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A categorical concept of completion of objects

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Abstract. We introduce the concept of firm classes of morphisms as basis for the axiomatic study of completions of objects in arbitrary categories. Results on objects injective with respect to given morphism classes are included. In a finitely well-complete category, firm classes are precisely the coessential first factors of morphism factorization structures.

Keywords: firm reflection, (sub-)firm class, injective object, (co)-essential morphism

Classification: Primary 18A40, 18A32, 18G05; Secondary 54B30, 54E15

0. Introduction.

Many authors have defined, in various categorical or precategorical contexts, the notions of complete object and of completion of objects. On the one hand one wants a clear and simple theory of sufficient scope to cover the standard examples in algebra and in general topology. Many contributions of this kind occur in the literature on reflections, e.g. in [Kennison 65], [Herrlich 68], [Herrlich Strecker 79], [Herrlich 82], [Adámek Herrlich Strecker 90], and also in the literature on relative injectivity, e.g. in [Maranda 64], [Pumplün 72], [Tholen 81], [Kiss Márki Pröhle Tholen 83], [Herrlich 86], [Adámek Herrlich Strecker 90].

On the other hand, however, we are interested in a notion with peculiarly strong consequences in addition to having the properties of reflectiveness and injectivity. Contributions of this kind, fewer in number, start with the intriguing precategorical paper of Garrett Birkhoff [Birkhoff 37] and include papers by P.D. Bacsich [Bacsich 73], R.-E. Hoffmann [Hoffmann 76], and, recently, the present authors with H. Herrlich [Brümmer Giuli Herrlich 89].

We take as primary motivation the exemplary behaviour of the usual completion in the category $\mathbf{X} = \mathbf{Unif}_0$ of Hausdorff uniform spaces with uniformly continuous maps. Let $\mathcal{U}$ denote the class of all dense uniform embeddings in this category. As is well known, every space X in $\mathbf{X}$ has a reflection $r_X : X \to RX$ into the full subcategory $\mathbf{R}$ of complete spaces, with $r_X \in \mathcal{U}$. The property known as $\mathcal{U}$-reflectiveness of $\mathbf{R}$ says that for any $f : X \to A$ in $\mathbf{X}$ with $A \in \mathbf{R}$ there is a unique $\mathbf{X}$-morphism $f^* : RX \to A$ with $f^* r_X = f$, and that also $r_X \in \mathcal{U}$. There is an additional property, generally known as "uniqueness of completions", namely that

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if $f : X \to A$ is any completion of X, i.e. if the above f belongs to $\mathcal{U}$ with $A \in \mathbf{R}$, then the morphism f^* is an isomorphism in $\mathbf{X}$. (This property is not implied by $\mathcal{U}$-reflectiveness, as one sees by changing $\mathbf{X}$ to the category of Tychonoff spaces with $\mathbf{R}$ the compact Hausdorff spaces, $\mathcal{U}$ the dense embeddings: a space can have inequivalent Hausdorff compactifications.)

We observe that the "uniqueness of completions" is equivalent to the property:

"whenever $u : X \to Y$ is in $\mathcal{U}$, then $Ru : RX \to RY$ is an isomorphism"

where R denotes the reflection functor. Equivalently,

$$\mathcal{U} \subset L(\mathbf{R})$$

where $L(\mathbf{R})$ denotes the class of all those $\mathbf{X}$-morphisms u for which Ru is iso.

However, the example $\mathbf{Unif}_0$ has the generally unnoticed stronger feature that $\mathcal{U} = L(\mathbf{R})$. The present paper serves to show that this feature, abstracted, gives a particularly simple theory with strong consequences and is situated at the top of a hierarchy of completion notions that have, partly, been much considered in the literature.

Henceforth $\mathbf{X}$ will be an arbitrary category and $\mathcal{U}$ any class of $\mathbf{X}$-morphisms, closed under composition and also closed under composition with isomorphisms. Already in the case $\mathcal{U} \subset L(\mathbf{R})$ we shall prove that $\mathbf{R} = \mathrm{Inj}\,\mathcal{U}$, the class of $\mathcal{U}$-injective objects (Corollary 1.5). We shall call the class $\mathcal{U}$ *firm* if and only if $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}$-reflective and $\mathcal{U} = L(\mathbf{R})$, and are then forced to call $\mathcal{U}$ subfirm if and only if $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}$-reflective and $\mathcal{U} \subset L(\mathbf{R})$. Among the weaker notions in our hierarchy, naturally, are the properties "$\mathbf{X}$ has $\mathcal{U}$-injective hulls" and "$\mathbf{X}$ has enough $\mathcal{U}$-injectives". If $\mathcal{U}$ is a subclass of the epimorphisms, the latter property is equivalent to subfirmness of $\mathcal{U}$. The general result is that $\mathcal{U}$ is subfirm if and only if $\mathbf{X}$ has enough $\mathcal{U}$-injectives and each $\mathcal{U}$-morphism is $\mathrm{Inj}\,\mathcal{U}$-dense (Theorem 1.6).

We show that, for subfirm $\mathcal{U}$, firmness is equivalent to coessentiality, i.e. each $\mathcal{U}$-morphism is $\mathcal{U}$-coessential (Proposition 1.11). For any given full reflective subcategory $\mathbf{R}$ of $\mathbf{X}$, $L(\mathbf{R})$ is the largest corresponding subfirm class; the smallest is just the class $S(\mathbf{R})$ of reflection morphisms. Moreover, $L(\mathbf{R})$ is the intersection of the well-known Maranda class $M(\mathbf{R})$ with the class of $\mathbf{R}$-dense $\mathbf{X}$-morphisms (Proposition 3.3).

Two of the major results of Cassidy, Hébert and Kelly [Cassidy Hébert Kelly 85] can be restated as the following nice characterization of firmness in case $\mathbf{X}$ is finitely complete and has intersections of arbitrary classes of strong monomorphisms (Theorem 3.4):

$\mathcal{U}$ is firm if and only if $\mathcal{U}$ is coessential and first factor of a factorization structure for $\mathbf{X}$-morphisms.

From a result of Bousfield [Bousfield 77] we then deduce an internal characterization of firmness (Theorem 3.5).

In our paper [Brümmer Giuli Herrlich 89] with Horst Herrlich, the main concern was with subfirm $\mathcal{U}$-reflections where $\mathcal{U}$ was a class of embeddings (suitably axiomatized) which were epimorphic. In that setting there were strong consequences, e.g. that a subfirm $\mathcal{U}$-reflection functor preserved embeddings and preserved products [Brümmer Giuli Herrlich 89, results 2.3, 2.5, 2.7]. The preconceived choice of $\mathcal{U}$ (i.e., epimorphic embeddings) resulted in a relative rarity of examples. In the present paper, the generality which we allow $\mathcal{U}$ yields an abundance of examples. In most of these examples, $\mathcal{U}$ still consists of embeddings, but these may be special (e.g. the C^*-embeddings and C-embeddings of general topology); and $\mathcal{U}$ may or may not consist of epimorphisms. We devote Section 2 of this paper to the interplay between subfirmness in $\mathbf{X}$ and subfirmness in certain reflective subcategories of $\mathbf{X}$, and thereby extend some of the results about T_0-subcategories in [Brümmer Giuli Herrlich 89].

For general categorical notions we refer to [Adámek Herrlich Strecker 90] or [Herrlich Strecker 79]. The present paper is self-contained and does not depend on [Brümmer Giuli Herrlich 89].

We are grateful to Walter Tholen for showing us the characterization of firmness in Theorem 3.4 below, and to Hubertus Bargenda for a conversation which led to the discovery of Theorem 1.6.

1. (Sub-)firm $\mathcal{U}$-reflections and $\mathcal{U}$-injective objects.

Let $\mathbf{X}$ be any category. Subcategories will always be assumed to be full and isomorphism-closed. If $\mathbf{R}$ is a reflective subcategory of $\mathbf{X}$ we will denote by R the reflection functor and by $r_X : X \to RX$ the $\mathbf{R}$-reflection morphism of the object X, and, for each $\mathbf{X}$-morphism $f : X \to Y$ with $Y \in \mathbf{R}$, we will denote by $f^* : RX \to Y$ the unique morphism such that $f^* r_X = f$.

If $\mathcal{U}$ is a given class of $\mathbf{X}$-morphisms one says $\mathbf{R}$ is *$\mathcal{U}$-reflective* if $r_X \in \mathcal{U}$ for every $\mathbf{X}$-object X.

We impose on $\mathcal{U}$ the following standing assumptions:

(α) $\mathcal{U}$ is closed under composition;
(β) $\mathcal{U}$ is closed under composition with isomorphisms on both sides.

Definition 1.1. (a) A subcategory $\mathbf{R}$ of $\mathbf{X}$ is called *subfirmly $\mathcal{U}$-reflective* in $\mathbf{X}$ if it is $\mathcal{U}$-reflective in $\mathbf{X}$ and the following condition is fulfilled:

(sfirm) $\qquad f^*$ is an isomorphism whenever f is in $\mathcal{U}$ with codomain in $\mathbf{R}$.

If $\mathbf{R}$ is $\mathcal{U}$-reflective, by property (α), (sfirm) is equivalent to

(sfirm') $\qquad Rf$ is an isomorphism whenever f is in $\mathcal{U}$.

(b) A class $\mathcal{U}$ of $\mathbf{X}$-morphisms is called *subfirm* if there exists a subfirmly $\mathcal{U}$-reflective subcategory $\mathbf{R}$ of $\mathbf{X}$.
If moreover the following condition holds:

(firm) $\qquad Rf$ is an isomorphism if and only if f is in $\mathcal{U}$, then the class $\mathcal{U}$ is called *firm*, and we say $\mathbf{R}$ is a *firmly $\mathcal{U}$-reflective subcategory of* $\mathbf{X}$.

For a given reflective subcategory $\mathbf{R}$ of $\mathbf{X}$ set

$$L(\mathbf{R}) = \{f \in \operatorname{Mor}\mathbf{X} \mid Rf \in \operatorname{Iso}\mathbf{X}\}.$$

The proof of the following result is trivial.

Proposition 1.2. *A family $\mathcal{U}$ of $\mathbf{X}$-morphisms is firm if and only if there exists a reflective subcategory $\mathbf{R}$ such that $\mathcal{U} = L(\mathbf{R})$.*

Recall that an $\mathbf{X}$-object J is called *$\mathcal{U}$-injective* if it is injective with respect to $\mathcal{U}$, that is: for each $u : X \to Y$ in $\mathcal{U}$ and $f : X \to J$ there exists $f' : Y \to J$ such that $f'u = f$ (f' is then called an *extension of f along u*). $\operatorname{Inj}\mathcal{U}$ will denote the full subcategory of all $\mathcal{U}$-injective objects of $\mathbf{X}$.

A morphism $u \in \mathcal{U}$ is called *$\mathcal{U}$-essential* if g belongs to $\mathcal{U}$ whenever gu belongs to $\mathcal{U}$. $\mathcal{U}^*$ will denote the family of all $\mathcal{U}$-essential morphisms.

A morphism $u \in \mathcal{U}$ is called *$\mathcal{U}$-coessential* if f belongs to $\mathcal{U}$ whenever uf belongs to $\mathcal{U}$. $\mathcal{U}_*$ will denote the family of all $\mathcal{U}$-coessential morphisms.

$\mathcal{U}$ is said to be an *essential* (respectively: *coessential*) *family* if $\mathcal{U} = \mathcal{U}^*$ (respectively: $\mathcal{U} = \mathcal{U}_*$).
Note that for each full and isomorphism-closed subcategory $\mathbf{A}$ of $\mathbf{X}$, $\operatorname{Mor}\mathbf{A}$ satisfies (α) and (β) and is both essential and coessential.

One says that the category $\mathbf{X}$ *has enough $\mathcal{U}$-injectives* if for every $\mathbf{X}$-object X there exists a $\mathcal{U}$-injective object Y and a morphism $u : X \to Y$ in $\mathcal{U}$.

If the previous property holds with $\mathcal{U}$-essential $u : X \to Y$, then one says that $\mathbf{X}$ *has $\mathcal{U}$-injective hulls.*

Let $\mathbf{X}'$ be a subcategory of $\mathbf{X}$. A morphism $f : X \to Y$ in $\mathbf{X}$ is called $\mathbf{X}'$-*dense* if $gf = hf$ implies $g = h$ whenever the common codomain of g and h is in $\mathbf{X}'$.

The universal property of reflections gives:

Lemma 1.3. *If $\mathbf{R}$ is reflective in $\mathbf{X}$ then every reflection morphism is $\mathbf{R}$-dense.*

The theorem below shows that subfirmness is strongly related to $\mathcal{U}$-injectivity.

Theorem 1.4. *If $\mathbf{R}$ is $\mathcal{U}$-reflective in $\mathbf{X}$ then the following conditions are equivalent:*

(i) *$\mathbf{R}$ is subfirmly $\mathcal{U}$-reflective in $\mathbf{X}$;*
(ii) *$\mathbf{R} = \operatorname{Inj}\mathcal{U}$ and every u in $\mathcal{U}$ is $\mathbf{R}$-dense;*
(iii) *$\mathbf{R} = \operatorname{Inj}\mathcal{U}$ and every u in $\mathcal{U}$ having codomain in $\mathbf{R}$ is $\mathbf{R}$-dense.*

Proof: (i) $\Rightarrow$ (ii). First we show that $\mathbf{R} \subset \operatorname{Inj}\mathcal{U}$: let $A \in \mathbf{R}$, $f : X \to A$ any morphism and $u : X \to Y$ a morphism in $\mathcal{U}$. By subfirmness, Ru is iso. Hence $f^*(Ru)^{-1}r_Y$ is an extension of f along u, consequently $A \in \operatorname{Inj}\mathcal{U}$.

To show that $\operatorname{Inj}\mathcal{U} \subset \mathbf{R}$ take any $X \in \operatorname{Inj}\mathcal{U}$. Then there is an extension e of the identity 1_X along r_X, i.e. $er_X = 1_X$. If we show that $r_X e = 1_{RX}$ then X, being isomorphic to RX, belongs to $\mathbf{R}$. Now $(r_X e)r_X = r_X(er_X) = r_X 1_X = 1_{RX} r_X$. Since RX belongs to $\mathbf{R}$, then $r_X e = 1_{RX}$, by Lemma 1.3.

It remains to show that every $u \in \mathcal{U}$ is $\mathbf{R}$-dense. To prove that assume, in the diagram below, $u \in \mathcal{U}$, $Z \in \mathbf{R}$ and $gu = hu$. Since $Z \in \mathrm{Inj}\,\mathcal{U} = \mathbf{R}$ and $r_Y \in \mathcal{U}$, there exist $g', h' : RY \to Z$ such that $g'r_Y = g$ and $h'r_Y = h$. So $g'(Ru)r_X = g'r_Y u = gu = hu = h'r_Y u = h'(Ru)r_X$. Now Ru is an isomorphism, by $u \in \mathcal{U}$, so that $g' = h'$, by $Z \in \mathbf{R}$ and Lemma 1.3. We conclude that $g = g'r_Y = h'r_Y = h$.

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "r_X"'] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, "r_Y"'] & Z \\
RX \arrow[r, "Ru"] & RY \arrow[ur, shift left, "g'"] \arrow[ur, shift right, "h'"'] &
\end{tikzcd}
$$

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). To prove subfirmness take any $u : X \to Y$ in $\mathcal{U}$ with $Y \in \mathbf{R}$. Then there is u^* such that $u^* r_X = u$ and by $RX \in \mathrm{Inj}\,\mathcal{U}$ (since $\mathrm{Inj}\,\mathcal{U} = \mathbf{R}$) there exists t such that $tu = r_X$. Now, by $Y \in \mathbf{R}$ and the assumption, u is $\mathbf{R}$-dense, so that from $u^*tu = u^*r_X = u = 1_Y u$ we deduce $u^*t = 1_Y$. On the other hand $(tu^*)r_X = t(u^*r_X) = tu = r_X = 1_{RX}r_X$ and $RX \in \mathbf{R}$, so that $tu^* = 1_{RX}$, by Lemma 1.3. Thus u^* is an isomorphism and $\mathbf{R}$ is subfirmly $\mathcal{U}$-reflective. $\square$

Corollary 1.5. *Given a morphism class $\mathcal{U}$ in a category $\mathbf{X}$, there exists at most one subfirmly $\mathcal{U}$-reflective subcategory $\mathbf{R}$ of $\mathbf{X}$. In that case $\mathbf{R} = \mathrm{Inj}\,\mathcal{U}$.*

It will be seen in Section 3 that, in contrast with the result above (which says that for every subfirm class $\mathcal{U}$ there is a unique reflective subcategory $\mathbf{R}$ which is subfirmly $\mathcal{U}$-reflective), for every reflective subcategory $\mathbf{R}$ of $\mathbf{X}$ there exists a non-empty interval of classes $\mathcal{U}$ such that $\mathbf{R}$ is subfirmly $\mathcal{U}$-reflective in $\mathbf{X}$. The largest class in that interval is $L(\mathbf{R})$.

We caution that the results 1.4 and 1.5 depend on the standing assumption (α) that $\mathcal{U}$ is compositive: in showing (i) $\Rightarrow$ (ii) we used $r_Y u \in \mathcal{U}$. Both of the standing assumptions (α) and (β) are repeatedly used throughout the paper.

As a standing notation, whenever dealing with (sub)firm $\mathcal{U}$, we shall denote the reflection functor to $\mathrm{Inj}\,\mathcal{U}$ by R, the reflection morphism by r_X, and sometimes for brevity $\mathrm{Inj}\,\mathcal{U}$ itself by $\mathbf{R}$.

Theorem 1.6. *The following conditions are equivalent:*

(i) $\mathrm{Inj}\,\mathcal{U}$ *is subfirm;*

(ii) $\mathbf{X}$ *has enough $\mathcal{U}$-injectives and every u in $\mathcal{U}$ is $\mathrm{Inj}\,\mathcal{U}$-dense;*

(iii) $\mathbf{X}$ *has enough $\mathcal{U}$-injectives and every u in $\mathcal{U}$ with $\mathcal{U}$-injective codomain is $\mathrm{Inj}\,\mathcal{U}$-dense.*

Proof: (i) $\Rightarrow$ (ii) follows from (i) $\Rightarrow$ (ii) of Theorem 1.4 and (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i) we show that $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}$-reflective (then the result follows from (iii) $\Rightarrow$ (i) of Theorem 1.4). Assuming (iii), for each $\mathbf{X}$-object X there exist a $\mathcal{U}$-injective object A and a $\mathcal{U}$-morphism $u : X \to A$. This is the $\mathrm{Inj}\,\mathcal{U}$-reflection of X. In fact, for each $f : X \to Y$ with Y $\mathcal{U}$-injective, by $u \in \mathcal{U}$ and $Y \in \mathrm{Inj}\,\mathcal{U}$, there exists $f' : A \to Y$ such that $f'u = f$. The uniqueness of f' follows from the fact that u is $\mathrm{Inj}\,\mathcal{U}$-dense. □

Example 1.7. Let $\mathbf{Ab}$ be the category of abelian groups and $\mathbf{R}$ be the category of divisible groups. Then it is well known that $\mathbf{R} = \mathrm{Inj}(\mathrm{Mono}\,\mathbf{Ab})$ and that every abelian group is a subgroup of a divisible group; in fact this is the $(\mathrm{Mono}\,\mathbf{Ab})$-injective hull. However $\mathbf{R}$ is not even reflective in $\mathbf{Ab}$.

We now have the following sufficient conditions for the existence of $\mathcal{U}$-injective hulls.

Proposition 1.8. (a) *If $\mathcal{U}$ is firm, then $\mathcal{U} = \mathcal{U}^*$ and $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}^*$-reflective in $\mathbf{X}$.*
(b) *If $\mathcal{U}$ is subfirm and $\mathcal{U}$ satisfies the condition:*

$$(*) \qquad \text{whenever } gf \in \mathrm{Iso}\,\mathbf{X} \text{ and } g \in \mathcal{U} \text{ then } f \in \mathcal{U},$$

then $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}^*$-reflective in $\mathbf{X}$.
(c) *If $\mathcal{U}$ is subfirm, then the following two conditions are equivalent:*

(i) *$\mathcal{U} \subset \mathrm{Mono}\,\mathbf{X}$;*
(ii) *Every $\mathrm{Inj}\,\mathcal{U}$-reflection morphism is in $\mathrm{Mono}\,\mathbf{X}$.*

Together with $\mathcal{U}$ subfirm, either condition implies that $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}^*$-reflective in $\mathbf{X}$.

Proof: (a) If $\mathcal{U}$ is firm then, by Proposition 1.2, $\mathcal{U} = L(\mathbf{R})$ for some reflective subcategory $\mathbf{R}$. Given u in $\mathcal{U}$ consider gu in $\mathcal{U}$. Then both Ru and $R(gu)$ are iso, so that Rg is iso, and then $g \in L(\mathbf{R})$, i.e. $g \in \mathcal{U}$, so that $u \in \mathcal{U}^*$. Thus $\mathcal{U} = \mathcal{U}^*$.

(b) To show that $r_X \in \mathcal{U}^*$, consider $t : RX \to Y$ with $tr_X \in \mathcal{U}$. Then $r_Y tr_X$ is in $\mathcal{U}$ with codomain in $\mathbf{R}$. Assuming $\mathcal{U}$ subfirm, we have the isomorphism $(r_Y tr_X)^* = r_Y t$. By condition $(*)$ then $t \in \mathcal{U}$ since $r_Y \in \mathcal{U}$. Thus $r_X \in \mathcal{U}^*$.

(c) Trivially (i) $\Rightarrow$ (ii). To show (ii) $\Rightarrow$ (i), assume $\mathcal{U}$ subfirm and let $u : X \to Y$ be in $\mathcal{U}$. Since $r_Y u$ is in $\mathcal{U}$ with codomain in $\mathbf{R} = \mathrm{Inj}\,\mathcal{U}$, it follows that $r_Y u$ is an $\mathbf{R}$-reflection and therefore mono by (ii). This shows that u is mono.

Again assume that $\mathcal{U}$ is subfirm and (ii) holds. To show $r_X \in \mathcal{U}^*$, consider any $g : RX \to Y$ with $gr_X \in \mathcal{U}$. Since both $r_X, gr_X \in L(\mathbf{R})$, we have $g \in L(\mathbf{R})$ by (a) above. Now $r_Y g$ is in $L(\mathbf{R})$ with codomain RY in $\mathbf{R}$. By the firmness of $L(\mathbf{R})$, $r_Y g$ is then an $\mathbf{R}$-reflection morphism. Since the domain RX of $r_Y g$ is in $\mathbf{R}$, $r_Y g$ is iso. Thus r_Y is a retraction, and by assumption (ii) it is mono, hence r_Y is iso, and so g is iso. Being an isomorphism between $\mathbf{R}$-objects, g is an $\mathbf{R}$-reflection and hence equivalent to r_{RX}. By definition, r_{RX} is in $\mathcal{U}$, and $\mathcal{U}$ is closed for composition with isos, so $g \in \mathcal{U}$. (We do not assume $\mathcal{U}$ to contain all isos.) Thus $r_X \in \mathcal{U}^*$. □

The following example shows that there exist categories $\mathbf{X}$ admitting classes $\mathcal{U}$ of morphisms such that: (a) $\mathcal{U}$ is closed under composition and under composition with isomorphisms, (b) $\mathcal{U}$ satisfies $(*)$ of 1.8, (c) $\mathrm{Inj}\,\mathcal{U}$ is $\mathcal{U}^*$-reflective, (d) $\mathcal{U}$ is not subfirm.

Example 1.9. Let $\mathbf{Top}_0$ be the category of T_0-spaces. It is well known that the subcategory **Sob** of sober spaces is subfirmly $\mathcal{V}$-reflective with $\mathcal{V}$ the class of all T_0-dense embeddings (cf. [Hoffmann 76], [Brümmer Giuli Herrlich 89]); in fact, $\mathcal{V}$ is firm. Consider now the class $\mathcal{U} = \mathcal{V} \cup \{f : T \to Y \mid T \text{ is a point and } Y \in \mathbf{Top}_0\}$. Then $\mathcal{U}$ trivially satisfies the conditions (a) and (b) above and $\mathrm{Inj}\,\mathcal{U} = \mathrm{Inj}\,\mathcal{V}$ (= **Sob**), so that $\mathrm{Inj}\,\mathcal{U}$ satisfies (c) also. Finally every $f : T \to Y$ with $Y \in \mathbf{Sob}$ is such that $f^* = f$, while f is iso iff Y is a point, so **Sob** is not subfirmly $\mathcal{U}$-reflective in $\mathbf{Top}_0$.

Theorem 1.10. *Let* $\mathbf{X}$ *be any category and* $\mathcal{U}$ *be a class of* $\mathbf{X}$*-morphisms satisfying the standing assumptions* (α) *and* (β)*. Then the implications shown in the following diagram are true. Moreover the diagram admits no arrows other than the ones displayed and those given by transitivity.*

$$\begin{tikzcd}
(1)\ \boxed{\mathcal{U} \text{ is firm}} \arrow[d, Rightarrow] & \\
(2)\ \boxed{\begin{array}{c}\mathcal{U} = \mathcal{U}^* \text{ and} \\ \mathrm{Inj}\,\mathcal{U} \text{ is } \mathcal{U}^*\text{-reflective}\end{array}} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
(3a)\ \boxed{\mathrm{Inj}\,\mathcal{U} \text{ is } \mathcal{U}^*\text{-reflective}} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (3b)\ \boxed{\mathcal{U} \text{ is subfirm}} \arrow[d, Rightarrow] \\
(4a)\ \boxed{\mathbf{X} \text{ has } \mathcal{U}\text{-injective hulls}} \arrow[d, Rightarrow] & (4b)\ \boxed{\mathrm{Inj}\,\mathcal{U} \text{ is } \mathcal{U}\text{-reflective}} \arrow[d, Rightarrow] \arrow[dl, Rightarrow] \\
(5a)\ \boxed{\mathbf{X} \text{ has enough } \mathcal{U}\text{-injectives}} & (5b)\ \boxed{\mathrm{Inj}\,\mathcal{U} \text{ is reflective}}
\end{tikzcd}$$

Proof: (1) $\Rightarrow$ (2): See 1.8 (a).
(2) $\Rightarrow$ (3b): Consider $f : X \to A$ with $f \in \mathcal{U}$ and $A \in \mathrm{Inj}\,\mathcal{U}$. We have a unique $f^* : RX \to A$ with $f^* r_X = f$. Since $RX \in \mathrm{Inj}\,\mathcal{U}$ and $f \in \mathcal{U}$, we have $s : A \to RX$ with $sf = r_X$. By the reflection property of r_X it follows that $sf^* = 1_{RX}$. Now $sf = r_X \in \mathcal{U}$ with $f \in \mathcal{U} = \mathcal{U}^*$, so that $s \in \mathcal{U}$ by the definition of $\mathcal{U}^*$. Since A is $\mathcal{U}$-injective, we then have $t : RX \to A$ with $ts = 1_A$. Thus s is inverse to f^*, and $\mathcal{U}$ is subfirm.

The other implications displayed are trivial. We turn to the non-implications.
(2) $\not\Rightarrow$ (1): Example 3.7 or (if $\mathcal{U}$ is required to consist of embeddings) the examples 3.8 and 3.11.
(3a) $\not\Rightarrow$ (3b): Example 1.9.
(4a) $\not\Rightarrow$ (5b): Example 1.7, with $\mathbf{X} = \mathbf{Ab}$ and $\mathcal{U} = \mathrm{Mono}\,\mathbf{Ab}$.

(3b) $\not\Rightarrow$ (4a): Let $\mathbf{X} = \mathbf{Set}$, with $\mathbf{R}$ the reflective subcategory of all singletons. $L(\mathbf{R}) = \mathrm{Mor}\,\mathbf{X}$. As in Section 3 below, let $S(\mathbf{R})$ denote the class of all $\mathbf{R}$-reflection morphisms, and let $\mathcal{U} = S(\mathbf{R}) \cup \{f \in \mathrm{Mor}\,\mathbf{X} : \mathrm{dom}\, f = \emptyset \text{ and } |\,\mathrm{codom}\, f| = 2\}$. Then $\mathcal{U}$ satisfies (α) and (β), so that by Theorem 3.2 (c) below, $\mathcal{U}$ is subfirm. The reflection map $r : \emptyset \to \{0\}$ is not $\mathcal{U}$-essential because the inclusion $f : \{0\} \to \{0, 1\}$ has $fr \in \mathcal{U}$ but $f \notin \mathcal{U}$. Thus (3a) is not satisfied. However it is a general fact, easily proved, that (3b) together with (4a) implies (3a). Hence this example satisfies (3b) but not (4a).
(5b) $\not\Rightarrow$ (5a): $\mathbf{X} = \mathbf{Haus}$ and $\mathcal{U} = \mathrm{Emb}\,\mathbf{X}$. Then $\mathrm{Inj}\,\mathcal{U}$ is the subcategory of all singletons [Adámek Herrlich Strecker 90, p. 140, 9.3. (4) (g)]. $\mathrm{Inj}\,\mathcal{U}$ is reflective but $\mathbf{Haus}$ does not have enough $\mathcal{U}$-injectives.

The above five counterexamples suffice to show that the above implication diagram admits no further arrows; in fact one sees from the diagram that:
(3a) $\not\Rightarrow$ (3b) entails (3a) $\not\Rightarrow$ (2) and (4b) $\not\Rightarrow$ (3b);
(4a) $\not\Rightarrow$ (5b) entails (4a) $\not\Rightarrow$ (3a), (4a) $\not\Rightarrow$ (3b), (4a) $\not\Rightarrow$ (4b), (5a) $\not\Rightarrow$ (4b), (5a) $\not\Rightarrow$ (5b);
(3b) $\not\Rightarrow$ (4a) entails (5a) $\not\Rightarrow$ (4a), (3b) $\not\Rightarrow$ (2), (3b) $\not\Rightarrow$ (3a), (4b) $\not\Rightarrow$ (3a), (4b) $\not\Rightarrow$ (4a);
(5b) $\not\Rightarrow$ (5a) entails (5b) $\not\Rightarrow$ (4b), (5b) $\not\Rightarrow$ (4a). □

Proposition 1.11. *$\mathcal{U}$ is firm if and only if $\mathcal{U}$ is subfirm and coessential.*

PROOF: Assume $\mathcal{U}$ firm. Then $\mathcal{U} = L(\mathbf{R})$ for some reflective subcategory $\mathbf{R}$ of $\mathbf{X}$. (cf. Proposition 1.2). Now $L(\mathbf{R})$ is coessential since u, uf in $L(\mathbf{R})$ imply that both Ru and $R(uf) = (Ru)(Rf)$ are isomorphisms, so that (Rf is an isomorphism, hence) f belongs to $\mathcal{U}$.

Conversely, if $\mathcal{U}$ is subfirm then $\mathcal{U} \subset L(\mathbf{R})$ for some reflective subcategory $\mathbf{R}$ of $\mathbf{X}$. If $f : X \to Y$ is in $L(\mathbf{R})$, then Rf being an isomorphism $(Rf)r_X = r_Y f$ is in $\mathcal{U}$ since r_X is in $\mathcal{U}$. Then if $\mathcal{U}$ is also coessential, $r_Y f$ and r_Y in $\mathcal{U}$ give f in $\mathcal{U}$. □

Lemma 1.12. *Let $\mathbf{R}$ be any monoreflective subcategory of $\mathbf{X}$. Then $L(\mathbf{R}) \subset \mathrm{Mono}\,\mathbf{X} \cap \mathrm{Epi}\,\mathbf{X}$.*

PROOF: From Proposition 1.8 (c) we have $L(\mathbf{R}) \subset \mathrm{Mono}\,\mathbf{X}$. For $f \in L(\mathbf{R})$, let $sf = tf$ with $s, t : Y \to Z$. Since Rf is iso, $Rs = Rt$. Then $r_Z s = (Rs)r_Y = (Rt)r_Y = r_Z t$, and since r_Z is mono, $s = t$. Thus f is epi. □

Corollary 1.13. *If $\mathcal{U}$ is subfirm such that $\mathrm{Inj}\,\mathcal{U}$ is monoreflective in $\mathbf{X}$, then $\mathcal{U} \subset \mathrm{Mono}\,\mathbf{X} \cap \mathrm{Epi}\,\mathbf{X}$.*

Proposition 1.14. (a) *Let $\mathcal{S} \subset \mathrm{Mono}\,\mathbf{X}$ be coessential and such that $\mathcal{U} = \mathcal{S} \cap \mathrm{Epi}\,\mathbf{X}$ satisfies the standing assumptions (α) and (β). If $\mathcal{U}$ is subfirm, then $\mathcal{U}$ is firm.*
(b) *In any concrete category $\mathbf{X}$ over $\mathbf{Set}$, let $\mathcal{U}$ be the class of all epimorphic embeddings. If $\mathcal{U}$ is subfirm, then $\mathcal{U}$ is firm.*

PROOF: (a) Let $\mathcal{U} = \mathcal{S} \cap \mathrm{Epi}\,\mathbf{X}$ be subfirm. We show that $\mathcal{U}$ is coessential. Consider $gf \in \mathcal{U}$ with $g \in \mathcal{U}$. Then $f \in \mathcal{S}$. Since $R(gf)$ and Rg are iso, Rf is iso, so that by Lemma 1.12 f is epi, so $f \in \mathcal{U}$. Thus $\mathcal{U}$ is coessential, and by 1.11 $\mathcal{U}$ is firm.
(b) This is immediate from (a). □

Example 1.15. In contrast to 1.14 (b), we have a subfirm but not firm class $\mathcal{U}$ of epimorphic embeddings in $\mathbf{X} = \mathbf{Top_0}$. Let $\mathbf{R} = \mathbf{Sob}$. Let $\mathbb{N}$ be the non sober T_0-space whose points are the natural numbers and whose non-void open sets are all sets of the form $\{n, n+1, n+2, \dots\}$. Let K be the topological coproduct of three copies on $\mathbb{N}$. Then the sobrification remainder $RK - K$ consists of three points $\infty_1, \infty_2, \infty_3$. Let $K_1 = K \cup \{\infty_1\}$, $K_2 = K \cup \{\infty_1, \infty_2\}$, both taken as subspaces of RK, and consider the inclusion maps $f_1 : K \to K_1$, $f_2 : K \to K_2$. Let $S(\mathbf{Sob})$ be the class of all reflection maps in $\mathbf{Top_0}$ to $\mathbf{Sob}$, and let $\mathcal{U}$ be the closure of the class $S(\mathbf{Sob}) \cup \{f_1, f_2\}$ under composition with isomorphisms. Then $\mathcal{U}$ satisfies the standing assumptions (α) and (β); for (α) one uses that $RK_1 = RK_2 = RK$. Also $\mathcal{U}$ is subfirm by Theorem 3.2 (c) because $S(\mathbf{Sob}) \subset \mathcal{U} \subset L(\mathbf{Sob})$. However, $\mathcal{U} = \mathcal{U}^*$ because $f_1 \in \mathcal{U}^*$. Indeed, the inclusion $m : K_1 \to K_2$ is not in $\mathcal{U}$ while $mf_1 = f_2$. Thus $\mathcal{U}$ satisfies (3b) but not (2) of Theorem 1.10. (By Proposition 1.8 (c), however, $\mathcal{U}$ satisfies (3a).)

Example 1.16. Let $\mathbf{X} = \mathbf{TopGrp_0}$, the category of Hausdorff topological groups and continuous group homomorphisms. $\operatorname{Emb}\mathbf{X}$ consists of those group homomorphisms which are topological embeddings; we call these the *embeddings* in $\mathbf{X}$. Let $\mathcal{U}$ be the class of all dense (in the topological sense) embeddings in $\mathbf{X}$. It is well known that the groups which are complete with respect to the central (=two-sided) uniformity form a $\mathcal{U}$-reflective subcategory, say $\mathbf{R}$, of $\mathbf{X}$, the reflector R giving the uniform completion in the central uniformity. (For the history of this result, see [Nummela 80].) We claim that $\mathbf{R}$ is firmly $\mathcal{U}$-reflective in $\mathbf{X}$. For let $f : G \to A$ be in $\mathcal{U}$ with A in $\mathbf{R}$. Clearly f is a dense uniform embedding for the central uniformities, so that the map $f^* : RG \to A$ is a uniform isomorphism and thus an $\mathbf{X}$-isomorphism. This establishes the subfirmness of $\mathcal{U}$. Now let $g : X \to Y$ be any $\mathbf{X}$-morphism with Rg iso. Then $r_Y g \in \mathcal{U}$. Clearly $g \in \operatorname{Emb}\mathbf{X}$. Viewing X, Y and RY just as topological spaces, with r_Y a dense embedding, we easily see that g is dense, so that $g \in \mathcal{U}$. Thus $\mathcal{U}$ is firm. We do not know whether $\operatorname{Emb}\mathbf{X} \cap \operatorname{Epi}\mathbf{X}$ is firm, because it is not known whether the latter class coincides with $\mathcal{U}$.

Remark 1.17. In the paper [Brümmer Giuli Herrlich 89] the present authors and H. Herrlich gave a large number of examples of concrete categories $\mathbf{X}$ over $\mathbf{Set}$ in which the class $\mathcal{U}$ of epimorphic embeddings was (in effect) shown to be subfirm. By the above result 1.14 (b), in all those examples $\mathcal{U}$ was in fact firm. To help the reader of that paper, we point out a shift in terminology: we considered there a class $\mathcal{S}$ of $\mathbf{X}$-morphisms under the standing assumptions ((S_1), (S_2), (S_3) in that paper) which held for embeddings but were slightly different from the assumptions on $\mathcal{S}$ in the present 1.14 (a); and what we there called "$\mathcal{S}$-firmly epireflective subcategory", is precisely what we now call "subfirmly $\mathcal{S} \cap \operatorname{Epi}\mathbf{X}$-reflective subcategory". Thus the old term "firm" is now replaced by "subfirm", which fortunately coincides with our new "firm" in all the most natural concrete examples.

2. $\mathcal{U}$-injectivity in subcategories.

If $\mathbf{T}$ is a full subcategory of a category $\mathbf{X}$ and $\mathcal{U}$ is a class of $\mathbf{X}$-morphisms we denote $\mathcal{U} \cap \operatorname{Mor}\mathbf{T}$ simply by $\mathcal{U}_{\mathbf{T}}$.

It is clear that if $\mathbf{R}$ is a subfirmly $\mathcal{U}$-reflective subcategory of $\mathbf{X}$ then $\mathbf{R}$ is subfirmly $\mathcal{U}_{\mathbf{T}}$-reflective in $\mathbf{T}$ for each subcategory $\mathbf{T}$ of $\mathbf{X}$ containing $\mathbf{R}$. Conversely assume $\mathbf{T}$ reflective in $\mathbf{X}$ and $\mathbf{R}$ subfirmly $\mathcal{V}$-reflective in $\mathbf{T}$ for a suitable class $\mathcal{V} \subset \operatorname{Mor}\mathbf{T}$. Then there exists a class $\mathcal{U}$ of $\mathbf{X}$-morphisms determined by $\mathcal{V}$ such that $\mathbf{R}$ is subfirmly $\mathcal{U}$-reflective in $\mathbf{X}$ and $\mathcal{V} = \mathcal{U}_{\mathbf{T}}$. This result is a particular case of the following:

Proposition 2.1. *Let $\mathbf{R} \subset \mathbf{T}$ be two reflective subcategories of a given category $\mathbf{X}$. Moreover, let $\mathcal{U}$ be a class of $\mathbf{X}$-morphisms such that:*

(a) *$\mathbf{T}$ is $\mathcal{U}$-reflective;*
(b) *if $u : X \to Y$ is in $\mathcal{U}$ and $Y \in \mathbf{T}$ then the morphism u^* such that $u^* t_X = u$ lies in $\mathcal{U}$.*

Then, if $\mathbf{R}$ is subfirmly $\mathcal{U}_{\mathbf{T}}$-reflective in $\mathbf{T}$, it is subfirmly $\mathcal{U}$-reflective in $\mathbf{X}$.

Proof: Clearly $\mathbf{R}$ is $\mathcal{U}$-reflective in $\mathbf{X}$. To prove subfirmness in $\mathbf{X}$, let $u : X \to A$ be an $\mathbf{X}$-morphism in $\mathcal{U}$ with $A \in \mathbf{R}$. By $\mathbf{R} \subset \mathbf{T}$ and property (b), the morphism u^*, satisfying $u^* t_X = u$, lies in $\mathcal{U}$. Then by the subfirmness of $\mathbf{R}$ in $\mathbf{T}$ the morphism $(u^*)^*$, satisfying $((u^*)^*) r_{TX} = u^*$ is an isomorphism. Thus $\mathbf{R}$ is subfirmly $\mathcal{U}$-reflective in $\mathbf{X}$.

□

Let $\mathbf{T}$ be a reflective subcategory of $\mathbf{X}$ and $\mathcal{V}$ be a class of $\mathbf{T}$-morphisms. Set

$$\mathcal{V}(\mathbf{T}) = \{f \in \operatorname{Mor}\mathbf{X} : Tf \in \mathcal{V}\}.$$

The morphisms in $\mathcal{V}(\mathbf{T})$ will be called *weak $\mathcal{V}$-morphisms*.

Corollary 2.2. *Let $\mathbf{R} \subset \mathbf{T}$ be respectively a reflective and an epireflective subcategory of a given category $\mathbf{X}$ and $\mathcal{V}$ be a class of $\mathbf{T}$-morphisms containing the isomorphisms and such that $\mathbf{R}$ is subfirmly $\mathcal{V}$-reflective in $\mathbf{T}$. Then $\mathbf{R}$ is subfirmly $\mathcal{V}(\mathbf{T})$-reflective in $\mathbf{X}$. Moreover, $\mathbf{R}$ is firmly $\mathcal{V}$-reflective in $\mathbf{T}$ if and only if it is firmly $\mathcal{V}(\mathbf{T})$-reflective in $\mathbf{X}$.*

Proof: Clearly $\mathcal{V}(\mathbf{T}) \cap \operatorname{Mor}\mathbf{T} = \mathcal{V}$, so that it is sufficient to show that, with $\mathcal{U} = \mathcal{V}(\mathbf{T})$, the conditions (a) and (b) of Proposition 2.1 are satisfied. Now, for each $\mathbf{X}$-object X, $T(t_X)$ is iso and $\mathcal{V}$ contains isos, so that $\mathbf{T}$ is $\mathcal{V}(\mathbf{T})$-reflective. On the other hand, if $u : X \to Y$ is in $\mathcal{V}(\mathbf{T})$ and $Y \in \mathbf{T}$ then $u^* = Tu$, thus u^* belongs to $\mathcal{V}$ which is contained in $\mathcal{V}(\mathbf{T})$. For the last statement we need to show that $\mathcal{V}$ is coessential if and only if $\mathcal{V}(\mathbf{T})$ is coessential, by Proposition 1.11. Now coessentiality of $\mathcal{V}$ implies coessentiality of $\mathcal{V}(\mathbf{T})$ by functoriality of T, and the reverse implication follows from $\mathcal{V}(\mathbf{T}) \cap \operatorname{Mor}\mathbf{T} = \mathcal{V}$. □

Corollary 2.3 ([Brümmer Giuli Herrlich 89, Proposition 3.6])**.** *Let $\mathbf{X}$ be a universal topological category* (*in the sense of* [Marny 79]) *and let $T_0\mathbf{X}$ be the full subcategory of $\mathbf{X}$ consisting of all T_0-objects. Then a reflective subcategory $\mathbf{R}$ of $\mathbf{X}$ is firmly* ($\operatorname{Mor} T_0\mathbf{X} \cap \operatorname{Emb}\mathbf{X} \cap \{T_0\text{-}dense\}$)*-reflective in $T_0\mathbf{X}$ if and only if $\mathbf{R}$ is firmly* ($\{Initial\} \cap \{T_0\text{-}dense\}$)*-reflective in $\mathbf{X}$.*

Proof: The proof follows from the following facts: (a) $T_0\mathbf{X}$ is epireflective in $\mathbf{X}$; (b) if $\mathbf{X}$ is universally topological then every $T_0\mathbf{X}$-reflection morphism is initial (cf.

[Marny 79]); (c) every initial morphism with domain a T_0-object is a monomorphism (hence an embedding). □

Example 2.4. Let $\mathbf{X}$ be one of the categories **Top** (topological spaces), $frm{-}eTop$ (bitopological spaces), **Unif** (uniform spaces), **Prox** (proximity spaces), **Qun** (quasi uniform spaces). Then $\mathbf{X}$ is a universal topological category. It is also well known that the corresponding subcategories $T_0\mathbf{X}$ admit as firm class the family $\mathcal{V} = \mathrm{Mor}\, T_0\mathbf{X} \cap \mathrm{Emb}\,\mathbf{X} \cap \{T_0\text{-dense}\}$ (see e.g. [Brümmer Giuli Herrlich 89] and Remark 1.15 above). Thus, by Corollary 2.3, $\{\text{Initial}\} \cap \{T_0\text{-dense}\}$ is a firm class in $\mathbf{X}$.

Example 2.5. A projection space is a pair $(X, \{\alpha_n\})$ consisting of a set X and a sequence of maps $\alpha_n : X \to X$ subject to the condition $\alpha_n\alpha_m = \alpha_{\mathrm{Min}(n,m)}$ for all $n, m \in \mathbb{N}$. A projection space $(X, \{\alpha_n\})$ is called separated if $\alpha_n(x) = \alpha_n(y)$ for each $n \in \mathbb{N}$ implies $x = y$. A projection map $f : (X, \{\alpha_n\}) \to (Y, \{\beta_n\})$ is a map $f : X \to Y$ satisfying the condition $\beta_n f = f\alpha_n$ for each $n \in \mathbb{N}$. A projection map f is called s-dense if $\beta_n(y) \in f(X)$ for each $y \in Y$ and $n \in \mathbb{N}$. **PRO** will denote the category of projection spaces and projection maps. $\mathbf{PRO}_s$ will denote the corresponding full subcategory of separated projection spaces.

A sequence (x_n) is called a Cauchy sequence if $\alpha_n(x_{n+1}) = x_n$, for each $n \in \mathbb{N}$, and a sequence (x_n) is said to converge to a point x if $\alpha_n(x) = x_n$, for each $n \in \mathbb{N}$. A projection space is called complete if every Cauchy sequence converges.

It is shown in [Giuli 89] that the full subcategory $\mathbf{CPRO}_s$ of complete separated projection spaces is firmly $\mathcal{V}$-reflective in $\mathbf{PRO}_s$ with $\mathcal{V}$ the class of all s-dense monomorphisms (firmly epireflective in the terminology of [Brümmer Giuli Herrlich 89], since the s-dense $\mathbf{PRO}_s$-morphisms and the $\mathbf{PRO}_s$-epimorphisms coincide; cf. [Giuli 89, Corollary 5.2 (b)]).

Let us denote by $\mathcal{U}$ the family of all s-dense projection maps $f : (X, \{\alpha_n\}) \to (Y, \{\beta_n\})$ in **PRO** satisfying the condition

$$f(x) = f(x') \text{ implies } \alpha_n(x) = \alpha_n(x'), \quad \text{for each } n \in \mathbb{N}.$$

Note that, for each projection space $(X, \{\alpha_n\})$, TX is the quotient of X obtained by the relation $x \sim x'$ if $\alpha_n(x) = \alpha_n(x')$ for each $n \in \mathbb{N}$, and that t_X is the corresponding quotient map. Then it is easy to see that $\mathcal{U} = \mathcal{V}(\mathbf{PRO}_s)$, so that Corollary 2.2 applies: $\mathbf{CPRO}_s$ is firmly $\mathcal{U}$-reflective in **PRO**.

Example 2.6. It is known that the category $\mathbf{R}$ of all divisible, torsion-free abelian groups is firmly $\mathcal{V}$-reflective in the category $\mathbf{T}$ of all torsion-free abelian groups, with $\mathcal{V}$ the class of all $\mathbf{T}$-dense monomorphisms. Now $\mathbf{T}$ is epireflective in **Ab**, the category of all abelian groups, so that, by Corollary 2.2, setting $\mathcal{U} = \{f \in \mathrm{Mor}\,\mathbf{Ab} : Tf \in \mathcal{V}\}$, we obtain that $\mathbf{R}$ is firmly $\mathcal{U}$-reflective in **Ab**.

In the results above we associate to every class $\mathcal{V}$ of $\mathbf{T}$-morphisms ($\mathbf{T}$ epireflective in $\mathbf{X}$) a class $\mathcal{U} = \mathcal{V}(\mathbf{T})$ of $\mathbf{X}$-morphisms such that $\mathcal{V}(\mathbf{T})_{\mathbf{T}} = \mathcal{V}$ and $\mathrm{Inj}\,\mathcal{U} = \mathrm{Inj}\,\mathcal{V} = \mathbf{R}$. It should be noted that, for a given class $\mathcal{U}$ of $\mathbf{X}$-morphisms, $\mathrm{Inj}\,\mathcal{U} \cap \mathrm{Ob}\,\mathbf{T} \subset \mathrm{Inj}\,\mathcal{U}_{\mathbf{T}}$ and that inclusion may be proper as the following example shows:

Example 2.7. Let $\mathbf{X} = R$-**Mod** be the category of left modules over a unitary ring R. For a given idempotent, cohereditary radical $\mathbf{r}$, let $\mathbf{T} = \mathcal{F}_{\mathbf{r}}$ be the class of all $\mathbf{r}$-torsion free modules and $\mathcal{U}$ be the class of all $\mathbf{T}$-dense monomorphisms of $\mathbf{X}$. Since $\mathbf{r}$ is cohereditary, the epimorphisms in $\mathbf{T}$ are surjective (cf. [Dikranjan Giuli 90]), so that every $\mathbf{T}$-dense monomorphism of $\mathbf{T}$ is an isomorphism, and consequently $\operatorname{Inj}\mathcal{U}_{\mathbf{T}} = \mathbf{T}$. Assume now that $\mathbf{r}$ is even non hereditary. Then there exists a non $\mathcal{U}$-injective $\mathbf{r}$-torsion free module by [Dikranjan Giuli 91, Proposition 4.4 (a)], consequently $\operatorname{Inj}\mathcal{U} \cap \operatorname{Ob}\mathbf{T}$ is properly contained in $\operatorname{Inj}\mathcal{U}_{\mathbf{T}} = \mathbf{T}$.

3. Characterizations of (sub)-firm classes.

For each reflective subcategory $\mathbf{R}$ of $\mathbf{X}$ let $S(\mathbf{R})$ denote the class of all $\mathbf{R}$-reflection morphisms in $\mathbf{X}$. $S(\mathbf{R})$ is closed under composition and under composition with isomorphisms and it is essential. By definition, $S(\mathbf{R}) \subset L(\mathbf{R})$ and also by definition, the following holds:

Proposition 3.1. $\mathbf{R}$ *is subfirmly* $S(\mathbf{R})$*-reflective in* $\mathbf{X}$ *and* $S(\mathbf{R})$ *is the smallest subfirm class corresponding to the reflective subcategory* $\mathbf{R}$.

Let $\mathbb{SBF}(\mathbf{X})$ be the conglomerate of all subfirm classes of $\mathbf{X}$ and $\mathbb{R}efl(\mathbf{X})$ be the conglomerate of all reflective subcategories of $\mathbf{X}$. Then the association $\mathcal{U} \to \operatorname{Inj}\mathcal{U}$ defines a map

$$I : \mathbb{SBF}(\mathbf{X}) \longrightarrow \mathbb{R}efl(\mathbf{X}).$$

Moreover, by Proposition 2.1 and Proposition 3.1 the associations $\mathbf{R} \to L(\mathbf{R})$ and $\mathbf{R} \to S(\mathbf{R})$ define two maps

$$L, S : \mathbb{R}efl(\mathbf{X}) \longrightarrow \mathbb{SBF}(\mathbf{X}).$$

Theorem 3.2. (a) *L and S are sections of I, i.e.* $IL = 1_{\mathbb{R}efl(\mathbf{X})} = IS$.

(b) *For each reflective subcategory* $\mathbf{R}$ *of* $\mathbf{X}$, $L(\mathbf{R})$ *is the largest and* $S(\mathbf{R})$ *is the smallest subfirm class such that* $I(L(\mathbf{R})) = \mathbf{R} = I(S(\mathbf{R}))$.

(c) *A class* $\mathcal{U}$ *of* $\mathbf{X}$*-morphisms is subfirm in* $\mathbf{X}$ *if and only if* $\operatorname{Inj}\mathcal{U}$ *is reflective in* $\mathbf{X}$ *and* $S(\operatorname{Inj}\mathcal{U}) \subset \mathcal{U} \subset L(\operatorname{Inj}\mathcal{U})$.

Proof: (a) Use the first part of the proof of (i) $\Rightarrow$ (ii) in Theorem 1.4 with $\mathcal{U} = L(\mathbf{R})$ and $\mathcal{U} = S(\mathbf{R})$ respectively.

(b) Trivial.

(c) One implication follows from (b). For the other, note that $S(\operatorname{Inj}\mathcal{U}) \subset \mathcal{U}$ says that $\operatorname{Inj}\mathcal{U}$ is $\mathcal{U}$-reflective and $\mathcal{U} \subset L(\operatorname{Inj}\mathcal{U})$ ensures the property (sfirm') in Definition 1.1, so that $\mathcal{U}$ is subfirm. □

The class $L(\mathbf{R})$ is strongly related with another class introduced by Maranda (cf. [Maranda 64]). For each class $\mathbf{P}$ of $\mathbf{X}$-objects set

$$M(\mathbf{P}) = \{(u : X \to Y) \in \operatorname{Mor}\mathbf{X} \mid \text{ every } f : X \to P,\ P \in \mathbf{P}, \text{ has an extension along } u\}.$$

$M(\mathbf{P})$ is called the *Maranda class* associated to $\mathbf{P}$. If $f : X \to A$ is an $\mathbf{X}$-morphism with codomain in a reflective subcategory $\mathbf{R}$ of $\mathbf{X}$ and $u : X \to Y$ belongs to $L(\mathbf{R})$, equivalently Ru is iso, then an extension of f along u is given by $f^*(Ru)^{-1}r_Y$. Thus, for each reflective subcategory $\mathbf{R}$, $L(\mathbf{R}) \subset M(\mathbf{R})$. The precise relation between the above two classes is as follows.

Proposition 3.3. *Let $\mathbf{R}$ be a reflective subcategory of $\mathbf{X}$. A morphism in $\mathbf{X}$ belongs to $L(\mathbf{R})$ if and only if it is $\mathbf{R}$-dense and belongs to the Maranda class $M(\mathbf{R})$.*

Proof: We know that $L(\mathbf{R}) \subset M(\mathbf{R})$. Moreover, $L(\mathbf{R}) \subset \{\mathbf{R}\text{-dense}\}$ is a special case of Theorem 1.4 because $\mathbf{R} = \operatorname{Inj} L(\mathbf{R})$ is firmly $L(\mathbf{R})$-reflective in $\mathbf{X}$. Thus $L(\mathbf{R}) \subset M(\mathbf{R}) \cap \{\mathbf{R}\text{-dense}\}$.

Now consider $f \in M(\mathbf{R}) \cap \{\mathbf{R}\text{-dense}\}$. To show $f \in L(\mathbf{R})$, we have to show that Rf is an isomorphism. By definition of $M(\mathbf{R})$, there exists a morphism d with $df = r_X$. Then $(Rf)d = r_Y$ because $RY \in \mathbf{R}$ and f is $\mathbf{R}$-dense.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r_X"'] & Y \arrow[d, "r_Y"] \arrow[dl, "d"'] \\
RX \arrow[r, shift left, "Rf"] & RY \arrow[l, shift left, "d^*"]
\end{tikzcd}$$

The $\mathbf{R}$-reflection gives d^* with $d^*r_Y = d$. Now $d^*(Rf)r_X = d^*r_Yf = df = r_X = 1_{RX}r_X$. Cancel r_X by the reflection property because the codomain of $d^*(Rf)$ and of 1_{RX} is in $\mathbf{R}$. Thus $d^*(Rf) = 1_{RX}$. Also $(Rf)d^*r_Y = (Rf)d = r_Y = 1_{RY}r_Y$ and cancel r_Y for the same reason. Thus $(Rf)d^* = 1_{RY}$ and so Rf is an isomorphism. □

The following characterization of firm classes follows from results of Cassidy, Hébert and Kelly [Cassidy Hébert Kelly 85], as Walter Tholen pointed out to us. First we note that the term "factorization system" in [Cassidy Hébert Kelly 85] means the same as "factorization structure for morphisms" in [Adámek Herrlich Strecker 90]. As in [Cassidy Hébert Kelly 85], for a morphism class $\mathcal{E}$ we denote $\mathcal{E}^{\downarrow}$ the class of those m such that for all $e \in \mathcal{E}$ and all p, q with $qe = mp$ there exists a unique d with $de = p$ and $md = q$.

Theorem 3.4. *Let $\mathbf{X}$ be finitely complete and have intersections of arbitrary classes of strong monomorphisms. Then a class $\mathcal{U}$ of $\mathbf{X}$-morphisms is firm if and only if $\mathcal{U}$ is coessential and $(\mathcal{U}, \mathcal{U}^{\downarrow})$ is a factorization structure for* $\operatorname{Mor}\mathbf{X}$.

Proof: Let $\mathcal{U}$ be firm. This means there is a reflective subcategory $\mathbf{R}$ of $\mathbf{X}$ such that $\mathcal{U} = L(\mathbf{R})$. By [Cassidy Hébert Kelly 85, p. 290], $(L(\mathbf{R}), L(\mathbf{R})^{\downarrow})$ is always a "prefactorization system" of $\mathbf{X}$, and by [Cassidy Hébert Kelly 85, Theorem 2.3] $L(\mathbf{R})$ is coessential. Under the given completeness assumption on $\mathbf{X}$, [Cassidy Hébert Kelly 85, Corollary 3.4] states that $(L(\mathbf{R}), L(\mathbf{R})^{\downarrow})$ is a factorization structure for $\operatorname{Mor}\mathbf{X}$.

Conversely, let $(\mathcal{U},\mathcal{U}^{\downarrow})$ be a factorization structure for $\operatorname{Mor}\mathbf{X}$. Assuming only that $\mathbf{X}$ has a terminal object $\mathbf{l}$, by [Cassidy Hébert Kelly 85, p. 290] we have a reflective subcategory $\mathbf{R}$ consisting of those objects A for which the morphism $A \to \mathbf{l}$ belongs to $\mathcal{U}^{\downarrow}$. Then, assuming $\mathcal{U}$ also to be coessential, by [Cassidy Hébert Kelly 85, Theorem 2.3], $\mathcal{U} = L(\mathbf{R})$, i.e. $\mathcal{U}$ is firm. □

The above result together with the characterization theorem for factorization systems given in [Bousfield 77, Theorem 3.1] gives:

Theorem 3.5. *Assume $\mathbf{X}$ cocomplete, finitely complete and with arbitrary intersections of strong monomorphisms. Then a class $\mathcal{U}$ of $\mathbf{X}$-morphisms is firm if and only if the following properties are fulfilled:*

(i) $\operatorname{Iso}\mathbf{X} \subset \mathcal{U}$;
(ii) *$\mathcal{U}$ is closed under composition;*
(iii) *$\mathcal{U}$ is coessential;*
(iv) *$\mathcal{U}$ is closed under the formation of colimits;*
(v) *(Solution set condition.) Each $\mathbf{X}$-morphism f has a set of factorizations $f = g_i u_i$, $i \in I$ with each u_i in $\mathcal{U}$ and such that for each factorization $f = gu$ with $u \in \mathcal{U}$ there is some $j \in I$ and some morphism h such that $u_j = hu$ and $g = g_j h$.*

Example 3.6. For each category $\mathbf{X}$ both $\operatorname{Iso}\mathbf{X}$ and $\operatorname{Mor}\mathbf{X}$ are coessential and first factors of factorization structures. $\operatorname{Iso}\mathbf{X}$ is firm (in fact $L(\mathbf{X}) = \operatorname{Iso}\mathbf{X}$). In contrast $\operatorname{Mor}\mathbf{X}$ need not be firm: take the multiplicative monoid $\mathbb{Z}$ of integers as (morphism class of a) category $\mathbf{X}$. Then the unique reflective subcategory of $\mathbf{X}$ is $\mathbf{R} = \mathbf{X}$ and the corresponding firm class is $\{-1, 1\} = \operatorname{Iso}\mathbf{X}$. In particular, $\operatorname{Mor}\mathbf{X} = \mathbb{Z}$ is not firm. If $\mathbf{X}$ has terminal objects then $\operatorname{Mor}\mathbf{X}$ is firm: in such a case $\operatorname{Mor}\mathbf{X} = L(\mathbf{R})$, with $\mathbf{R}$ the reflective subcategory whose objects are the terminal objects.

Example 3.7. It is known that the category $\mathbf{X} = \mathbf{Set}$ admits precisely four factorization structures (see [Adámek Herrlich Strecker 90, Example 14.2 (4)]). The first factors are respectively $\operatorname{Iso}\mathbf{X}$, $\operatorname{Mor}\mathbf{X}$, $\operatorname{Epi}\mathbf{X}$, and $\mathcal{U} = \{f : X \to Y \mid X = \emptyset \Rightarrow Y = \emptyset\}$. Only $\operatorname{Epi}\mathbf{X}$ is not coessential. Thus, by Theorem 3.4, $\mathbf{Set}$ admits exactly three firm classes: $\operatorname{Iso}\mathbf{X}$, $\operatorname{Mor}\mathbf{X}$ and $\mathcal{U}$. The corresponding reflective subcategories are $\mathbf{R}_1 = \mathbf{Set}$, $\mathbf{R}_2 = \{\text{singletons}\}$ and $\mathbf{R}_3 = \{\text{singletons}\} \cup \{\emptyset\}$. For the smallest subfirm classes we have: $S(\mathbf{R}_1) = \operatorname{Iso}\mathbf{X}$, $S(\mathbf{R}_2) = \{\text{maps into singletons}\}$, $S(\mathbf{R}_3) = (S(\mathbf{R}_2) - \{\emptyset \to \{\emptyset\}\}) \cup \{\emptyset \to \emptyset\}$. Then a family $\mathcal{U} \neq \operatorname{Iso}\mathbf{X}$ is subfirm either if it contains all maps into singletons or if it contains the empty map and all maps from a non empty set into singletons, and does not contain maps from the empty set into non empty sets.

The subfirm and firm classes above remain such for each topological category $\mathbf{X}$.

Example 3.8. Let $\mathbf{X} = \mathbf{Tych}$ be the category of Tychonoff spaces. For a reflective subcategory $\mathbf{R}$ of **Tych** the following conditions are equivalent:

(i) $\mathbf{R}$ contains the closed unit interval;
(ii) $S(\mathbf{R})$ consists of embeddings;
(iii) $S(\mathbf{R})$ consists of dense embeddings;
(iv) $S(\mathbf{R})$ consists of dense C^*-embeddings;
(v) $\mathbf{R}$ is bireflective;
(vi) $\mathbf{R}$ is monoreflective;
(vii) $L(\mathbf{R})$ consists of dense C^*-embeddings;
(viii) Every $\mathcal{U} \in I^{-1}(\mathbf{R})$ consists of dense C^*-embeddings.

Proof: (i) $\Rightarrow$ (ii): Since $[0,1]$ is in $\mathbf{R}$, $\mathbf{R}$ is closed under products, and every Tychonoff space X admits an embedding k in a product of closed unit intervals, it follows that the reflection r_X is an embedding because it is a first factor of k.

(ii) $\Rightarrow$ (iii): Every embedding is mono and by a general categorical result every monoreflective subcategory is bireflective. Moreover the epimorphisms in **Tych** are precisely the dense maps.

(iii) $\Rightarrow$ (v): We have observed that epi $=$ dense in **Tych**.

(v) $\Leftrightarrow$ (vi): This is a general categorical result.

(v) $\Rightarrow$ (i): The reflection map $r_{[0,1]}$ is a dense monomorphism with compact domain, hence iso, and then $[0,1]$ belongs to $\mathbf{R}$.

(iii) $\Leftrightarrow$ (iv): Use (iii) $\Rightarrow$ (i).

(vii) $\Rightarrow$ (viii): $\mathcal{U} \subset L(\mathbf{R})$ (cf. Theorem 3.2 (a) and (c)).

(viii) $\Rightarrow$ (iv): $S(\mathbf{R}) \in I^{-1}(\mathbf{R})$ (cf. Proposition 3.1).

(iv) $\Rightarrow$ (vii): Let $(f : X \to Y) \in L(\mathbf{R})$, that is, Rf is an isomorphism. Then since $r_Y f = (Rf) r_Y$, $r_Y f$ is a dense C^*-embedding by (iv). Then f as first factor of a C^*-embedding is such, too. Furthermore f is dense by the fact that $[0,1]$-dense $=$ dense $=$ **Tych**-epimorphism, the equivalence (iv) $\Leftrightarrow$ (i) and Proposition 3.3. □

Assume that P is a cogenerator of a reflective subcategory $\mathbf{R}$ of **Tych**, that is: X belongs to $\mathbf{R}$ if and only if X can be embedded as a closed subspace in a product of copies of P. Then $L(\mathbf{R}) = M(\{P\}) \cap \{\text{dense}\}$. In fact, by a standard argument, $M(\{P\}) = M(\mathbf{P})$ where $\mathbf{P}$ is the class of all products of copies of P. Moreover, $M(\mathbf{P}) \cap \{\text{dense}\} = M(\mathbf{R}) \cap \{\text{dense}\}$ ($= L(\mathbf{R})$, by Proposition 3.3). To prove the last statement, consider $u : X \to Y$ in $M(\mathbf{P}) \cap \{\text{dense}\}$ and $f : X \to A$ with A in $\mathbf{R}$. Then denoting by $k : A \to Q$ a closed embedding into a product of copies of P, we have an extension of kf along u. Since u is dense and k is closed, this extension can be restricted to A and this restriction is actually an extension of f along u. Thus $M(\mathbf{P}) \cap \{\text{dense}\} \subset M(\mathbf{R}) \cap \{\text{dense}\}$ and the reverse inclusion follows from $\mathbf{P} \subset \mathbf{R}$.

Particular instances of the results given above are:

(a) $\mathbf{R} = \{\text{compact Hausdorff spaces}\}$ and $P = [0,1]$. Then $L(\mathbf{R}) = \{\text{dense } C^*\text{-embeddings}\}$;

(b) $\mathbf{R} = \{\text{realcompact spaces}\}$ and $P =$ real line. Then $L(\mathbf{R}) = \{\text{dense } C\text{-embeddings}\}$;

(c) k infinite cardinal, $\mathbf{R} = \{k\text{-compact spaces in the sense of [Herrlich 67]}\}$ and $P =$ cogenerator described in [Hušek 69]. Then $L(\mathbf{R}) = \{\text{dense } C(k)\text{-embeddings}\}$, where an embedding $u : X \to Y$ is $C(k)$-embedding if every continuous map $f : X \to P$ has an extension along u.

Example 3.9. Let $\mathbf{R}$ be the subcategory of the topologically complete spaces in **Tych**. Then $L(\mathbf{R})$ consists of all those maps which are dense uniform embeddings with respect to the fine uniformity. To prove this, consider the fine uniformity functor $\Phi : \mathbf{Tych} \to \mathbf{Unif}_0$. Let $u : X \to Y$ be such that $\Phi u : \Phi X \to \Phi Y$ is a dense uniform embedding. Consider $f : X \to A$ with A topologically complete, i.e. ΦA complete. Let $\eta_{\Phi Y} : \Phi Y \to \gamma\Phi Y$ be the T_0-uniform completion of ΦY. Then $\eta_{\Phi Y}\Phi u$ is a dense uniform embedding into a complete uniform space, hence it is a completion of ΦX. Thus there exists a uniformly continuous map $g : \gamma\Phi Y \to \Phi A$ with $g\eta_{\Phi Y}\Phi u = \Phi f$. Letting T denote the functor which assigns the underlying topology, we have $T(g\eta_{\Phi Y})u = f$, which shows that $u \in M(\mathbf{R})$ and hence by density $u \in L(\mathbf{R})$. Conversely let $u : X \to Y$ belong to $L(\mathbf{R})$. Since $R = T\gamma\Phi$, $T\gamma\Phi u$ is a homeomorphism. Note that the codomain of the uniformly continuous map $\gamma\Phi u$ is the fine space $\gamma\Phi Y = \Phi RY$, and that $\gamma\Phi u$ has a continuous inverse which therefore is uniformly continuous. Thus $\gamma\Phi u$ is a uniform isomorphism. Then $r_{\Phi Y}\Phi u = (\gamma\Phi u)r_{\Phi X}$ is a uniform embedding, and therefore Φu is a uniform embedding. Further, u being in $L(\mathbf{R})$ is $\mathbf{R}$-dense by Proposition 3.3, and hence dense.

Example 3.10. For $\mathbf{X} = \mathbf{Tych}$, with $\mathbf{R} = \{\text{compact Hausdorff}\}$, we have seen that the firm class $L(\mathbf{R}) = \{\text{dense } C^*\text{-embeddings}\}$. The corresponding factorization structure has second factor $L(\mathbf{R})^{\downarrow} = \{\text{perfect maps}\}$ [Adámek Herrlich Strecker 90, Example 14.2 (6)].

Example 3.11. To exhibit a subfirm class $\mathcal{U}$ which is contained strictly between $S(\mathbf{R})$ and $L(\mathbf{R})$, take again $\mathbf{X} = \mathbf{Tych}$ and $\mathbf{R} = \{\text{compact Hausdorff}\}$. Let $\mathcal{U}$ consist of all dense C^*-embeddings with realcompact codomain. Clearly $\mathcal{U}$ satisfies our standing assumptions (α) and (β). Since $L(\mathbf{R}) = \{\text{dense } C^*\text{-embeddings}\}$ by 3.8, it is clear that the $\mathcal{U}$-maps are $\mathcal{U}$-essential. Thus $\mathcal{U}$ is subfirm with $\mathcal{U} = \mathcal{U}^*$, but not firm.

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