in the infinite wedge $p_i\sigma \in K(p_iz)$ for all i and $\nabla\sigma = [\nabla z]$ or $[\phi]$. We shall show that X is $\overline{T_0}$ at p. Suppose $[x] \in K(p)$ and $[p] \in K(x)$ for some $x \neq p$. Let $\sigma = [(x, p, p, \dots)]$ and $z = (p, x, p, p, \dots)$. Clearly $p_1\sigma = [x] \in K(p_iz = p)$, $p_2\sigma = [p] \in K(p_2z = x)$, $p_i\sigma = [p] \in K(p_iz = p)$ for all $i \geq 3$, and $\nabla\sigma = [\nabla z = x]$. Since p is closed, $\sigma = [z]$, a contradiction since $x \neq p$. Hence X must be $\overline{T_0}$ at p by 2.1. Conversely, we must show if X is $\overline{T_0}$ at p, then p is closed. Let $x_i = (p, p, \dots, x, p, \dots)$ denote any point in the wedge with $x \neq p$ and x in the i-th place. If σ satisfies $p_i\sigma \in K(p_ix_i = x)$, $p_n\sigma \in K(p_nx_i = p)$ for all $n \neq i$, and $\nabla\sigma = [\phi]$ or $[\nabla x_i = x]$, then it can be easily seen that $\sigma = [\phi]$ or $[x_j]$ for some j or $\sigma \supset \bigcap_{k=1}^{n}[x_{ik}]$ (since $\nabla\sigma = [\phi]$ or $[x]$). We must show that $\sigma = [\phi]$ or $[x_i]$. If $\sigma = [x_j]$ for some $j \neq i$, then $p_i[x_j] = [p] \in K(p_ix_i = x)$ and $p_j[x_j] = [x] \in K(p_jx_i) = [x] \in K(p_jx_i = p)$, a contradiction since X is $\overline{T_0}$ at p. If $[\phi] \neq \sigma \neq \bigcap_{k=1}^{n}[x_{ik}]$ and $\sigma \supset \bigcap_{k=1}^{n}[x_{ik}]$, then it follows easily (see the proof of 2.1) that $\sigma = \bigcap_{k=1}^{m}[x_{ik}]$ for some $m < n$. If $\sigma = \bigcap_{k=1}^{m}[x_{ik}]$, then by 1.15, $p_i\sigma = [x] \cap [p]$ and $p_i\sigma \in K(p_ix_i = x)$ by assumption if $i = i_k$ and $m > 1$ and $p_i\sigma = [p] \in K(p_ix_i = x)$ if $i \neq i_k$ and $m \geq 1$, $p_j\sigma = [x] \cap [p]$ and $p_j\sigma \in K(p_jx_i = p)$ by assumption if $i \neq j = i_k$ and $m > 1$, and consequently $[x] \in [p]$ and $[p] \in K(x)$, a contradiction. Hence $\sigma = [\phi]$ or $[x_i]$. If $p_i\sigma \in K(p)$ for all i and $\nabla\sigma = [\phi]$ or $[p]$, then $\sigma = [\phi]$ or $[(p, p, \dots)]$ since $\nabla^{-1}(p) = (p, p, \dots)$. If $\sigma = [\phi]$ or $[x_i]$, then $p_i\sigma = [\phi]$ or $[x]$ which are in $K(x)$, $p_n\sigma = [\phi]$ or $[p]$ which are in $K(p)$ for all $n \neq i$, and $\nabla\sigma = [\phi]$ or $[x]$. Hence p is closed. □

Theorem 2.11. *P in B is closed for $X = (B, K)$ in $ConFCO$ iff X is $\overline{T_0}$ at p, i.e. by 2.3 for each $x \neq p$, $[x] \cap [p] \notin K$.*

Proof: By 2.10, p is closed iff X is $\overline{T_0}$ at p. Since K is constant and initial lifts are the same, by 2.3 X is $\overline{T_0}$ at p iff for each $x \neq p$, $[x] \cap [p] \notin K$. □

Theorem 2.12. *Points are always closed in X for X in CP.*

Proof: This follows from 2.6 and Definition 1.12 since discreteness is reflective. □

Theorem 2.13. *p in B is closed for X in $P\,Born$, $P\,Born^*$, $Born$ or $ConLFCO$ iff $B = \{p\}$.*

Proof:
Case 1. Suppose p in B is closed for $X = (B, F)$ in $P\,Born$, $P\,Born^*$ or $Born$, i.e. by [11, p. 530], 1.13, and 1.12, for any subset W in the infinite wedge, $p_i W \in F$ for all i and $\triangledown W$ is finite subset of B iff W is finite. We must show that $B = \{p\}$. If $B \neq \{p\}$, then there exists x in B such that $x \neq p$. Let $W = \vee_p^\infty \{x_i\}$ and note that $p_i W = \{x, p\} \in F$ for all i, and $\triangledown W = \{x\}$. This is a contradiction since W is not a finite set.

Case 2. Suppose p is closed in $X = (B, K)$ for X in $ConLFCO$, i.e. by Definition 1.12 and [11, p. 530], for any filter σ on the wedge $p_i\sigma \in K$ for all i and $\triangledown\sigma$ contains a finite set iff σ contains a finite set. We shall show that $B = \{p\}$. If $B \neq \{p\}$, then there exists x in B such that $x \neq p$. Let $\sigma = \bigcap_{i=1}^\infty [x_i]$ and note that $p_j\sigma = [x] \cap [p] \in K$ for each j and $\triangledown\sigma = [x]$. But σ does not contain a finite set since σ is generated by the infinite set $\{x_1, x_2, \dots, x_n, \dots\}$. This is a contradiction to the fact that p is closed.

Conversely, if $B = \{p\}$, then clearly the infinite wedge is just one point and consequently p is closed. □

3. Closed and strongly closed subobjects.

Let $U : E \to$ Sets be a topological category over Sets, X an object in E, and F a nonempty subset of UX. In this section, we will characterize closed and strongly closed nonempty F of UX in the topological categories discussed in 1. As an application, we will derive a characterization of the separation properties, namely ST_2 and ΔT_2, defined in 1.12. Let $q : X \to X/F$ be the quotient map defined in 1.12, i.e. q is the final lift of the U-sink $B = UX \to B/F$, identifying F to a point $*$.

Theorem 3.1. *Let $X = (B, K)$ be in FCO or $LFCO$. $\phi \neq F \subset B$ is strongly closed iff for any $a \in B$ if $a \notin F$, then $[a] \notin K(c)$ for all $c \in F$ and if $\alpha \in K(a)$, then $\alpha \cup [F]$ is improper.*

Proof: F is strongly closed iff by 1.12 X/F is T_1 at $*$ iff by 2.2 for each $a \neq *$ in B/F, $[a] \notin K'(*)$ and $[*] \notin K'(a)$ where K' is defined as in [14, p. 1375] iff by 1.14 and 1.12 for any $a \in B$ if $a \notin F$, then $[a] \notin K(c)$ for all for all $c \in F$ and if $\alpha \in K(a)$, then $\alpha \cup [F]$ is improper. □

Theorem 3.2. *Let $X = (B, K)$ be in FCO or LFCO. $\phi \neq F \subset B$ is closed iff for any $a \notin F$, if there exists $\alpha \in K(a)$ such that $\alpha \cup [F]$ is proper, then $[a] \notin K(c)$ for all $c \in F$.*

Proof: F is closed iff by 1.12 $*$ is closed in B/F iff by 2.10 X/F is $\overline{T_0}$ at $*$ iff by 2.1 for each $a \neq *$ in B/F $[a] \notin K'(*)$ or $[*] \notin K'(a)$ iff by 1.14 and 1.12 for any $a \in F$ if there exists $\alpha \in K(a)$ such that $\alpha \cup [F]$ is proper, then $[a] \notin K(c)$ for all $c \in F$. □

Lemma 3.3. *Let $\phi \neq F \subset B$, $q : B \to B/F$ be the identification map that identifies F to a point $*$, σ be a filter on B, and $a \in B$ with $a \notin F$. $[a] \cap [*] = q([a] \cap [F]) \supset q\sigma$ iff $\sigma \cup [F]$ is proper and $\sigma \subset [a]$.*

Proof: First note that by letting $\alpha = [a] \cap [F]$ in 1.16 and noting that $\alpha \cup [F] = [F]$ is proper (since $F \neq \phi$). Hence by 1.16 (2), $q([a] \cap [F]) \supset q\sigma$ iff $\sigma \cup [F]$ is proper and $\sigma \cap [F] \subset [a] \cap [F]$. We will show that $\sigma \cap [F] \subset [a] \cap [F]$ iff $\sigma \subset [a]$. If $\sigma \subset [a]$, then clearly $\sigma \cap [F] \subset [a] \cap [F]$. Conversely, if $\sigma \cap [F] \subset [a] \cap [F]$ and $\sigma \not\subset [a]$, then there exists V in σ such that $a \notin V$. Since $a \notin F$, $a \notin V \cup F$ and consequently $F \cup V \in \sigma \cap [F] \subset [a] \cap [F] \subset [a]$, i.e. $F \cup V$ contains a, a contradiction. This completes the proof. □

Theorem 3.4. *Let $X = (B, K)$ be in ConFCO. $\phi \neq F \subset B$ is strongly closed iff for each $a \in B$ with $a \notin F$ and for all $\alpha \in K$, $\alpha \cup [F]$ is improper or $\alpha \not\subset [a]$.*

Proof: F is strongly closed iff by 1.12 X/F is T_1 at $*$ iff by 2.4 for each $a \neq *$ in B/F, $[a] \cap [*] \notin K'$ iff by [14] $[a] \cap [*] = q([a] \cap [F]) \not\supset q\alpha$ for all $\alpha \in K$ iff by 3.3 $[a] \not\supset \alpha$ or $\alpha \cup [F]$ is improper. □

Theorem 3.5. *Let $X = (B, K)$ be in ConFCO. $\phi \neq F \subset B$ is closed iff for each $a \notin F$ and for any $\alpha \in K$, $\alpha \cup [F]$ is improper or $\alpha \not\subset [a]$.*

Proof: F is closed iff by 1.12 $*$ is closed in B/F iff by 2.11 X/F is $\overline{T_0}$ iff by 2.3 for each $a \neq *$ in B/F, $[a] \cap [*] \notin K'$ iff by 3.3 and [14] $a \notin F$ and $\alpha \in K$ $\alpha \cup [F]$ is improper or $\alpha \not\supset [a]$. □

Theorem 3.6. *Let $X = (B, K)$ be in ConFCO. Every $\phi \neq F \subset B$ is strongly closed.*

Proof: F is strongly closed iff by 1.12 X/F is T_1 at $*$. However, since by 2.7 X/F is always T_1 at $*$, F is strongly closed. □

Theorem 3.7. *Let $X = (B, K)$ be in ConFCO. Every $\phi \neq F \subset B$ is closed iff $F = B$.*

Proof: F is closed iff by 1.12 $*$ is closed in B/F iff by 2.13 $B/F = \{*\}$ iff, by definition of B/F, $F = B$. □

Theorem 3.8. *Let $X = (B, K)$ be in CP. Each $F \neq \phi$ is closed and strongly closed.*

Proof: This follows from 2.6 and Definition 1.12. □

Theorem 3.9. *Let $X = (B, G)$ be in one of $P\,Born$, $P\,Born^*$ or $Born$. Each $F \neq \phi$ is strongly closed.*

PROOF: F is strongly closed iff by 1.12 X/F is T_1 at $*$. However, since by 2.8 X/F is always T_1 at $*$, F is strongly closed. □

Theorem 3.10. *Let $X = (B, G)$ be in one of $P\,Born$, $P\,Born^*$ or $Born$. Each $F \neq \phi$ is closed iff $B = F$.*

PROOF: F is closed iff by 1.12 $*$ is closed in B/F iff by 2.13 $B/F = *$ iff, by definition of B/F, $F = B$. □

Now as an application of the notions of the closedness, we will characterize the separation properties, ST_2 and ΔT_2, defined in 1.12.

Corollary 3.11. *Let $X = (B, K)$ be in $ConLFCO$ or let $X = (B, F)$ be in $P\,Born$, $P\,Born^*$ or $Born$. X is ΔT_2 iff B a point or ϕ.*

PROOF: X is ΔT_2 iff by 1.12 Δ, the diagonal, is closed in B^2 iff by 3.7 or 3.10 letting $F = \Delta$, we get $\Delta = B^2$ iff (clearly) B is a point or ϕ for $X \in ConLFCO$, $P\,Born$, $P\,Born^*$ or $Born$. □

Corollary 3.12. *All X in $ConLFCO$, $P\,Born$, $P\,Born^*$, $Born$ or CP are ST_2.*

PROOF: This follows from definition 1.12 and Theorems 3.6, 3.8, and 3.9 (by letting $F = \Delta$). □

Corollary 3.13. *All object in CP are ΔT_2.*

PROOF: Combine 3.8 (let $F = \Delta$) and Definition 1.12. □

Corollary 3.14. *$X = (B, K)$ in FCO or $LFCO$ is ST_2 iff for any distinct pair of points x and y in B, $K(x) \cap K(y) = \{[\phi]\}$.*

PROOF: X is ST_2 iff by 1.12 Δ is strongly closed iff by 3.1, letting $F = \Delta$, for each $x \neq y$ in B, $[(x, y)] \notin K(a, a)$ for all $a \in B$ and for any $\alpha \in K(x, y)$, $\alpha \cup [\Delta]$ is improper. □

Claim. If $x \neq y$, then $\alpha \cup [\Delta]$ is improper for $\alpha \in K(x, y)$ iff $K(x) \cap K(y) = \{[\phi]\}$. Recall, by definition, $\alpha \in K(x, y)$ iff $\pi_1\alpha \in K(x)$ and $\pi_2\alpha \in K(y)$. Suppose for $x \neq y$, $\alpha \cup [\Delta]$ is improper for all $\alpha \in K(x, y)$ and there exists a proper filter $\beta \in K(x) \cap K(y)$. Let $\sigma = \pi_1^{-1}\beta \cup \pi_2^{-1}\beta$ and note that $\pi_1\sigma = \beta \in K(x)$ and $\pi_2\sigma = \beta \in K(y)$ and consequently $\sigma \in K(x, y)$. Hence $\sigma \cup [\Delta]$ is improper, i.e. there exists $V \in \sigma$ such that $V \cap \Delta = \phi$. But $V \in \sigma$ implies that there exists $U \in \beta$ such that $V \supset \pi_1^{-1}U \cap \pi_2^{-1}U = U^2$. Since $V \cap \Delta$ is empty, it follows that $U^2 \cap \Delta$ is empty and consequently $U = \phi \in \beta$, a contradiction since β is proper. Hence $K(x) \cap K(y) = \{[\phi]\}$ for $x \neq y$. Conversely, suppose $K(x) \cap K(y) = \{[\phi]\}$ for $x \neq y$ and there exists $\alpha \in K(x, y)$ such that $\alpha \cup [\Delta]$ is proper. $\alpha \in K(x, y)$ implies $\pi_1\alpha \in K(x)$ and $\pi_2\alpha \in K(y)$. Let $\beta = \pi_1^{-1}\pi_1\alpha \cup \pi_2^{-1}\pi_2\alpha$ and note that $\beta \in K(x, y)$ (since $\pi_1\beta = \pi_1\alpha \in K(x)$ and $\pi_2\beta = \pi_2\alpha \in K(y)$) and by 3.2 (1) in [2], $\beta \subset \alpha$. Since $\alpha \cup [\Delta]$ is proper, it follows that $\beta \cup [\Delta]$ is proper and consequently for any $V \in \beta$, $V \cap \Delta \neq \phi$. But $V \in \beta$ implies $V \supset V_1 \times V_2$ for some $V_1 \in \pi_1\alpha$

and $V_2 \in \pi_2\alpha$. Hence $(V_1 \times V_2) \cap \Delta \neq \phi$. Note that $(V_1 \times V_2) \cap \Delta \neq \phi$ iff $V_1 \cap V_2 \neq \phi$. Since $V_1 \cap V_2 \in \pi_1\alpha \cup \pi_2\alpha$, it follows that $\pi_1\alpha \cup \pi_2\alpha$ is proper and in $K(x) \cap K(y)$, a contradiction. This proves the claim. If X is ST_2, then by the claim, $K(x) \cap K(y) = \{[\phi]\}$ for $x \neq y$. If for $x \neq y$, $K(x) \cap K(y) = \{[\phi]\}$, then we must show that X is ST_2. By claim, $\alpha \cup [\Delta]$ is improper for all $\alpha \in K(x, y)$, $x \neq y$. If $[(x, y)] \in K(a, a)$, then $a \neq y$ (since $x \neq y$) and consequently $[y] \in K(x) \cap K(y)$, a contradiction. If $a \neq x$, then $[x] \in K(x) \cap K(a)$ (since $[x] \in K(a)$), a contradiction. This completes the proof.

Corollary 3.15. *$X = (B, K)$ in FCO or LFCO is ΔT_2 iff $x \neq y$,. then $[x] \notin K(y)$.*

Proof: Suppose X is ΔT_2, i.e. by 1.12 Δ is closed and suppose $[x] \in K(y)$ for some $x \neq y$. Let $F = \Delta$ in 3.2 and let $\beta = [(x, y)]$ and note that $\beta \in K(y, y)$, since $\pi_1\beta = [x] \in K(y)$ and $\pi_2\beta = [y] \in K(y)$. Furthermore, $[x] \in K(y)$ implies $K(x) \cap K(y) \neq \{[\phi]\}$ for some $x \neq y$. Hence, by the claim in the proof of 3.14, $\alpha \cup [\Delta]$ is proper for some $\alpha \in K(x, y)$, $x \neq y$. However, we also have $[(x, y)] \in K(y, y)$. This is a contradiction since Δ is closed (3.2).

Conversely, suppose that for all $x, y \in X$ if $x \neq y$, then $[x] \notin K(y)$. We will show that Δ is closed, i.e. by 3.2 for any $(x, y) \notin \Delta$, i.e. $x \neq y$ if there exists $\alpha \in K(a, a)$, $\alpha \in B$ such that $\alpha \cup [\Delta]$ is proper, then $[(x, y)] \notin K(a, a)$. If $[(x, y)] \in K(a, a)$, then $\pi_1[(x, y)] = [x] \in K(a)$ and $\pi_2[(x, y)] = [y] \in K(a)$. If $a = x$, then $a \neq y$ since $x \neq y$. Note that $[y] \in K(a)$, a contradiction.

If $a \neq x$, then $[x] \in K(a)$, a contradiction. Hence Δ is closed, i.e. X is ΔT_2. □

Corollary 3.16. *$X = (B, K)$ in ConFCO is ST_2 or ΔT_2 iff for each pair of distinct points x and y in B and for any $\alpha, \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.*

Proof: This follows easily from 1.12, 3.4 and 3.5 by letting $F = \Delta$. □

We can infer the following results:

1. The notions of closedness and strongly closedness in general are independent of each other; in *FCO* and *LFCO*, strongly closedness implies closedness (Theorems 3.1 and 3.2) and in *P Born*, closedness implies strongly closedness (Theorems 3.9 and 3.10) but the converse of each of these implications is not true. They could be equal also (see Theorem 3.8).

2. Generally speaking, T_2 structures, ST_2 and ΔT_2, are independent of each other. In *FCO*, ST_2 implies ΔT_2 (Corollaries 3.14 and 3.15) but the converse is not true. In *Born*, ΔT_2 implies ST_2 (Corollaries 3.11 and 3.12) but the converse is not true. In *CP*, they are equivalent (Corollaries 3.12 and 3.13).

3. We have some relationships among our T_2's and Nel's T_2 [14]. In *LFCO*, his T_2 is equivalent to our ST_2 and his T_2 implies our ΔT_2.

4. The subcategory of *FCO* (*LFCO*) determined by those objects X which satisfy ST_2 or ΔT_2 is a quotient reflective in *FCO* (*LFCO*). Hence, they are cartesian closed initially structured categories [14], and monotopological [16].

5. In general, for any topological category our notions of T_0' and Others T_0''s and separatedness are independent of each other. For example, in *CP* their T_0

$(X = (A, B))$, is T_0 iff B is a point or empty set (this follows from definition and indiscrete structure, which is $A = B$, on A) implies our T_0's (all objects in CP are $\overline{T_0}$ and T_0', which follows from 2.6 and Definition 1.12), but the converse is not true.

In $ConSCO$, the category of constant stack convergence spaces where in 1.5 "filters" are replaced by "stacks" [2] or [15], $X = (B, K)$ is T_0 in their sense iff for any distinct pair of points x and y in B the stack $[x] \cap [y]$ is not in K. This follows from definition and indiscrete structure on B, which is $K = STK(B)$, the set of all stacks on B. (This is a special case of 2.2 (2) of [16] where "filters" are replaced by "stacks".) $X = (B, K)$ is $\overline{T_0}$ iff B is a point or empty set. To see this, if B is not a point and empty set, then let $\sigma = \pi_{11}^{-1}[x] \cup \pi_{21}^{-1}[x] \cup \pi_{12}^{-1}[x] \cup [(x, y)_1 \cup (x, y)_2]$. Note that σ is proper and it follows from 1.13 and [2] that $\pi_{11}\sigma = [x] = \pi_{12}\sigma = \pi_{21}\sigma$ is in K and $\nabla\sigma \supset [(x, y)]$. Since X is $\overline{T_0}$, $\sigma \supset [z]$ for some z in the wedge. But this is a contradiction since X is $\overline{T_0}$ and σ contains no singletons. The converse is clear. Hence, our $\overline{T_0}$ implies their T_0 and separatedness but the converse is not true.

The separated objects of $Born$ are those with at most one point ([16, p. 322]) and all objects of $Born$ are T_0' (this follows from 2.5 and 1.12). Hence, separatedness implies our T_0' but the converse is not true in general.

We now show that our T_0' implies their T_0 and separatedness. Consider the topological category of stack convergence spaces, SCO, where in 1.3 "filters" are replaced by "stacks" [2] or [15]. If $X = (B, K)$ in SCO is T_0', then X is discrete, i.e. for x in B, $K(x) = \{\alpha \mid \alpha \supset [x]\}$. Suppose X is T_0' and X is not discrete. Hence there exists a stack α in $K(x)$ such that it does not contain $[x]$. Let σ_1 is in $K^2(x, y)$, where K^2 is a product structure on B^2. Let $\sigma = i_1\sigma_1 \cup [(x, y)_2]$ with $x \neq y$ (since X is not discrete). Note that $\nabla\sigma \supset [(x, y)]$ and $\sigma \supset i_1\sigma_1$ with $i_1(x, y) = (x, y)_1$. But σ does not contain $[(x, y)_1]$, a contradiction since X is T_0'.

$X = (B, K)$ in SCO is T_0 iff for each distinct pair of points x and y in B, $[x] \cap [y]$ is not in $K(x)$ or $K(y)$. This is a special case of 2.2 (2) of [16, p. 318], where filters are replaced by stacks. Hence, it follows that in SCO, our T_0' implies T_0 and consequently separatedness but the converse is not true as it can be seen by taking $B = \{x, y\}$, two-point set and $K(x) = \{[x], [x] \cap [y],\ PB = [\phi],\ [x] \cup [y]\}$, and $K(y) = \{[y],\ PB,\ [x] \cup [y]\}$.

In [3], for an arbitrary topological category, it has been shown that $\overline{T_0}$ implies T_0' but the converse is not true, in general. Our notions of T_0's do make sense for topological category over a topos, too. Also, two of other T_2 structures appeared in [1] are defined in terms of T_0 structures.

6. For topological spaces, we have [13] X is T_1 iff all points of X are closed. For an arbitrary topological category, this is not true in general. For example, in $Born$ X is always T_1 (it follows from 2.5 and 1.12). Hence by 2.10 closed points imply X is T_1 but the converse is not true. In FCO, $X = (B, K)$ is T_1 iff for any distinct pair of points x and y in B, $[x]$ is not in $K(y)$ (the proof is similar to the proof of 2.2). Thus, T_1 implies that the points are closed (Theorem 2.10) but the converse is not true.

7. It is possible to have all the separation properties defined in 1.12 to be equivalent. This occurs, for example, in CP.

8. Except for ΔT_2, all of the other separation properties defined in 1.12 are equivalent in *Born*.

9. ΔT_2 is equivalent to T_1 and implies $\overline{T_0}$ in *FCO* and *LFCO*. Our $\overline{T_0}$ is equivalent to Schwarz's T_0 ([16, Proposition 2.1]) in case of *LFCO*.

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Erciyes University, Department of Mathematics, 38093 Kayseri, Turkey

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