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ON COMPLETIONS OF LINEARLY ORDERED GROUPS

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Summary. Each lattice ordered group G can be associated with a class $C(G)$ of lattice ordered groups which are in a certain sense generated by G (for a thorough definition cf. below). In this note we investigate the relations between $C(G)$ and the completion of G, where G is a linearly ordered group.

Keywords: linearly ordered group, closed l-subgroup, completion of a linearly ordered group

AMS Subject Classification: 06F15.

1. INTRODUCTION

For a lattice ordered group G we denote by $m(G)$ the completion of G (in the sense of [4], Chap. V, § 10; in [4], the notation G_D was used). This notion was studied in [3] and [6] for the abelian case, and in [2] for the non-abelian case.

Clearly we have $m(m(G)) = m(G)$. If $m(G) = G$, then G will be said to be m-complete. In the case when G is archimedean, $m(G)$ coincides with the Dedekind completion of $d(G)$ of G (cf., e.g., [1], Chap. XIII, § 13).

An l-subgroup G_1 of G is said to be closed if, whenever $\{x_i\}_{i\in I} \subseteq G_1$, $x \in G$ and the relation $x = \sup \{x_i\}_{i\in I}$ is valid in G, then $x \in G_1$. In such a case the corresponding dual condition also holds.

If G is an l-subgroup of a lattice ordered group H such that for each closed l-subgroup H_1 of H with $G \subseteq H_1$ the relation $H_1 = H$ is valid, then we say that H is a c-completion of G, or that G c-generates H.

We denote by $C(G)$ the class of all lattice ordered groups H such that

(i) H is m-complete;

(ii) G c-generates H.

In general, even in the case when G is archimedean, $C(G)$ can contain infinitely many (in fact, a proper class of) mutually non-isomorphic lattice ordered groups (cf. [8], [9]). More thoroughly: there exists an archimedean lattice ordered group G such that for each cardinal α there is $H \in C(G)$ with card $H \geqq \alpha$. Hence, in a certain sense, $C(G)$ can be "extremally large".

A similar situation occurs in the theory of Boolean algebras [5] and of vector lattices [7].

It is obvious that $m(G) \in C(G)$ for each lattice ordered group G. We shall show that if G is a linearly ordered group, then the class $C(G)$ is "extremally small"; namely, the following result will be proved:

$(*)$ Let G be a linearly ordered group. Then each element of $C(G)$ is isomorphic to $m(G)$.

This generalizes a result from [9] (Proposition 3.4) concerning archimedean linearly ordered groups.

2. PROOF OF $(*)$

Let G be a linearly ordered group. If $G = \{0\}$, then $(*)$ obviously holds. In what follows, G denotes a nonzero linearly ordered group and H a fixed element of $C(G)$.

Let T_1 be the system of all elements $h \in H$ such that $h = \bigvee_{i \in I} x_i$ for some subset $\{x_i\}_{i \in I}$ of G. Next, let T_2 have the dual meaning and $T = T_1 \cup T_2$.

The following assertion is obvious.

2.1. Lemma. *Let $\{h_j\}_{j \in J} \subseteq T_1$, $\{h'_k\}_{k \in K} \subseteq T_2$, $h, h' \in H$, $h = \bigvee_{j \in J} h_j$ and $h' = \bigwedge_{k \in K} h'_k$. Then $h \in T_1$ and $h' \in T_2$. Further, $G \subseteq T_1 \cap T_2$.*

2.2. Lemma. *Let $\emptyset \neq \{h_j\}_{j \in J} \subseteq T_1$, $h \in H$, $h = \bigwedge_{j \in J} h_j$. Then $h \in T$.*

Proof. If there exists $j \in J$ such that $h_j = h$, then $h \in T_1 \subseteq T$. Suppose that $h_j > h$ for each $j \in J$.

Let $j \in J$. There are elements $g_{ij} \in G$ $(i \in I(j))$ such that $h_j = \bigvee_{i \in I(j)} g_{ij}$. If $g_{ij} \leqq h$ for each $i \in I(j)$, then $h_j \leqq h$, which is a contradiction. Hence there is $f(j) \in I(j)$ such that $h < g_{f(j),j} \leqq h_j$. Therefore $h = \bigwedge_{j \in J} g_{f(j),j}$ and thus $h \in T_2 \subseteq T$.

2.3. Lemma. *Let $\emptyset \neq \{h_j\}_{j \in J} \subseteq T$, $h \in H$, $h = \bigwedge_{j \in J} h_j$. Then $h \in T$.*

Proof. Denote $J_1 = \{j \in J : h_j \in T_1\}$, $J_2 = \{j \in J : h_j \in T_2\}$. If $J_1 = \emptyset$, then in view of 2.1 we have $h \in T_2$. If $J_2 = \emptyset$, then 2.2 yields that $h \in T$.

Suppose that $J_1 \neq \emptyset \neq J_2$. For each $j \in J_2$ there is $\{g_{ij}\}_{i \in I(j)} \subseteq G$ such that $h_j = \bigwedge_{i \in I(j)} g_{ij}$. Then one of the following conditions is valid:

(i) $\bigwedge_{j \in J_2, i \in I(j)} g_{ij} = h$;

(ii) there exists $c \in H$ with $h < c$ such that $g_{ij} > c$ for each $j \in J_2$ and each $i \in I(j)$.

Assume that (i) holds. Then $h \in T_2$. Next, suppose that (ii) is valid. Then the set $J_3 = \{j \in J_1 : h_j < c\}$ is nonempty and $h = \bigwedge_{j \in J_3} h_j$. Thus 2.2 yields that $h \in T$.

Analogously we can verify that the assertion dual to 2.3 also holds. Hence we have

2.4. Corollary. *T is a closed sublattice of H.*

2.5. Lemma. *T is a subgroup of H.*

Proof. If $k \in \{1, 2\}$ and $x, y \in T_k$, then clearly $x + y \in T_k$. Let $x \in T_1$ and $y \in T_2$. Hence there are $\{x_i\}_{i \in I} \subseteq G$ and $\{y_j\}_{j \in J} \subseteq G$ such that $x = \bigvee_{i \in I} x_i$ and $y = \bigwedge_{j \in J} y_j$. Then

$$x + y = (\bigvee_{i \in I} x_i) + (\bigwedge_{j \in J} y_j) = \bigwedge_{j \in J}((\bigvee_{i \in I} x_i) + y_j) =$$
$$= \bigwedge_{j \in J} \bigvee_{i \in I}(x_i + y_j).$$

Thus in view of 2.2 we infer that $x + y$ belongs to T. Next, if $x \in T_1$, then $-x \in T_2$; analogously from $x \in T_2$ we obtain that $-x \in T_1$.

2.6. Lemma. $T = H$.

Proof. This follows from $H \in C(G)$ and from Lemmas 2.1, 2.4 and 2.5.

2.7. Lemma. $T_1 = T_2$.

Proof. It suffices to verify that $T_1 \subseteq T_2$. By way of contradiction, assume that there is $h \in T_1$ such that h does not belong to T_2. Hence $h \notin G$ and there is an element $c \in H$ with $h < c$ such that no element g' of G satisfies the relation $h < g' < c$.

Suppose that $h' \in H$ and $h < h' < c$. Then neither $h' \in T_1$ nor $h' \in T_2$ can be valid, which contradicts 2.6. Therefore the interval $[h, c]$ of H is a prime interval. By applying the translation $\psi(t) = t + (-c + h)$ (where t runs over H) we obtain that $[\psi(h), \psi(c)]$ is a prime interval in H as well; clearly $\psi(c) = h$. This shows that h does not belong to T_1, which is a contradiction.

2.8. Lemma. *Let $h \in H$. Then there are $X, Y \subseteq G$ such that* $\sup X = h = \inf Y$ *holds in H.*

Proof. This is a consequence of 2.6 and 2.8.

Now let us investigate the relations between the lattice ordered groups $m(G)$ and H.

Let $t \in m(G)$. Put $X_1 = \{g \in G: g \leqq t\}$ and $Y_1 = \{g \in G: g \geqq t\}$. Then in view of 1.3 in [2], the relations

(a) $\bigwedge(y - x: x \in X_1,\ y \in Y_1) = 0,$

(b) $\bigwedge(-x + y: x \in X_1,\ y \in Y_1) = 0$

are valid in G.

Let $h_0 \in H$, $h_0 \leqq y - x$ for each $x \in X_1$ and each $y \in Y_1$. If $h_0 > 0$, then in view of 2.8 there is $x_0 \in G$ with $0 < x_0 \leqq h_0$. Hence $x_0 \leqq y - x$ for each $x \in X_1$ and each $y \in Y_1$, which contradicts (a). Thus the condition

(a_1) $\bigwedge(y - x: x \in X_1,\ y \in Y_1) = 0$

is valid in H.

Similarly, the condition

(b_1) $\bigwedge(-x + y: x \in X_1,\ y \in Y_1) = 0$

holds in H.

Next, by virtue of the conditions $(a_1), (b_1)$ and in view of the fact that H is m-complete there is $h_1 \in H$ such that the relations

$$\sup X_1 = h_1 = \inf Y_1$$

hold in H. Put $\varphi(t) = h_1$.

It is easy to verify that for $t_1, t_2 \in H$ the equivalence

$$t_1 \leqq t_2 \Leftrightarrow \varphi(t_1) \leqq \varphi(t_2)$$

is valid.

Let $h \in H$. Next, let X and Y be as in 2.8. Again, because $m(H) = H$ and in view of 1.3 in [2] the conditions (a_1) and (b_1) hold for X and Y in H. This yields that the conditions (a) and (b) are valid in G for X and Y. Thus there is $t_1 \in m(G)$ such that

$$\sup X = t_1 = \inf Y$$

is valid in $m(G)$. Then we clearly have $\varphi(t_1) = h$. Hence φ is a surjection.

Let $t_2, t_3 \in m(G)$. There are $X_2, X_3 \subseteq G$ such that the relations

$$t_2 = \sup X_2 \quad \text{and} \quad t_3 = \sup X_3$$

hold in $m(G)$. This yields that

$$\varphi(t_2) = \sup X_2 \quad \text{and} \quad \varphi(t_3) = \sup X_3$$

are valid in H. Next, we obtain that the relation

$$t_2 + t_3 = \sup \{x_2 + x_3 \colon x_2 \in X_2 \text{ and } x_3 \in X_3\}$$

holds in $m(G)$ and that

$$\varphi(t_2) + \varphi(t_3) = \sup \{x_2 + x_3 \colon x_2 \in X_2 \text{ and } x_3 \in X_3\}$$

is valid in H. Thus

$$\varphi(t_2 + t_3) = \varphi(t_2) + \varphi(t_3)\,.$$

Clearly $\varphi(g) = g$ for each $g \in G$.

Summarizing, we have the following result (which implies that (*) holds):

2.9. Theorem. *Let G be a linearly ordered group and let $H \in C(G)$. Then there is an isomorphism φ of $m(G)$ onto H such that $\varphi(g) = g$ for each $g \in G$.*

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Súhrn

O ZÚPLNENIACH LIEÁRNE USPORIADANÝCH GRÚP

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Každej zväzove usporiadanej grupe G prislúcha trieda $C(G)$ zväzove usporiadaných grúp, ktoré sú v istom zmysle vytvorené grupou G. V tejto poznámke sa vyšetrujú vzťahy medzi $C(G)$ a zúplnením G v případe, že G je lineárne usporiadaná grupa.

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