Vladimír Volenec
GS-quasigroups

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GS-QUASIGROUPS

VLADIMIR VOLENEC, Zagreb

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Summary. A quasigroup $(Q, \cdot)$ is said to be a GS-quasigroup iff $aa = a$, $a(ab \cdot c) \cdot c = b$, $a . (a . bc) c = b$ for any $a, b, c \in Q$. A geometrical terminology can be introduced in any GS-quasigroup followed by some "geometrical" results. There is a GS-quasigroup $(Q, \cdot)$ iff there is a commutative group $(Q, +)$ and its automorphism φ such that the identity $(\varphi \circ \varphi)(a) - \varphi(a) - a = 0$ holds.

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1. INTRODUCTION

First of all, let us prove a lemma:

Lemma. *In any cancellative groupoid $(Q, \cdot)$ the identities*

(1) $$a(ab \cdot c) \cdot c = b\,, \quad a \cdot (a \cdot bc)\, c = b$$

are equivalent. Any cancellative groupoid with these two identities is a quasigroup.

Proof. According to (1) we have the identity $[a \cdot (a \cdot bc)\, c]\, c = bc$, which yields (1)'. Conversely, by (1)' we obtain $a\,[a(ab \cdot c) \cdot c] = ab$, which implies (1). For every $a, b \in Q$ there are $x, y \in Q$ such that $ax = b$ and $ya = b$. Indeed, we can take $x = (a \cdot ba)\, a$ and $y = a(ab \cdot a)$ because of (1)' and (1).

A quasigroup $(Q, \cdot)$ is said to be a *golden section quasigroup* or shortly a *GS-quasigroup* iff it satisfies the identities (1), (1)' and moreover the identity of *idempotency*

(2) $$aa = a\,.$$

Example 1. Let $(G, +)$ be a commutative group which possesses an automorphism φ such that

(3) $$(\varphi \circ \varphi)(a) - \varphi(a) - a = 0\,.$$

If we define an operation $\cdot$ on the set G by

(4) $$ab = a + \varphi(b - a)\,,$$

then $(G, \cdot)$ is a GS-quasigroup. Let us prove this statement. For any $a, b \in G$ the equations $ax = b$ and $ya = b$ are equivalent, because of (4), to the equations $a + \varphi(x - a) = b$ and $y + \varphi(a) - \varphi(y) = b$. The first equation has the unique solution $x = a + \varphi^{-1}(b - a)$ and the second equation can be written in the form $\varphi(y) + (\varphi \circ \varphi)(a) - (\varphi \circ \varphi)(y) = \varphi(b)$, i.e. by (3) in the form $(\varphi \circ \varphi)(a) - y = \varphi(b)$, and has the unique solution $y = (\varphi \circ \varphi)(a) - \varphi(b)$. Obviously (4) implies (2). By virtue of (4) we obtain after some arrangements

$$ab \cdot c = (\varphi \circ \varphi)(a) - 2\varphi(a) + a - (\varphi \circ \varphi)(b) + \varphi(b) + \varphi(c) .$$

Because of (3) this becomes

$$ab \cdot c = 2a - \varphi(a) - b + \varphi(c) .$$

Therefore, we have

$$a(ab \cdot c) \cdot c = 2a - \varphi(a) - [2a - \varphi(a) - b + \varphi(c)] + \varphi(c) = b .$$

We shall show later that Example 1 is a characteristic example of GS-quasigroups, i.e. that any GS-quasigroup can be derived from a commutative group as in Example 1.

Example 2. Let $(F, +, \cdot)$ be a field in which the equation

$$(5) \qquad q^2 - q - 1 = 0$$

has a solution q and let $*$ be an operation on the set Q defined by

$$(6) \qquad a * b = (1 - q)\, a + qb .$$

The identity $\varphi(a) = qa$ obviously defines an automorphism φ of the commutative group $(F, +)$ and (5) implies (3). The equality (6) can be written in the form $a * b = a + \varphi(b - a)$ and the result of Example 1 implies that $(F, *)$ is a GS-quasigroup.

Example 3. Let $(C, +, \cdot)$ be the field of complex numbers and $*$ an operation on the set C defined by (6), where $q = \frac{1}{2}(1 + \sqrt{5})$ or $q = \frac{1}{2}(1 - \sqrt{5})$. Then the equality (5) holds and the result of Example 2 implies that $(C, *)$ is a GS-quasigroup. This quasigroup has a beautiful geometrical interpretation which provides motivation for studying the GS-quasigroups and defining geometrical notions in them. Let us regard complex numbers as points of the Euclidean plane. For any two different points a, b the equality (6) can be written in the form

$$\frac{a * b - a}{b - a} = q ,$$

which means that the point $a * b$ divides the pair a, b in the ratio q. If $q = \frac{1}{2}(1 + \sqrt{5})$ or $q = \frac{1}{2}(1 - \sqrt{5})$, then the point b or a divides the pair $a, a * b$ or the pair $b, a * b$, respectively, in the ratio of the golden section, which justifies the term of GS-quasigroups. Any identity in the GS-quasigroup $(C, *)$ can be interpreted as a geometrical theorem which, of course, can be proved directly, but the theory of GS-quasigroups gives a better insight into the mutual relations of such theorems. For example, Figure 1 gives an illustration of the identity (1) in the quasigroup $(C, *)$ with $q =$

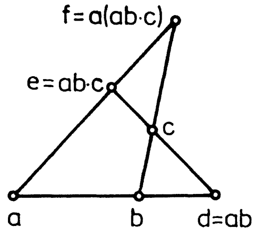

Fig. 1

$= \frac{1}{2}(1 + \sqrt{5})$ [and also of the identity (1)′ in the quasigroup $(C, *)$ with $q = \frac{1}{2}(1 - \sqrt{5})$], where the sign $\cdot$ is used instead of the sign $*$ (we shall use the same in all figures). All figures shall be represented in the above-mentioned quasigroup. Nevertheless, if we interchange the role of the elements x and y in all "products" of the form $x * y$, then we obtain in the same figures illustrations of the quasigroup $(C, *)$ with $q = \frac{1}{2}(1 - \sqrt{5})$.

2. ELEMENTARY PROPERTIES

The following result is obvious.

Theorem 1. *If the operation $\bullet$ on the set Q is defined by the equivalence*

$$a \bullet b = c \Leftrightarrow ba = c ,$$

i.e. by the identity $a \bullet b = ba$, then $(Q, \bullet)$ is a GS-quasigroup iff $(Q, \cdot)$ is a GS-quasigroup.

Further, we have

Theorem 2. *In any GS-quasigroup $(Q, \cdot)$ the mediality holds, i.e. we have the identity*

$$(7) \qquad ab \cdot cd = ac \cdot bd .$$

Proof. We have successively

$$ac \cdot (ab \cdot cd)\, d =^{(1)'} a[ab \cdot (ab \cdot cd)\, d] \cdot (ab \cdot cd)\, d =^{(1)} b =^{(1)'}$$
$$= ac \cdot (ac \cdot bd)\, d\,,$$

which yields (7).

Corollary. *In any GS-quasigroup* $(Q, \cdot)$ *the elasticity and left and right distributivity hold, i.e. we have the identities*

(8) $$ab \cdot a = a \cdot ba\,,$$

(9) $$a \cdot bc = ab \cdot ac\,, \quad ab \cdot c = ac \cdot bc\,, \qquad (9)'$$

Proof. Follows by (7) and (2).

Because of Theorem 2 we can apply all results of [3].

Theorem 3. *In any GS-quasigroup* $(Q, \cdot)$ *the identities*

(10) $$a(ab \cdot b) = b\,, \quad (b \cdot ba)\, a = b\,, \qquad (10)'$$

(11) $$a(ab \cdot c) = b \cdot bc\,, \quad (c \cdot ba)\, a = cb \cdot b \qquad (11)'$$

and the equivalencies

(12) $$ab = c \Leftrightarrow a = c \cdot cb\,, \quad ab = c \Leftrightarrow b = ac \cdot c \qquad (12)'$$

hold.

Proof. We have successively

$$a(ab \cdot c) \cdot c =^{(1)} b =^{(1)} b(bb \cdot c) \cdot c =^{(2)} (b \cdot bc)\, c\,,$$

which implies (11). Now, (10) follows from (11) because of (2). The identities (10)′ and (11)′ follow from (10) and (11) by Theorem 1. Moreover, by (10)′ and (10) we have $(c \cdot cb)\, b = c$ nad $a(ac \cdot c) = c$ and therefore the equality $ab = c$ is equivalent to $a = c \cdot cb$ and $b = ac \cdot c$.

In the sequel, let $(Q, \cdot)$ by any GS-quasigroup.

Theorem 4. *Any three of the four equalities*

(13) $$ab = d\,,$$

(14) $$ae = f\,,$$

(15) $$dc = e\,,$$

(16) $$fc = b$$

imply the remaining equality (Fig. 1).

Proof. The substitutions $b \leftrightarrow e$, $d \leftrightarrow f$ imply the substitutions (13) $\leftrightarrow$ (14) and (15) $\leftrightarrow$ (16). Therefore, it is sufficient to prove the implications (14) & (15) & (16) $\Rightarrow$

$\Rightarrow (13)$ and $(13) \& (14) \& (16) \Rightarrow (15)$. However, we have successively

$$ab =^{(16)} a \cdot fc =^{(14)} a(ae \cdot c) =^{(15)} a \cdot (a \cdot dc)\, c =^{(1)'} d\,,$$

$$dc =^{(13)} ab \cdot c =^{(16)} (a \cdot fc)\, c =^{(14)} a(ae \cdot c) \cdot c =^{(1)} e\,.$$

Theorem 5. *Any two of the four equalities*

(17) $\qquad ab = c\,,$

(18) $\qquad dc = b\,,$

(19) $\qquad ac = d\,,$

(20) $\qquad db = a$

imply the remaining two equalities (Fig. 2).

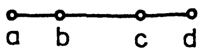

Fig. 2

Proof. The substitutions $a \leftrightarrow d$, $b \leftrightarrow c$ imply the substitutions $(17) \leftrightarrow (18)$ and $(19) \leftrightarrow (20)$. Therefore, it is sufficient to prove the first of the two implications

(21) $\qquad (17) \& (18) \Rightarrow (19)\,,$

(22) $\qquad (17) \& (18) \Rightarrow (20)\,;$

for the proof of the implication $(19) \& (20) \Rightarrow (17) \& (18)$ it suffices to prove the implication

(23) $\qquad (19) \& (20) \Rightarrow (18)$

and for the proof of the implications

(24) $\qquad (17) \& (19) \Rightarrow (18) \& (20)\,,$

(25) $\qquad (17) \& (20) \Rightarrow (18) \& (19)\,,$

$(18) \& (20) \Rightarrow (17) \& (19)$ and $(18) \& (19) \Rightarrow (17) \& (20)$ it is sufficient to prove the implications (24) and (25). We have successively

$$ac =^{(2)} a \cdot cc =^{(17)} a(ab \cdot c) =^{(18)} a \cdot (a \cdot dc)\, c =^{(1)'} d\,,$$

$$dc =^{(2)} dd \cdot c =^{(19)} (d \cdot ac)\, c =^{(20)} d(db \cdot c) \cdot c =^{(1)} b\,,$$

which proves the implications (21) and (23). Further, we obtain

$$dc \cdot b =^{(19)} (ac \cdot c)\, b =^{(17)} (a \cdot ab)\,(ab) \cdot b =^{(9)} a(ab \cdot b) \cdot b =^{(1)}$$
$$= b =^{(2)} bb\,,$$
$$d \cdot ac =^{(17)} d(a \cdot ab) =^{(20)} d \cdot (db)\,(db \cdot b) =^{(9)'} d \cdot (d \cdot db)\, b =^{(1)'}$$
$$= d =^{(2)} dd\,,$$

which implies the equalities (18) and (19), i.e. the implications (17) & (19) $\Rightarrow$ (18) and (17) & (20) $\Rightarrow$ (19) hold. The first of these implications together with (22) proves (24) and the second together with (23) proves (25).

3. PARALLELOGRAMS

In any GS-quasigroup $(Q, \cdot)$ we shall introduce a geometrical terminology motivated by Example 3. The elements of the set Q are called *points*.

We shall say that the points a, b, c, d form a *parallelogram* and write $\mathrm{Par}(a, b, c, d)$ iff there are two points p and q such that $ap = bq$ and $dp = cq$ [3, Corollary 1].

In [3] it was proved that (Q, Par) is a *parallelogram space*, i.e the quaternary relation $\mathrm{Par} \subset Q^4$ has the following properties:

1° For any three points a, b, c there is exactly one point d such that $\mathrm{Par}(a, b, c, d)$.

2° If (e, f, g, h) is any cyclic permutation of (a, b, c, d) or of (d, c, b, a), then $\mathrm{Par}(a, b, c, d)$ implies $\mathrm{Par}(e, f, g, h)$.

3° $\mathrm{Par}(a, b, c, d)$ and $\mathrm{Par}(c, d, e, f)$ imply $\mathrm{Par}(a, b, f, e)$.

Let us prove

Theorem 6. *For any points a, b, c we have* $\mathrm{Par}(a, b, c, a \cdot b(ca \cdot a))$ (Fig. 3).

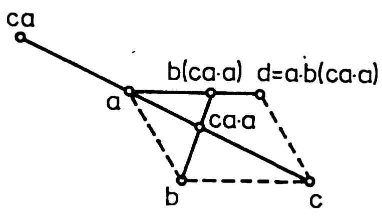

Fig. 3

Proof. It is sufficient to prove the equalities $ap = bq$, $[a \cdot b(ca \cdot a)]\, p = cq$ with $p = ab \cdot b$, $q = b$. We have successively

$$a(ab \cdot b) =^{(10)} b =^{(2)} bb\,,$$
$$[a \cdot b(ca \cdot a)]\,(ab \cdot b) =^{(7)} (a \cdot ab)\,[b(ca \cdot a) \cdot b] =^{(8)}$$

$$= (a \cdot ab)\,[b \cdot (ca \cdot a)\,b] =^{(7)} ab \cdot [ab \cdot (ca \cdot a)\,b] =^{(9)'}$$
$$= [a \cdot a(ca \cdot a)]\,b =^{(8)} [a \cdot (a \cdot ca)\,a]\,b =^{(10)'} cb\,.$$

By virtue of 1° Theorem 6 gives an alternative definition of parallelograms:

$$(26) \qquad \mathrm{Par}(a, b, c, d) \Leftrightarrow d = a \cdot b(ca \cdot a)\,.$$

On the other hand, we can start with this definition (26) and prove the properties 1°–3°. The property 1° is obvious. Further, let $\mathrm{Par}(a, b, c, d)$, i.e. $a \cdot b(ca \cdot a) = d$. For the proof of 2° it is sufficient to prove $\mathrm{Par}(b, c, d, a)$ and $\mathrm{Par}(c, b, a, d)$, i.e. $b \cdot c(db \cdot b) = a$ and $c \cdot b(ac \cdot c) = d$. However, we have successively

$$b[b \cdot c(db \cdot b)] =^{(9)} b[bc \cdot b(db \cdot b)] =^{(8)} b[bc \cdot (b \cdot db)\,b] =^{(11)'}$$
$$= b \cdot (bc)\,(bd \cdot d) =^{(9)} (b \cdot bc) \cdot b(bd \cdot d) =^{(10)} (b \cdot bc)\,d =$$
$$= (b \cdot bc)\,[a \cdot b(ca \cdot a)] =^{(7)} ba \cdot [bc \cdot b(ca \cdot a)] =^{(9)}$$
$$= b[a \cdot c(ca \cdot a)] =^{(10)} b \cdot aa =^{(2)} ba\,,$$

which implies $b \cdot c(db \cdot b) = a$, and we obtain

$$c \cdot b(ac \cdot c) =^{(9)} cb \cdot c(ac \cdot c) =^{(8)} cb \cdot (c \cdot ac)\,c =^{(9)'}$$
$$= [c \cdot (c \cdot ac)\,c]\,[b \cdot (c \cdot ac)\,c] =^{(1)'} a[b \cdot (c \cdot ac)\,c] =^{(11)'}$$
$$= a \cdot b(ca \cdot a) = d\,.$$

Now, let $\mathrm{Par}(a, b, c, d)$ and $\mathrm{Par}(c, d, e, f)$, i.e. $a \cdot b(ca \cdot a) = d$, $c \cdot d(ec \cdot c) = f$. Then

$$f = c \cdot d(ec \cdot c) =^{(9)} cd \cdot c(ec \cdot c) =^{(8)} cd \cdot (c \cdot ec)\,c =^{(11)'}$$
$$= cd \cdot (ce \cdot e) =^{(7)} (c \cdot ce) \cdot de =^{(11)} e(ec \cdot e) \cdot de =^{(8)}$$
$$= (e \cdot ec)\,e \cdot de =^{(9)'} (e \cdot ec)\,d \cdot e = (e \cdot ec)\,[a \cdot b(ca \cdot a)] \cdot e =^{(7)}$$
$$= (ea)\,[ec \cdot b(ca \cdot a)] \cdot e =^{(7)} (ea)\,[eb \cdot c(ca \cdot a)] \cdot e =^{(10)}$$
$$= (ea)\,(eb \cdot a) \cdot e =^{(9)'} (e \cdot eb)\,a \cdot e =^{(9)'} (e \cdot eb)\,e \cdot ae =^{(8)}$$
$$= e(eb \cdot e) \cdot ae =^{(11)} (b \cdot be) \cdot ae =^{(7)} ba \cdot (be \cdot e) =^{(9)'}$$
$$= b(be \cdot e) \cdot a(be \cdot e) =^{(10)} e \cdot a(be \cdot e)\,,$$

i.e. $\mathrm{Par}(e, a, b, f)$, wherefrom by 2° we obtain $\mathrm{Par}(a, b, f, e)$.

Let us prove some theorems about parallelograms.

Theorem 7. *For any points a, b we have* $\mathrm{Par}(a, a, b, b)$.

Proof. We have

$$a \cdot a(ba \cdot a) =^{(8)} a \cdot (a \cdot ba)\,a =^{(1)'} b\,.$$

Theorem 8. *Any two of the three statements* $\mathrm{Par}(a, b, c, d)$, $\mathrm{Par}(e, f, g, h)$ *and* $\mathrm{Par}(ae, bf, cg, dh)$ *imply the remaining statement.*

Proof. By (7) we obtain successively

$$ae \cdot [bf \cdot (cg \cdot ae)(ae)] = ae \cdot [bf \cdot (ca \cdot ge)(ae)] =$$
$$= ae \cdot [bf \cdot (ca \cdot a)(ge \cdot e)] = ae \cdot [b(ca \cdot a) \cdot f(ge \cdot e)] =$$
$$= [a \cdot b(ca \cdot a)] [e \cdot f(ge \cdot e)]$$

and it becomes obvious that any two of the three equalities $a \cdot b(ca \cdot a) = d$, $e \cdot f(ge \cdot e) = h$ and $ae \cdot [bf \cdot (cg \cdot ae)(ae)] = dh$ imply the remaining equality.

Theorem 9. *For any points* a, b, c, d *we have* $\mathrm{Par}(ab, cb, cd, ad)$.

Proof. According to Theorem 7 and the property 2° we have $\mathrm{Par}(a, c, c, a)$ and $\mathrm{Par}(b, b, d, d)$, wherefrom by Theorem 8 $\mathrm{Par}(ab, cb, cd, ad)$ follows.

Theorem 10. *For any point* p *the statements* $\mathrm{Par}(a, b, c, d)$, $\mathrm{Par}(ap, bp, cp, dp)$ and $\mathrm{Par}(pa, pb, pc, pd)$ are equivalent.

Proof. By Theorem 7 we have $\mathrm{Par}(p, p, p, p)$ and the statement of our theorem follows by Theorem 8.

Theorem 11. *If* a, b, c *are any three points and* $d = ac$, $e = ab$, $f = ec$, $g = df$, *then* $\mathrm{Par}(a, b, d, f)$, $\mathrm{Par}(b, e, f, g)$, $\mathrm{Par}(a, e, d, g)$ *hold* (Fig. 4).

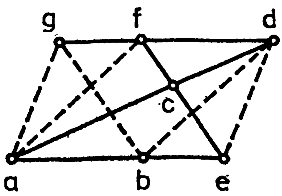

Fig. 4

Proof. We must prove the statements $\mathrm{Par}(a, b, ac, ab \cdot c)$, $\mathrm{Par}(b, ab, ab \cdot c, (a \cdot ab)\, c)$, $\mathrm{Par}(a, ab, ac, (a \cdot ab)\, c)$ because of $f = ec = ab \cdot c$ and

$$g = df = ac \cdot (ab \cdot c) =^{(9)'} (a \cdot ab)\, c\,.$$

However, we have successively

$$a[b \cdot (ac \cdot a)\, a] =^{(9)} ab \cdot [a \cdot (ac \cdot a)\, a] =^{(8)} ab \cdot [a(ac \cdot a) \cdot a] =^{(1)}$$
$$= ab \cdot c\,,$$
$$b \cdot (ab)\,[(ab \cdot c)\, b \cdot b] =^{(9)} b \cdot [ab \cdot (ab \cdot c)\, b]\,(ab \cdot b) =^{(11)}$$
$$= b \cdot (c \cdot cb)\,(ab \cdot b) =^{(9)} b(c \cdot cb) \cdot b(ab \cdot b) =^{(8)}$$
$$= b(c \cdot cb) \cdot (b \cdot ab)\, b =^{(7)} b(b \cdot ab) \cdot (c \cdot cb)\, b =^{(10)'}$$
$$= b(b \cdot ab) \cdot c =^{(8)} b(ba \cdot b) \cdot c =^{(11)} (a \cdot ab)\, c\,,$$
$$a[ab \cdot (ac \cdot a)\, a] =^{(8)} a[ab \cdot (a \cdot ca)\, a] =^{(11)'} a \cdot (ab)\,(ac \cdot c) =^{(9)}$$
$$= (a \cdot ab) \cdot a(ac \cdot c) =^{(10)} (a \cdot ab)\, c\,.$$

Now, if $c = ab = e$, then we have two equalities $ab = c$ and $ac = d$ of Theorem 5 and $f = ec = cc = c$ because of (2). Therefore $\mathrm{Par}(a, b, d, f)$ implies the following theorem:

Theorem 12. *By the hypothesis of Theorem* 5 *we have* $\mathrm{Par}(a, b, d, c)$ (Fig. 2).

Corollary. *For any points* a, b *we have* $\mathrm{Par}(a, b, a \cdot ab, ab)$.

4. MIDPOINTS

We shall say that b is a *midpoint* of the pair of points a, c and write $M(a, b, c)$ iff $\mathrm{Par}(a, b, c, b)$.

The properties 1°, 2° and Theorem 7 immediately imply

Theorem 13. *For any points* a, b *there is exactly one point* c *such that* $M(a, b, c)$. $M(a, b, c)$ *implies* $M(c, b, a)$. *For any point* a *we have* $M(a, a, a)$.

Theorem 14. *The statement* $M(a, b, c)$ *holds iff* $c = ba \cdot b$.

Proof. $\mathrm{Par}(a, b, c, b)$ is equivalent with $\mathrm{Par}(b, a, b, c)$, i.e. with $c = b \cdot a(bb \cdot b)$. Because of (2) and (8) this equality can be written in the form $c = ba \cdot b$.

Theorem 15. *From* $ae = c$, $af = b$, $cg = f$, $M(b, d, c)$, $hg = d$ *it follows that* $bg = e$, $dh = a$, $M(a, h, g)$ (Fig. 5).

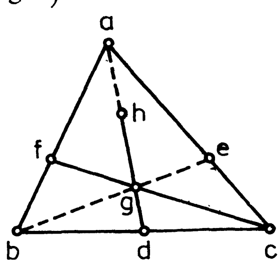

Fig. 5

Proof. From $ae = c$, $af = b$, $cg = f$ and Theorem 4 we conclude $bg = e$. Further, we obtain

$$a[(hg \cdot h)(bg) \cdot g] =^{(7)} a[(hg \cdot b)(hg) \cdot g] =$$
$$= a \cdot (db \cdot d) g = a \cdot cg = af = b =^{(1)'} a \cdot (a \cdot bg) g ,$$

which implies $hg \cdot h = a$. This equality proves the statement $M(g, h, a)$ and the equality $dh = a$.

In the case of the quasigroup $(C, *)$ Theorem 15 proves a result from [1]:

If two cevians divide the opposite sides (from the common vertex) of a triangle in the ratio of golden section, then their intersection divides them in the same ratio and the midpoint of the common vertex and this intersection divides in the same ratio the third cevian through the intersection.

We will say that $(Q, \cdot)$ is a *GS-quasigroup with unique halving* iff for any two points a, c there is exactly one point b such that $M(a, b, c)$ holds.

D. Vakarelov [2] has axiomatized the notion of the central symmetry by an idempotent medial quasigroup with the operation $\square$ such that the identity $(a \square b) \square \square\, b = a$ holds. Therefore, the next theorem naturally holds.

Theorem 16. *If $(Q, \cdot)$ is a GS-quasigroup with unique halving and $\square$ the operation on the set Q defined by $a \square b = ba\,.\,b$, then $(Q, \square)$ is a quasigroup of Vakarelov.*

Proof. By Theorem 14 we have the equivalence

$$a \square b = c \Leftrightarrow M(a, b, c)\,.$$

However, $(Q, \cdot)$ is a GS-quasigroup with unique halving and Theorem 13 holds. Therefore, it follows that $(Q, \square)$ is an idempotent quasigroup. Moreover, we have

$$(a \square b) \square (c \square d) = (dc \cdot d)(ba \cdot b) \cdot (dc \cdot d) =^{(7)}$$
$$= (dc \cdot ba)(db) \cdot (dc \cdot d) =^{(7)} (db \cdot ca)(db) \cdot (dc \cdot d) =^{(7)}$$
$$= (db \cdot ca)(dc) \cdot (db \cdot d) =^{(7)} (db \cdot d)(ca \cdot c) \cdot (db \cdot d) =$$
$$= (a \square c) \square (b \square d)\,,$$
$$(a \square b) \square b = b(ba \cdot b) \cdot b =^{(1)} a\,.$$

Corollary. *If $(Q, \cdot)$ is an arbitrary GS-quasigroup, then $(Q, \square)$ is a left quasigroup of Vakarelov.*

5. CHARACTERIZATION OF GS-QUASIGROUPS

Let us return to any GS-quasigroup $(Q, \cdot)$. Let O be any given point. We define an addition of points by the equivalence

$$(27) \qquad a + b = c \Leftrightarrow \mathrm{Par}(O, a, c, b)\,,$$

i.e. $\mathrm{Par}(O, a, a + b, b)$ for any points a, b.

In [3] it is proved that $(Q, +)$ is a commutative group with the neutral element O.

Theorem 17. *The mapping φ: $Q \to Q$ defined by $\varphi(a) = Oa$ is an automorphism of the group $(Q, +)$ such that the identity (3) holds.*

Proof. For any points a, b we have $\mathrm{Par}(O, a, a + b, b)$, which by Theorem 10 yields $\mathrm{Par}(OO, Oa, O(a + b), Ob)$, i.e. $\mathrm{Par}(O, Oa, O(a + b), Ob)$ because of (2). Therefore, by (27) we have $\varphi(a + b) = \varphi(a) + \varphi(b)$. Accoding to Corollary of Theorem 12 for any point a we have $\mathrm{Par}(O, a, O \cdot Oa, Oa)$, i.e. by (27) we have the equality $a + Oa = O \cdot Oa$, which can be written in the form (3).

Theorem 18. *For any points a, b the equality (4) holds, where φ is the mapping defined by $\varphi(a) = Oa$.*

Proof. By Theorem 9 we have $\mathrm{Par}(OO, aO, ab, Ob)$ and because of (2) we have $\mathrm{Par}(O, aO, ab, Ob)$, i.e. by (27) we obtain the equality

$$(28) \qquad ab = aO + Ob\,.$$

This equality and (2) immediately imply $aO = a - Oa$, which substituted back into (28) gives $ab = a - Oa + Ob$. According to Theorem 17 this equality can be written in the form (4).

From Theorems 17 and 18 it follows that any GS-quasigroup can be derived as in Example 1, i.e. we have the following theorem.

Theorem 19. *There is a GS-quasigroup $(Q, \cdot)$ iff there is a commutative group $(Q, +)$ and its automorphism φ such that the identity (3) holds. If a commutative group $(Q, +)$ and its automorphism φ with the identity (3) are given, then the operation $\cdot$ is defined by (4), and if a GS-quasigroup $(Q, \cdot)$ and a element $O \in Q$ are given, then the operation $+$ is defined by*

$$(29) \qquad a + b = \psi^{-1}(a)\,.\,\varphi^{-1}(b)$$

and O is the neutral element of the group $(Q, +)$, where φ, ψ are the bijections of the set Q defined by $\varphi(a) = Oa$, $\psi(a) = aO$.

Indeed, the identity (29) follows from (28) if we substitute the variables a and b by $\psi^{-1}(a)$ and $\varphi^{-1}(b)$.

Theorem 19 gives a more precise version (in the case of a GS-quasigroup) of the well-known Toyoda's theorem about medial quasigroups. Moreover, here we have its "geometrical" proof.

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GS-QUASIGRUPY

VLADIMIR VOLENEC

Kvazigrupa $(Q, \cdot)$ se nazývá GS-kvazigrupa, jestliže platí $aa = a$, $a(ab \cdot c) \cdot c = b$, $a \cdot (a \cdot bc)\, c = b$ pro všechna $a, b, c \in Q$. V každé kvazigrupě lze zavést geometrickou terminologii, která vede ke „geometrickým" výsledkům. GS-kvazigrupa $(Q, \cdot)$ existuje právě když existuje komutativní grupa $(Q, +)$ a její automorfizmus φ takový, že platí identita $(\varphi \cdot \varphi)(a) - \varphi(a) - = 0$.

Author's address: Department of Mathematics, University of Zagreb, p.o. box 187, 41001 Zagreb, Yugoslavia.