Milan Matoušek

Orthomodular lattices with fully nontrivial commutators

Commentationes Mathematicae Universitatis Carolinae, Vol. 33 (1992), No. 1, 25--32

Persistent URL: `http://dml.cz/dmlcz/118466`

Orthomodular lattices with fully nontrivial commutators

Milan Matoušek

Abstract. An orthomodular lattice L is said to have fully nontrivial commutator if the commutator of any pair $x, y \in L$ is different from zero. In this note we consider the class of all orthomodular lattices with fully nontrivial commutators. We show that this class forms a quasivariety, we describe it in terms of quasiidentities and situate important types of orthomodular lattices (free lattices, Hilbertian lattices, etc.) within this class. We also show that the quasivariety in question is not a variety answering thus the question implicitly posed in [4].

Keywords: orthomodular lattice, commutator, quasivariety

Classification: 06C15, 08C15

0. Preliminaries.

Let us first recall basic notions as we shall use them in the sequel (see e.g. [1], [6], [7], [9], etc.). Let us consider the orthomodular lattices as algebras of the type $\Delta = \{\wedge, \vee, ^{\perp}, \mathbf{0}, \mathbf{1}\}$, where the operations are subject to the standard axioms. Further, let us denote by OML the class of all orthomodular lattices, by I the class of all one-point OMLs, and by BA the class of all Boolean algebras. Thus, $I \subset BA \subset OML$.

Let $\mathbf{V} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots\}$ be the set of variables. As usual, instead of $\mathbf{x}_1, \mathbf{x}_2$ we shall sometimes write $\mathbf{x}, \mathbf{y}$. Further, let $\mathbf{T}$ denote the set of all $\Delta-terms$. Supposing $t \in \mathbf{T}$, let us denote by $var(t)$ the set of all variables occurring in t. If $n \in \mathbf{N}$ is a natural number, we shall use the notation $\mathbf{T}_n = \{t \in \mathbf{T}; var(t) \subseteq \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}\}$. For the sake of transparency, let us sometimes write $t(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ instead of $t \in \mathbf{T}_n$. If $t \in \mathbf{T}_n$ and $t_1, t_2, \dots, t_n \in \mathbf{T}$, the formula $t[\mathbf{x}_1 \to t_1, \mathbf{x}_2 \to t_2, \dots, \mathbf{x}_n \to t_n]$ will denote the term which comes into existence by substituting every $\mathbf{x}_i$ in t by t_i $(i = 1, 2, \dots, n)$.

Suppose that $L \in OML$. Then a mapping $v : \mathbf{V} \to L$ will be called a valuation in L. Obviously, the mapping v can be naturally extended over $\mathbf{T}$. Thus, we obtain $v_{\mathbf{T}} : \mathbf{T} \to L$. If $t \in \mathbf{T}_n$ and $v(\mathbf{x}_i) = a_i$ $(i = 1, 2, \dots, n)$, then the element $v_T(t)$ will be denoted by $t_L(a_1, a_2, \dots, a_n)$. So, every term $t \in \mathbf{T}_n$ determines a mapping $t_L : L^n \to L$.

The identities are words of the type $s \cong t$, where $s, t \in \mathbf{T}$ (the sign $\cong$ stands for the relation which transforms in a given $L \in OML$ to an equality). The

The author would like to thank Doc. P. Pták for the introduction to the problem and Dr. M. Navara for his interest in this research

quasiidentities are words of the type $(s_1 \cong t_1 \& \dots \& s_n \cong t_n) \Rightarrow s \cong t$, where $s_1, \dots, s_n, t_1, \dots, t_n, s, t \in \mathbf{T}$.

The semantic validity will be denoted by the symbol $\models$. If $s, t \in T$ and $OML \models s \cong t$, then we shall write $s \sim t$. If $OML \models s \leq t$, then we shall write $s \leq t$.

If α_1 (resp. α_2) is an identity or a quasiidentity, we say that α_1 is equivalent to α_2 if the following logical equivalence is valid in any $L \in OML : L \models \alpha_1 \iff L \models \alpha_2$.

The next simple proposition shows that in OML we may restrict ourselves to identities and quasiidentities of a very special type.

Proposition 0.1.

(a) *Every identity is equivalent to an identity of the form* $s \cong \mathbf{0}$.
(b) *Every identity is equivalent to a quasiidentity.*
(c) *Every quasiidentity is equivalent to a quasiidentity of the form* $s \cong \mathbf{0} \Rightarrow t \cong \mathbf{0}$.

The proof is elementary. □

Suppose that Ω denotes a set of quasiidentities in OML. Put $Mod(\Omega) = \{L \in OML; L \models \alpha \text{ for any } \alpha \in \Omega\}$. Further, if $\mathcal{A} \subseteq OML$ then $Id(\mathcal{A})$ will stand for the set of all identities which are valid in any $L \in \mathcal{A}$.

By a variety (resp. by a quasivariety) in OML we call any class of OML of the type $Mod(\Omega)$, where Ω is a set of identities (resp. a set of quasiidentities). A quasivariety which is not a variety will be called a proper quasivariety.

The following statement recalls a famous result of universal algebras (see [6]).

Proposition 0.2. *Suppose that $\mathcal{Q}$ is a subclass of OML. Then $\mathcal{Q}$ is a quasivariety if and only if it is closed under the formation of subalgebras, products, ultraproducts, isomorphic algebras and it contains a trivial algebra.*

A quasivariety $\mathcal{Q}$ is a variety if and only if it is closed under the formation of epimorphic images.

Let us now recall a central notion of our concern in this note. Put

$$c(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{x} \wedge \mathbf{y}^{\perp}) \vee (\mathbf{x}^{\perp} \wedge \mathbf{y}) \vee (\mathbf{x}^{\perp} \wedge \mathbf{y}^{\perp}).$$

Then the term $c(\mathbf{x}, \mathbf{y})$ is called the commutator of $\mathbf{x}, \mathbf{y}$.

Let us first observe the following useful property of the commutator (see [1]). (Following the language of the lattice theory, a set $P \subseteq L$ is called a p-filter in L if P is filter in L and $a \in P$ implies $x \vee (x^{\perp} \wedge a) \in P$ for all $x \in L$ (see [1, p. 182], [7, p. 75]). A p-ideal in L is defined dually.)

Proposition 0.3. *Suppose that $L \in OML$. Put*

$$P = \{z \in L; \exists a_1, \dots, a_n, b_1, \dots, b_n \in L : \wedge_{i=1}^{n} c(a_i, b_i) \leq z\}.$$

Then P is a p-filter in L.

The proof is simple. Let us call the latter p-filter P the commutator p-filter in L. □

In the conclusion of preliminaries, let us agree to reserve the symbol F_2 for the free orthomodular lattice over the set $\{x, y\}$. It should be noted that F_2 has 96 elements and that its representation is treated in detail in [1, p. 82]. Obviously, for any term $s \in \mathbf{T}_2$ there exists exactly one term $t \in F_2$ such that $s \sim t$.

1. The quasivariety NC.

Put

$$NC = \{L \in OML; \forall x, y \in L : c_L(x, y) \neq \mathbf{0}_L\} \cup I,$$

where I is the class of all one-point $OMLs$ and the indices indicate the belongness to the respective L.

We shall now examine the class NC. (It should be noted that the notion of commutator has already proved its significance in the study of $OMLs$ — see e.g. [2], [3], [9], etc. Here we add some universal algebra aspects.)

Proposition 1.1.

(a) *Every free orthomodular lattice belongs to NC.*
(b) *BA is a proper subclass of NC.*

Proof: (a) Let F be a free orthomodular lattice over a set M ($M \neq \emptyset$). Let P be the commutator p-filter in F. The factor morphism $\delta : F \to F/P$ is the reflection of F into BA (see [7, p. 299]). Then F/P is a free Boolean algebra over $\delta(M)$ ($\delta(M) \neq \emptyset$). If there were two elements $x, y \in F$ such that $c_F(x, y) = \mathbf{0}_F$, then $P = F$ and therefore F/P is a singleton. This is a contradiction since F/P was a free Boolean algebra. Thus, for all elements $x, y \in F$ we have $c_F(x, y) \neq \mathbf{0}_F$. It follows that $F \in NC$.

(b) Suppose that $B \in BA$ with $card(B) > 1$. Then for any couple $x, y \in B$ we have $c_B(x, y) = \mathbf{1}_B$. Thus, $BA \subseteq NC$. On the other hand, the free orthomodular lattice F_2 belongs to NC but it does not belong to BA. □

Corollary 1.2. *The class NC is a proper quasivariety of OML.*

Proof: By our definition of NC, we immediately see that

$$NC = Mod(\{c(\mathbf{x}, \mathbf{y}) \cong \mathbf{0} \Rightarrow \mathbf{1} \cong \mathbf{0}\}).$$

It follows that NC is a quasivariety. We have to show that NC is not a variety. Take the lattice MO_2 of [7] (recall that MO_2 is the horizontal sum of two 4-point Boolean algebras). One easily checks that $MO_2 \notin NC$. Since MO_2 has two generators, it is a homomorphic image of F_2. But F_2 belongs to NC and therefore NC is not a variety. □

An important class of $OMLs$ is formed by set-representable $OML's$ (see e.g. [9]). Since F_2 is set-representable, which does not seem to be explicitly known but can be established easily, and since every Boolean algebra is set-representable, Proposition 1.1 naturally suggests whether all set-representable $OMLs$ belong to NC. This is however not the case as the following example shows.

Example 1.3. Let X be a four-point set and let L denote the collection of all subsets of X with an even number of points. Then $L \in OML$ but $L \notin NC$. (The proof is simple.) $\square$

Another type of $OMLs$ are Hilbertian lattices. Let H be a Hilbert space over reals and let $L(H)$ denote the (orthomodular) lattice of all closed subspaces of H. Since the lattice $L(H)$ plays an important role both in the lattice theory and the applications (see [5], [9], etc.) it may be desirable to situate $L(H)$ within the class NC.

Theorem 1.4. *The lattice $L(H)$ belongs to NC if and only if* $\dim(H)$ *is a* (*finite*) *odd number.*

Proof: Suppose first that $\dim H < \infty$ and $\dim H$ is an odd number. Assume that $A, B \in L(H)$. Since $\dim A + \dim A^{\perp} = \dim H$, we have either $\dim A > \dim A^{\perp}$ or $\dim A < \dim A^{\perp}$, and we have the analogous situation for B, too. Obviously, $c(A,B) = c(A,B^{\perp}) = c(A^{\perp},B) = c(A^{\perp},B^{\perp})$ (where commutator is considered in $L(H)$). We have to show that each of the latter commutators is different from 0.

Without any loss of generality, we may assume that $\dim A > \dim A^{\perp}$ and $\dim B > \dim B^{\perp}$. Then we obtain $\dim A + \dim B > \dim H$. Since

$$\dim(A \vee B) = \dim A + \dim B - \dim(A \cap B) \text{ and } \dim(A \vee B) \leq \dim H,$$

we infer that $\dim(A \cap B) > 0$. Thus, $A \cap B \neq \{0_H\}$ and therefore $c(A,B) \neq \mathbf{0}_{L(H)}$.

Suppose now that $\dim H < \infty$ and $\dim H$ is an even number. Thus, $\dim H = 2n$. Let $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ be an orthonormal base of H. Let A (resp. B) be the linear subspace of H generated by $a_1, a_2, \ldots, a_n$ (resp., generated by $a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n$). One can check easily that $A^{\perp}$ is generated by $b_1, b_2, \ldots, b_n$ and $B^{\perp}$ is generated by $a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n$. It implies that $A \cap B = A \cap B^{\perp} = A^{\perp} \cap B = A^{\perp} \cap B^{\perp} = \{0_H\}$.

Finally, if $\dim H$ is infinite, we can write H as a direct sum of (infinitely many) copies of $\mathbf{R}^2$. Since $L(\mathbf{R}^2) \in NC$ and the commutator of $L(H)$ is formed "coordinatewise", we immediately obtain that $L(H) \notin NC$. This completes the proof of Theorem 1.4. $\square$

2. Special terms in OML (the set $\mathbf{T}^{ex}$).

Let us denote by $\mathbf{T}^{ex}$ the set of all terms $t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ ($n \geq 1$) with the following property: Supposing that $f : K \to L$ is a surjective morphism in OML, then the equality $t_L(b_1, b_2, \ldots, b_n) = \mathbf{0}_L$ implies the existence of elements $a_1, a_2, \ldots, a_n \in K$ such that $f(a_i) = b_i$ ($i = 1, 2, \ldots, n$) and $t_K(a_1, a_2, \ldots, a_n) = \mathbf{0}_K$.

The following proposition indicates the significance of the class $\mathbf{T}^{ex}$:

Proposition 2.1. *Suppose that $s(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \in \mathbf{T}^{ex}$ and suppose that $t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is an arbitrary term. Put $Q = Mod(\{s \cong \mathbf{0} \Rightarrow t \cong \mathbf{0}\})$. Then Q is variety.*

Proof: Since $s \in \mathbf{T}^{ex}$, the class Q is closed under the formation of epimorphic images. The rest follows from Proposition 0.2. $\square$

In what follows in this paragraph we shall concentrate on the study of when a term belongs to $\mathbf{T}^{ex}$. We shall succeed in clarifying this question for the case of two variables.

We shall first prove some auxiliary results.

Lemma 2.2.

(a) *Suppose that $s, t \in \mathbf{T}$ and suppose that $s \sim t$. If $s \in \mathbf{T}^{ex}$, then $t \in \mathbf{T}^{ex}$.*

(b) *Suppose that $t(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathbf{T}$. Suppose that we are given natural numbers $i_1, i_2, \dots, i_m$ ($m \in \mathbf{N}$) such that $1 \le i_1 < i_2 < \dots < i_m \le n$. Put $s = t[\mathbf{x}_{i_1} \to {\mathbf{x}_{i_1}}^{\perp}, \mathbf{x}_{i_2} \to {\mathbf{x}_{i_2}}^{\perp}, \dots, \mathbf{x}_{i_m} \to {\mathbf{x}_{i_m}}^{\perp}]$. Then $s \in \mathbf{T}^{ex}$ if and only if $t \in \mathbf{T}^{ex}$.*

Proof: (a) Evident.

(b) Without any loss of generality, we may suppose that $s = t[\mathbf{x}_1 \to {\mathbf{x}_1}^{\perp}]$ (the general case can be obtained by induction). Suppose first that $t \in \mathbf{T}^{ex}$. Suppose further that we are given an epimorphism $f : K \to L$ and we have $s_L(b_1, b_2, \dots, b_n) = \mathbf{0}_L$ for some elements $b_1, b_2, \dots, b_n \in L$. Then $t_L({b_1}^{\perp}, b_2, \dots, b_n) = s_L(b_1, b_2, \dots, b_n) = \mathbf{0}_L$. Since $t \in \mathbf{T}^{ex}$, there exist elements ${a_1}^{\perp}, a_2, \dots, a_n \in K$ such that $f(a_i) = b_i$ ($i = 1, 2, \dots, n$) and $t_K({a_1}^{\perp}, a_2, \dots, a_n) = \mathbf{0}_K$. It follows that $s_K(a_1, a_2, \dots, a_n) = t_K({a_1}^{\perp}, a_2, \dots, a_n) = \mathbf{0}_K$.

Conversely, suppose that $s \in \mathbf{T}^{ex}$. Put $t_1 = s[\mathbf{x}_1 \to {\mathbf{x}_1}^{\perp}]$. According to the argument above, we have $t_1 \in \mathbf{T}^{ex}$. Since $t \sim t_1$, we infer that $t \in \mathbf{T}^{ex}$. This completes the proof of Lemma 2.2. $\square$

Lemma 2.3. *Suppose that $s(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathbf{T}$. Then the following three conditions are equivalent:*

(i) *$BA \models s \cong \mathbf{1}$,*

(ii) *Put*

$$c_n = \bigvee_{\epsilon \in \{1, \perp\}^{\{1, \dots, n\}}} ({\mathbf{x}_1}^{\epsilon(1)} \wedge {\mathbf{x}_2}^{\epsilon(2)} \wedge \dots \wedge {\mathbf{x}_n}^{\epsilon(n)}),$$

where ${\mathbf{x}_i}^1 = \mathbf{x}_i$. Then $OML \models c_n(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \le s(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$.

(iii) *If $B_1 = \{0, 1\}$ is a two-point Boolean algebra and $v : \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\} \to B_1$ is a mapping, then $s_{B_1}(v(\mathbf{x}_1), v(\mathbf{x}_2), \dots, v(\mathbf{x}_n)) = 1$.*

Proof: (i) $\Longleftrightarrow$ (ii) See [1, Ch. 7, Th. 2.19].

(i) $\Rightarrow$ (iii) Evident.

(iii) $\Rightarrow$ (i) The assumption (iii) means that $B_1 \models s \cong \mathbf{1}$. It is well known that $Id(B_1) = Id(BA)$. Thus, $s \cong \mathbf{1} \in Id(BA)$ which gives $BA \models s \cong \mathbf{1}$. $\square$

Let us now recall that the center of an orthomodular lattice L, $C(L)$, is the set of all absolutely commutative elements in L (see e.g. [1], [7]). As known, $C(L)$ is a Boolean subalgebra of L (in fact, $C(L)$ is the intersection of all maximal Boolean subalgebras of L).

Lemma 2.4. *Suppose $s(\mathbf{x}_1, \mathbf{x}_2) \in \mathbf{T}$. Then there exist a term $t(\mathbf{x}_1, \mathbf{x}_2) \in C(F_2)$ such that the following two conditions are satisfied:*

(i) $s(\mathbf{x}_1, \mathbf{x}_2) \leq t(\mathbf{x}_1, \mathbf{x}_2)$,
(ii) $OML \models s(\mathbf{x}_1, \mathbf{x}_2) \cong \mathbf{0} \Rightarrow t(\mathbf{x}_1, \mathbf{x}_2) \cong \mathbf{0}$.

PROOF: All the terms $u(\mathbf{x}_1, \mathbf{x}_2) \in F_2$ such that $OML \models s(\mathbf{x}_1, \mathbf{x}_2) \cong \mathbf{0} \Rightarrow u(\mathbf{x}_1, \mathbf{x}_2) \cong \mathbf{0}$, form a p-ideal, I, in F_2. As F_2 has only finite Boolean subalgebras, I has a maximal element, some $t(\mathbf{x}_1, \mathbf{x}_2)$. According to [7, Exercise 8, p. 117], this element belongs to $C(F_2)$. □

We are now ready to state one of the main results of this paper.

Theorem 2.5. *Suppose that $s(\mathbf{x}, \mathbf{y}) \in \mathbf{T}$. Then $s \in \mathbf{T}^{ex}$ if and only if $BA \not\models s \cong \mathbf{1}$.*

PROOF: Suppose first that $s \in \mathbf{T}^{ex}$. Consider a homomorphism of a two-point Boolean algebra B_1 onto a one-point Boolean algebra B_0. Then there is a mapping $v : \{\mathbf{x}, \mathbf{y}\} \to \{0, 1\}$ such that $s_{B_1}(v(\mathbf{x}), v(\mathbf{y})) = 0$. Since $0 \neq 1$, we see that $BA \not\models s \cong \mathbf{1}$ (Lemma 2.3).

Suppose now that $BA \not\models s \cong \mathbf{1}$. Then there is a valuation $v : \{\mathbf{x}, \mathbf{y}\} \to \{0, 1\}$ such that $s_{B_1}(v(\mathbf{x}), v(\mathbf{y})) = \mathbf{0}$. Without any loss of generality, let us assume that $v(\mathbf{x}) = v(\mathbf{y}) = 0$. (Obviously, if e.g. $v(\mathbf{x}) = 1$, then we could consider the term $s_1 = s[\mathbf{x} \to \mathbf{x}^\perp]$ (Lemma 2.2).)

Suppose that $t \in C(F_2)$ with $s \leq t$, $OML \models s \cong \mathbf{0} \Rightarrow t \cong \mathbf{0}$. We are going to show that $s[\mathbf{x} \to \mathbf{x} \wedge t^\perp, \mathbf{y} \to \mathbf{y} \wedge t^\perp] \sim \mathbf{0}$.

We have $t \in C(F_2)$ and therefore $F_2 = (t] \oplus (t^\perp]$. Let us write $L_1 = (t], L_2 = (t^\perp]$. Suppose that $\pi_i : F_2 \to L_i$ $(i = 1, 2)$ are the respective projections onto L_i (thus, $\pi_1(p) = p \wedge t, \pi_2(p) = p \wedge t^\perp$). Put $\sigma = s[\mathbf{x} \to \mathbf{x} \wedge t^\perp, \mathbf{y} \to \mathbf{y} \wedge t^\perp]$. We can (and shall) assume that $\sigma \in F_2$. We shall now prove that $\pi_1(\sigma) = \pi_2(\sigma) = \mathbf{0}$. Making use of $s(\mathbf{x}, \mathbf{y}) \leq t(\mathbf{x}, \mathbf{y})$, we have

$\pi_1(\sigma) = \pi_1(\sigma(\mathbf{x}, \mathbf{y})) = \sigma_{L_1}(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})) = \sigma_{L_1}(\mathbf{x} \wedge t, \mathbf{y} \wedge t) = s_{L_1}((\mathbf{x} \wedge t) \wedge t^\perp, (\mathbf{y} \wedge t) \wedge t^\perp) = s_{L_1}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$,

and

$\pi_2(\sigma) = \sigma_{L_2}(\mathbf{x} \wedge t^\perp, \mathbf{y} \wedge t^\perp) = s_{L_2}((\mathbf{x} \wedge t^\perp) \wedge t^\perp, (\mathbf{y} \wedge t^\perp) \wedge t^\perp) = s_{L_2}(\pi_2(\mathbf{x}), \pi_2(\mathbf{y})) = \pi_2(s(\mathbf{x}, \mathbf{y})) = s(\mathbf{x}, \mathbf{y}) \wedge t^\perp(\mathbf{x}, \mathbf{y}) = \mathbf{0}$,

while $s(\mathbf{x}, \mathbf{y}) \leq t(\mathbf{x}, \mathbf{y})$.

We shall now complete the proof by showing that $s \in \mathbf{T}^{ex}$. Suppose that $K, L \in OML$ and suppose that $f : K \to L$ is a surjective homomorphism. Suppose further that we have $s_L(b_1, b_2) = \mathbf{0}_L$ for $b_1, b_2 \in L$. Since $OML \models s \cong \mathbf{0} \Rightarrow t \cong \mathbf{0}$, we obtain $t_L(b_1, b_2) = \mathbf{0}_L$.

Take elements $d_1, d_2 \in K$ such that $f(d_i) = b_i$ $(i = 1, 2)$. Let K_1 be the orthomodular lattice generated in K by $\{d_1, d_2\}$. Then there is an epimorphism $g : F_2 \to K_1$ such that $g(\mathbf{x}) = d_1$ and $g(\mathbf{y}) = d_2$. Put $a_i = d_i \wedge {t_K}^\perp(d_1, d_2)$ $(i = 1, 2)$. Obviously $a_1, a_2 \in K_1$. Moreover, we obtain

$f(a_i) = f(d_i) \wedge (f({t_K}^\perp(d_1, d_2))) = b_i \wedge (t_L(f(d_1), f(d_2)))^\perp = b_i \wedge (t_L(b_1, b_2))^\perp = b_i \wedge \mathbf{1}_L = b_i$ $(i = 1, 2)$.

Utilizing the above part of the proof, we finally have

$s_K(a_1, a_2) = s_K(d_1 \wedge t^{\perp}(d_1, d_2), d_2 \wedge t^{\perp}(d_1, d_2)) = \sigma_K(d_1, d_2) = g(\sigma) = g(\mathbf{0}) = \mathbf{0}_K$.

This completes the proof of Theorem 2.5. □

Remark. In the paper [8], R. Mayet obtained the following result which is related to Theorem 2.5: Suppose that $t(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathbf{T}$ is a term with the following properties:

(i) the term t can be constructed from the terms of the set $\{\mathbf{x}_1, \mathbf{x}_1{}^{\perp}, \dots, \mathbf{x}_n, \mathbf{x}_n{}^{\perp}\}$ applying only the operation symbols $\wedge, \vee$,

(ii) at most one of the terms $\mathbf{x}_i, \mathbf{x}_i{}^{\perp}$ $(i = 1, 2, \dots, n)$ is a subterm of t.

Then $t \in \mathbf{T}^{ex}$.

Theorem 2.6. *The class NC is the smallest proper quasivariety in OML which can be described by means of a single quasiidentity in the variables $\mathbf{x}, \mathbf{y}$.*

Proof: Put $Q = Mod(s(\mathbf{x}, \mathbf{y}) \cong \mathbf{0} \Rightarrow t(\mathbf{x}, \mathbf{y}) \cong \mathbf{0})$. Suppose that Q is proper quasivariety. Since Q is not a variety, we have $s \notin \mathbf{T}^{ex}$. By Proposition 2.5 and Lemma 2.3, we infer that $c(\mathbf{x}, \mathbf{y}) \leq s(\mathbf{x}, \mathbf{y})$. Suppose that $L \in NC$. Further, suppose that $s_L(a, b) = \mathbf{0}_L$ for some elements $a, b \in L$. Since $c \leq s$, we see that $c_L(a, b) = \mathbf{0}_L$. It forces $\mathbf{1}_L = \mathbf{0}_L$ and therefore L is a one-point algebra. It implies that $t_L(a, b) = \mathbf{0}_L$. Thus, $L \in Q$ and the proof is complete. □

3. Related open questions.

In the conclusion of this note we would like to formulate two open questions.

1. Suppose that F is a finitely generated orthomodular lattice. Suppose that $x \in F$ and I_x is the smallest p-ideal in F containing x. When is I_x a principal ideal?

2. Can we generalize Theorem 2.5 for the case of the terms with an arbitrary numbers of variables?

References

[1] Beran L., *Orthomodular Lattices, Algebraic Approach*, D. Reidel, Dordrecht, 1985.

[2] Bruns G., Greechie R., *Some finiteness conditions for orthomodular lattices*, Canadian J. Math. **3** (1982), 535-549.

[3] Chevalier G., *Commutators and Decomposition of Orthomodular Lattices*, Order **6** (1989), 181-194.

[4] Godowski R., Pták P., *Classes of orthomodular lattices defined by the state conditions*, preprint.

[5] Gudder S., *Stochastic Methods in Quantum Mechanics*, Elsevier North Holland, Inc., 1979.

[6] Grätzer G., *Universal Algebra*, 2nd edition, Springer-Verlag, New York, 1979.

[7] Kalmbach G., *Orthomodular Lattices*, Academic Press, London, 1983.

[8] Mayet R., *Varieties of orthomodular lattices related to states*, Algebra Universalis, Vol. 20, No **3** (1987), 368-396.

[9] Pták P., Pulmannová S., *Orthomodular structures as quantum logics*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.
[10] Pulmannová S., *Commutators in orthomodular lattices*, Demonstratio Math. **18** (1985), 187-208.

Department of Logic, Philosophical Faculty of the Charles University, 116 42 Prague, Czechoslovakia

(*Received* June 17, 1991)