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Commentationes Mathematicae Universitatis Carolinae, Vol. 33 (1992), No. 1, 121--124

Persistent URL: <http://dml.cz/dmlcz/118477>

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A note on the Runge–Kutta method for stochastic differential equations

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Abstract. In the paper the convergence of a mixed Runge–Kutta method of the first and second orders to a strong solution of the Ito stochastic differential equation is studied under a monotonicity condition.

Keywords: stochastic differential equation, Runge–Kutta method, monotonicity and Lipschitz condition

Classification: 60H10, 65L05

1. Introduction.

We consider the Ito stochastic differential equation (SDE)

$$dX_t = f(t, X_t) dt + g(t, X_t) dW_t$$

with initial condition $X_0 = 0$, where $\{W_t\}$ is a standard Wiener process, $t \in [0, T]$, the random functions f, g are continuous functions of x , predictable with respect to (ω, t) and satisfying the linear growth condition, the function g satisfies the Lipschitz condition and f the monotonicity condition.

The convergence of Runge–Kutta (RK) approximation methods in the case when the function f is subjected to the stronger Lipschitz condition instead of the monotonicity condition is well known [1]. In [2] the convergence of the Euler scheme is proved provided f satisfies the monotonicity condition. The question is whether the RK approximation methods of higher orders can be applied under this condition. Our aim is to show a convergence of a generalized RK iterative scheme. In contradiction to the cited papers we use a second order RK approximation instead of the Euler scheme for the function $g(t, x)$. This result can be apparently generalized to the RK methods of any order n .

2. Results.

Let us have a sampling from $[0, T]$: $h = T/n$, $t_0 = 0$,

$$t_{i+1} = t_i + h, \quad i = 0, 1, \dots, n-1, \quad \text{and let } \Delta W_i = W_{t_{i+1}} - W_{t_i}.$$

Lemma. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\{\mathcal{F}_t, t \in \mathcal{R}_+\}$ a standard filtration. Assume that the functions $f : \Omega \times \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ and $g : \Omega \times \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ are continuous functions of $x \in \mathcal{R}$, predictable with respect to (ω, t) , that $g(\omega, \dots) \in \mathcal{C}^2$ for any ω , and that for every $\omega \in \Omega$, $t \geq 0$, $x, y \in \mathcal{R}$, they fulfil the conditions

- (i) $f^2(t, x) + g^2(t, x) \leq k_1(1 + x^2)$ — the linear growth condition,
- (ii) $2(x - y)(f(t, x) - f(t, y)) \leq k_2(x - y)^2$ — the monotonicity condition,
- (iii) $(g(t, x) - g(t, y))^2 \leq k_3(x - y)^2$ — the Lipschitz condition.

Assume further that for every $\omega \in \Omega$, $t \geq 0$, $x, y \in \mathcal{R}$,

- (iv) $\frac{\partial}{\partial x}g(t, x)$ satisfies the linear growth condition

$$\left(\frac{\partial}{\partial x}g(t, x)\right)^2 \leq k_4(1 + x^2),$$

- (v) $g(t, x)\frac{\partial}{\partial x}g(t, x)$ satisfies the Lipschitz condition

$$\left|g(t, x)\frac{\partial}{\partial x}g(t, x) - g(t, y)\frac{\partial}{\partial x}g(t, y)\right|^2 \leq k_5(x - y)^2,$$

- (vi) $\frac{\partial^2}{\partial x^2}g(t, x)$ and $\frac{\partial^2}{\partial t^2}g(t, x)$ are bounded.

Then the iterative scheme

$$(1) \quad \begin{aligned} \widehat{X}_{i+1} &= \widehat{X}_i + f(t_i, \widehat{X}_i)h + \frac{1}{2}(g(t_i, \widehat{X}_i) + g(t_{i+1}, \widetilde{X}_{i+1})) \Delta W_i, \\ \widetilde{X}_{i+1} &= \widehat{X}_i + f(t_i, \widehat{X}_i)h + g(t_i, \widehat{X}_i) \Delta W_i, \quad \widehat{X}_0 = 0, \quad i = 0, 1, \dots, n-1, \end{aligned}$$

converges in quadratic mean to the solution of the Stratonovich SDE

$$(2) \quad dY_t = f(t, Y_t) dt + g(t, Y_t) \circ dW_t, \quad Y_0 = 0,$$

that is

$$\max_{0 \leq i \leq n} \mathbf{E}|Y_{t_i} - \widehat{X}_i|^2 = \mathbf{O}(h^{1/2})$$

as h tends to 0.

PROOF: The Stratonovich SDE (2) is equivalent to the Ito SDE

$$dY_t = f(t, Y_t) dt + \frac{1}{2}g(t, Y_t)\frac{\partial}{\partial x}g(t, Y_t) dt + g(t, Y_t) dW_t, \quad Y_0 = 0.$$

We set $\widehat{Y}_0 = 0$ and

$$\widehat{Y}_{i+1} = \widehat{Y}_i + f(t_i, \widehat{Y}_i)h + \frac{1}{2}g(t_i, \widehat{Y}_i)\frac{\partial}{\partial x}g(t_i, \widehat{Y}_i)h + g(t_i, \widehat{Y}_i) \Delta W_i.$$

The results of [2] imply that $\mathbf{E} \max_{0 \leq i \leq n} |Y_i - \widehat{Y}_i|^2 = \mathbf{O}(h^{1/2})$. Since $|Y_i - \widehat{X}_i| \leq |Y_i - \widehat{Y}_i| + |\widehat{Y}_i - \widehat{X}_i|$, it is sufficient to show that $\mathbf{E} |\widehat{Y}_i - \widehat{X}_i|^2 = \mathbf{O}(h)$ for every $i = 0, 1, \dots, n$. Using the Taylor expansion of $g(t, x)$ we get

$$\begin{aligned} g(t_{i+1}, \widetilde{X}_{i+1}) &= g(t_i, \widehat{X}_i) + \frac{\partial}{\partial x} g(t_i, \widehat{X}_i) (f(t_i, \widehat{X}_i)h + g(t_i, \widehat{X}_i) \Delta W_i) + \\ &\quad + \frac{1}{2} \frac{\partial^2}{\partial x^2} g(t_i + \alpha_i h, \theta_i) (f(t_i, \widehat{X}_i)h + g(t_i, \widehat{X}_i) \Delta W_i)^2 + \\ &\quad + \frac{\partial}{\partial t} g(t_i, \widehat{X}_i)h + \frac{1}{2} \frac{\partial^2}{\partial t^2} g(t_i + \alpha_i h, \theta_i)h^2, \end{aligned}$$

where $\theta_i = \widehat{X}_i + \alpha_i(\widetilde{X}_{i+1} - \widehat{X}_i)$, $0 < \alpha_i < 1$.

We write the difference $\widehat{X}_{i+1} - \widehat{Y}_{i+1}$ in the form

$$\begin{aligned} \widehat{X}_{i+1} - \widehat{Y}_{i+1} &= \widehat{X}_i - \widehat{Y}_i + h(f(t_i, \widehat{X}_i) - f(t_i, \widehat{Y}_i)) + \Delta W_i (g(t_i, \widehat{X}_i) - g(t_i, \widehat{Y}_i)) + \\ &\quad + \frac{1}{2} h (g(t_i, \widehat{X}_i) \frac{\partial}{\partial x} g(t_i, \widehat{X}_i) - g(t_i, \widehat{Y}_i) \frac{\partial}{\partial x} g(t_i, \widehat{Y}_i)) + \\ &\quad + \frac{1}{2} g(t_i, \widehat{X}_i) \frac{\partial}{\partial x} g(t_i, \widehat{X}_i) (\Delta W_i^2 - h) + \frac{1}{2} f(t_i, \widehat{X}_i) \frac{\partial}{\partial x} g(t_i, \widehat{X}_i) h \Delta W_i + \\ &\quad + \frac{1}{4} \frac{\partial^2}{\partial x^2} g(t_{i+1} + \alpha_i h, \theta_i) (f(t_i, \widehat{X}_i)h + g(t_i, \widehat{X}_i) \Delta W_i)^2 \Delta W_i + \\ &\quad + \frac{1}{2} \frac{\partial}{\partial t} g(t_i, \widehat{X}_i) h \Delta W_i + \frac{1}{4} \frac{\partial^2}{\partial t^2} g(t_i + \alpha_i h, \theta_i) h^2 \Delta W_i. \end{aligned}$$

We square both sides of the equation and take the expectation. After estimating the members on the right-hand side, as e.g.

$$\mathbf{E}\{2(\widehat{X}_i - \widehat{Y}_i)h(f(t_i, \widehat{X}_i) - f(t_i, \widehat{Y}_i))\} \leq hk_2 \mathbf{E}(\widehat{X}_i - \widehat{Y}_i)^2$$

and

$$\begin{aligned} &\mathbf{E}\{2\Delta W_i (g(t_i, \widehat{X}_i) - g(t_i, \widehat{Y}_i)) \frac{1}{2} g(t_i, \widehat{X}_i) \frac{\partial}{\partial x} g(t_i, \widehat{X}_i) (\Delta W_i^2 - h)\} \leq \\ &\leq k_3^{\frac{1}{2}} \mathbf{E}\{|\widehat{X}_i - \widehat{Y}_i| (k_1(1 + \widehat{X}_i^2))^{\frac{1}{2}} (k_4(1 + \widehat{X}_i^2))^{\frac{1}{2}} \Delta W_i (\Delta W_i^2 - h)\} \leq \\ &\leq k \mathbf{E}\{|\widehat{X}_i - \widehat{Y}_i| (1 + \widehat{X}_i^2) \Delta W_i (\Delta W_i^2 - h) | \mathcal{F}_i\} \leq \\ &\leq k \mathbf{E}\{|\widehat{X}_i - \widehat{Y}_i| (1 + \widehat{X}_i^2) \mathbf{E}\{\Delta W_i (\Delta W_i^2 - h) | \mathcal{F}_i\}\} \leq \\ &\leq k \mathbf{E}\{|\widehat{X}_i - \widehat{Y}_i| (1 + \widehat{X}_i^2)\} \mathbf{E}\{\Delta W_i (\Delta W_i^2 - h)\} \leq \\ &\leq c(\mathbf{E}|\widehat{X}_i - \widehat{Y}_i|^2)^{\frac{1}{2}} \mathbf{E}\{\Delta W_i (\Delta W_i^2 - h)\} = 0 \end{aligned}$$

we obtain

$$\mathbf{E}|\widehat{Y}_{i+1} - \widehat{X}_{i+1}|^2 \leq \mathbf{E}|\widehat{Y}_i - \widehat{X}_i|^2 + C_1 h \mathbf{E}|\widehat{Y}_i - \widehat{X}_i|^2 + C_2 h^2.$$

Hence we conclude (see e.g. [2]) that $\mathbf{E}|\widehat{Y}_{i+1} - \widehat{X}_{i+1}|^2 = \mathbf{O}(h)$. This completes the proof of the lemma. \square

From the lemma one easily deduces the following

Theorem. *If $f(t, x)$ and $g(t, x)$ fulfil the conditions of Lemma then*

$$\sup_{0 \leq t \leq T} \mathbf{E}|Y_t - \widehat{X}_t|^2 = \mathbf{O}(h^{\frac{1}{2}}),$$

where we set

$$\begin{aligned} \widehat{X}_t &= \widehat{X}_i + \frac{t - t_i}{t_{i+1} - t_i} (\widehat{X}_{i+1} - \widehat{X}_i), \\ t_i &\leq t \leq t_{i+1}, \quad i = 0, 1, \dots, n-1, \end{aligned}$$

and \widehat{X}_i, Y_t are defined by (1), (2), respectively.

We note that the conclusions of Lemma and Theorem remain valid, if (iv), (v) are replaced by the assumption that $f(x, t)$, $g(x, t)$ and $\frac{\partial}{\partial x}g(t, x)$ are bounded.

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(Received October 26, 1990, revised October 10, 1991)