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ON LOCAL PROPERTIES OF GRAPHS AGAIN

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Abstract. In our earlier papers we considered two types of vertex neighborhoods but in this paper we deal mostly with the first one. Let v be a vertex of a graph G. By $N_1(v)$, the neighborhood of the first type, we mean the subgraph of G induced on the set of all vertices adjacent to v.

Let $\mathscr{C}_1$ be the class of all graphs G of order at least two with the property that for any two vertices of G, say u and v, we have $N_1(u) \not\cong N_1(v)$. First we show that for a given positive integer m there exists a positive integer $\varrho(m)$ such that for every graph $G^{(m)}$ of order m there is a graph G of order $\varrho(m)$ with $G \in \mathscr{C}_1$ and a vertex w of G with $N_1(w) \cong G^{(m)}$.

Further one can easily verify that for every integer $n \geqq 7$ there exists a locally connected graph of order n belonging to $\mathscr{C}_1$. It is known that the number of planar graphs belonging to $\mathscr{C}_1$ is finite. In the present paper we prove that there are exactly four outerplanar graphs in the class $\mathscr{C}_1$ (see Fig. 3). In [10] we defined a generalized outerplanar graph as a planar graph which can be embedded in the plane in such a way that at least one endvertex of each edge lies on the boundary of the same face. Next result of this paper states that if G is an outerplanar graph of order at least seven then $\bar{G}$ is not a generalized outerplanar graph. This statement strenghtens Theorem 11.12 in [4]. The generalized outerplanarity also allows us to interpose the following statement between Theorem 11.11 and Theorem 11.12 in [4]: Every generalized outerplanar graph with at least eight vertices has a nonplanar complement, and eight is the smallest such number. The final section of this paper shows how to use the so-called Ford circles to construct an infinite graph with the property that all vertex neighborhoods are isomorphic with a two-way infinite path.

Keywords: Graphs with nonisomorphic vertex neighborhoods, planarity, outerplanarity, generalized outerplanar graphs, Ford circles.

AMS classification: 05C 10.

I. AUXILIARY CONCEPTS

In this paper we consider undirected graphs without loops and multiple edges. All graphs except in Section IV are finite. If a graph G is given then the number of vertices of G is referred to as the *order* of G. If G_1 and G_2 are isomorphic then we write $G_1 \cong G_2$. Let $\bar{G}$ denote the complement of G. As usual, K_n stands for the complete graph and $K_{m,n}$ for the complete bipartite graph. By a *path* P_m $(m \geqq 2)$ we mean a tree on m vertices with exactly two vertices of degree one.

A block B of G is said to be *cyclic* if the order of B is at least three. Otherwise B is called *acyclic*. If B contains exactly one cut vertex of G then B is an *endblock* of G.

If X is a nonempty subset of the vertex set $V(G)$ then the subgraph $\langle X \rangle$ *induced by* X is the maximal subgraph of G with the vertex set X. If Y is a nonempty subset of the edge set $E(G)$ then the subgraph $\langle Y \rangle$ *induced by* Y is the graph whose vertex set consists of those vertices of G incident with at least one edge of Y and whose edge set is Y($\langle Y \rangle$ is also called the *reduction* of G to Y, see [14]).

If G is a graph with at most n vertices then by $K_n - G$ we mean the subgraph of K_n obtained by deleting all edges of G_0 where G_0 is a subgraph of K_n isomorphic with G. If the order of G is n then obviously $K_n - G = \bar{G}$. In what follows it is convenient to use the union $G_1 \cup G_2$ for the graph having two components G_1, G_2.

Behzad and Chartrand [1] called a graph *quasiperfect* if it has exactly two vertices u, v of the same degree. The vertices u and v are referred to as *exceptional* vertices. Some properties of quasiperfect graphs were described by Nebeský [6]. It is known that for every integer n, $n \geqq 2$, there exist two quasiperfect graphs of order n, one of them being connected, the other disconnected. If we denote the connected one by D_n then the disconnected one is $\bar{D}_n$. In D_n the exceptional vertices have degree $[\frac{1}{2}n]$ where $[x]$ means the integer part of x.

A subset Q of $V(G)$ is *independent* if no two vertices of Q are adjacent in G. The maximum value of $|Q|$ is called the *independence number* of G and is denoted by $\beta(G)$. The corresponding set, say $Q_{\max}$, is called the *maximum independent set.* In [11] we showed that

$$\beta(D_n) = [\tfrac{1}{2}(n + 1)] .$$

If n is even then $Q_{\max}$ of D_n consists of all vertices of degree less than $[\frac{1}{2}n]$ and of one of the exceptional vertices. If n is odd then $Q_{\max}$ consists of all vertices of degree less than $[\frac{1}{2}n]$ and of both exceptional vertices. In what follows we denote the set of all vertices of D_n not belonging to $Q_{\max}$ by R.

The aim of this paper is to continue the study of local properties of graphs. In [9] we considered two types of vertex neighborhoods. The first was taken from Zykov [16]. Let G be a (finite or infinite) graph. If v is a vertex of degree at least 1 in G then $N_1(v)$, the *neighborhood of the first type*, is the subgraph induced by the set of all vertices adjacent to v. If v is isolated then $N_1(v)$ is the empty graph. If u and v are two vertices of G with $N_1(u) \cong N_1(v)$ then u and v are said to be *of the same kind.* The neighborhoods of the second type will be considered later.

Lemma 1 which follows describes all neighborhoods $N_1(v)$ of D_n. It is clear that $N_1(v)$ is uniquely determined by the degree of v so that in Lemma 1 we can write $N_1(i)$ instead of $N_1(v)$ where i stands for the degree of v. The statement of Lemma 1 actually completes a result of Nebeský [6] where among other things the author describes the complete subgraphs of D_n.

Lemma 1. *If i is the degree of a vertex of D_n, $n \geqq 2$, then the following implications hold*:

(i) $1 \leqq i \leqq \left[\frac{n}{2}\right] \Rightarrow N_1(i) \cong K_i$,

(ii) $\left[\frac{n}{2}\right] < i \leqq n - 1 \Rightarrow N_1(i) \cong K_i - D_{2i-n+1}$.

Proof. We employ induction on n.

First let us prove (i). For $n = 2$ and $n = 3$ the statement (i) is obvious. Suppose that (i) holds for an integer $n \geqq 2$ and prove it for $n + 2$. For the vertex of degree 1 in D_{n+2} we obviously have $N_1(1) \cong K_1$. If we choose a vertex of degree j with

$$2 \leqq j \leqq \left[\frac{n+2}{2}\right]$$

then $N_1(j)$ in D_{n+2} can be obtained from $N_1(j-1)$ in D_n (i.e. from K_{j-1}) by adding a new vertex and joining it with every vertex in $N_1(j-1)$. Thus in D_{n+2} we have $N_1(j) \cong K_j$.

To prove (ii) we proceed as follows. For $n = 3$ and $n = 4$ the implication (ii) is clear. Assume that the statement holds for an integer $n \geqq 3$ and prove it for $n + 2$. It is obvious that in D_{n+2} the neighborhood of the vertex of degree $n + 1$ is $D_n \cup K_1$, which yields $K_{n+1} - D_{n+1}$. This expression can be viewed as $K_i - D_{2i-n+1}$ where we replace i by $n + 1$ and n by $n + 2$. If we choose in D_{n+2} a vertex of degree j with

$$\left[\frac{n+2}{2}\right] < j \leqq n$$

then its neighborhood $N_1(j)$ in D_{n+2} arises from $N_1(j-1)$ in D_n (by hypothesis, this is the graph $K_{j-1} - D_{2(j-1)-n+1}$) by adding a new vertex and joining it with every vertex of $N_1(j-1)$. In D_{n+2} we have

$$N_1(j) \cong K_j - D_{2(j-1)-n+1} \cong K_j - D_{2j-(n+2)+1}\,. \quad \square$$

II. GRAPHS WITH NON-ISOMORPHIC NEIGHBORHOODS

In [9] we considered connected graphs G of order at least 2 with the property that if x and y are two vertices of G then $N_1(x)$ and $N_1(y)$ are not isomorphic. Let $\mathscr{C}_1$ be the class of these graphs. It was shown that for a given integer n there exists a graph G of order n belonging to $\mathscr{C}_1$ if and only if $n \geqq 6$.

Let G_0 be a given graph and let G be a graph belonging to $\mathscr{C}_1$ and containing a vertex x whose neighborhood $N_1(x)$ is isomorphic with G_0. The graph G is said to be the *$\mathscr{C}_1$-realisation* of G_0. It is natural to ask whether every graph G_0 has a $\mathscr{C}_1$-realisation. In what follows it is shown that the answer is affirmative.

Theorem 1. *For every positive integer m there exists a positive integer $\varrho(m)$ such that every graph $G^{(m)}$ of order m has a $\mathscr{C}_1$-realisation G of order $\varrho(m)$.*

Proof. The cases $m = 1$ and $m = 2$ are trivial so that we can assume that $m \geqq 3$. Let $u^{(i)}$ $(i = 1, 2, \ldots, m)$ be the vertices of $G^{(m)}$.

(a) First let us assume that

$$(1) \qquad G^{(m)} \not\cong K_{m-1} \cup K_1$$

and construct a graph G of order $\varrho(m) = 3m^2 + 1$ as follows: Take the graph $D_{m(3m-1)}$, construct its maximum independent set $Q_{\max}$ and put

$$Q_{\max} = \bigcup_{i=1}^{m} Q_i \, , \quad |Q_i| = m + i - 1 \, .$$

Further, add both the graph $G^{(m)}$ disjoint with $D_{m(3m-1)}$ and the vertex x not belonging to $D_{m(3m-1)} \cup G^{(m)}$. Finally, join x with every $u^{(i)}$ and similarly every $u^{(i)}$ with every vertex from Q_i by an edge $(i = 1, 2, \ldots, m)$. This completes the construction of G.

To find that G belongs to $\mathscr{C}_1$ we compare its vertex neighborhoods with each other. Every $u^{(i)}$ has degree at least $m + 1$, thus $N_1(x) \not\cong N_1(u^{(i)})$. The degree of every $u^{(i)}$ is at most $3m - 1$. If we go through the vertices of $Q_{\max} \cup R$ having degrees $m + 1$ to $3m - 1$ inclusively we can see that each neighborhood has exactly one isolated vertex while in each $N_1(u^{(i)})$ we find at least m such vertices. Thus no $N_1(u^{(i)})$ is isomorphic with the neighborhood of a vertex from $Q_{\max} \cup R$. There is a unique vertex in $Q_{\max} \cup R$ having degree m. This vertex is of type $K_{m-1} \cup K_1$ but the construction is invented in such a way that x is not of this type. Finally, it is obvious that $N_1(u^{(i)}) \not\cong N_1(u^{(j)})$ for $i \neq j$ and by Lemma 1 we can conclude that no two vertices of $Q_{\max} \cup R$ have the same type.

(b) If (1) does not hold we take any graph of order m different from $G^{(m)}$ for which we construct the $\mathscr{C}_1$-realisation of order $\varrho(m) = 3m^2 + 1$ as described above. Clearly this graph is also a $\mathscr{C}_1$-realisation of $K_{m-1} \cup K_1$. □

If we slightly modify the above proof we can show that for a given m there exist infinitely many values $\varrho(m)$ with the property mentioned in Theorem 1. Thus we could seek the minimum value $\varrho_{\min}(m)$ if need be but the maximum value $\varrho_{\max}(m)$ does not exist. For small m we have

$$\varrho_{\min}(1) = 7 \, , \quad \varrho_{\min}(2) = 6 \, , \quad \varrho_{\min}(3) = 7 \, .$$

If for every vertex v of a given graph G the neighborhood $N_1(v)$ is connected then G is called *locally connected* (see [2] and [15]). Now we will be concerned with the following problem: Characterize all positive integers n for which there exists a graph of order n belonging to $\mathscr{C}_1$. It is already known that in $\mathscr{C}_1$ there is only one graph of order six but it is not locally connected.

Theorem 2. *For every integer $n \geqq 7$ there exists a locally connected graph G_n of order n belonging to $\mathscr{C}_1$.*

Sketch of proof. The cases $n = 7$ and $n = 8$ are depictured in Fig. 1 and Fig. 2. The complete proof by induction can be left to the reader. □

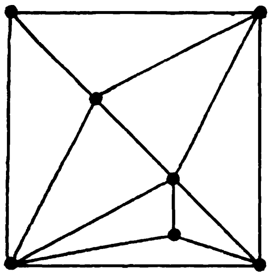

Fig. 1

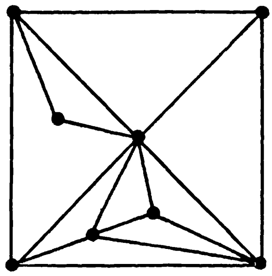

Fig. 2

If for every vertex v of G the neighborhood $N_1(v)$ is disconnected then G is said to be *locally disconnected*. The problem of locally disconnected graphs in $\mathscr{C}_1$ is left open here.

III. THE OUTERPLANARITY OF GRAPHS

In what follows we need the well-known concept of an outerplanar graph. A graph G is *outerplanar* if G can be embedded in the plane so that there exists a region Ω_1 determined by the embedding of G whose boundary contains every vertex of G. In [10] the class of all outerplanar graphs was denoted by $\mathscr{A}_1$.

In an earlier paper [8] we showed that the number of all planar graphs belonging to $\mathscr{C}_1$ is finite. The exact number of such graphs is not known but in [8] we proved that for every integer $n \in [6, 26]$ there exists at least one planar graph of order n belonging to $\mathscr{C}_1$. Since an outerplanar graph is a very special case of a planar graph we can expect that the cardinality of $\mathscr{A}_1 \cap \mathscr{C}_1$ is much smaller than the cardinality of the class of all planar graphs belonging to $\mathscr{C}_1$. This will be shown in the next theorem. Let us observe the structure of outerplanar graphs in detail before we proceed to Theorem 3.

An outerplanar cyclic block B can be drawn in the plane as a convex n-gon together with some diagonals without crossings. In this interpretation the vertices of B are the vertices of the n-gon and the edges are its sides or diagonals. The next terminology can be also borrowed from this model. A diagonal d is said to be *isolated* if there exists no diagonal d' of B so that d and d' have no vertex in common. Let d_0 separate B into two polygons M_1 and M_2. If there is no diagonal of B inside of M_i $(i = 1, 2)$ then d_0 is called *outer*. The *brink* of B is understood to be the set of all vertices of M_i. If B has no diagonals then the brink of B consists of all vertices of B.

Theorem 3. *There exist exactly four outerplanar graphs G belonging to* $\mathscr{C}_1$ (*see Fig.* 3).

Proof. Let $G \in \mathscr{A}_1 \cap \mathscr{C}_1$. It is clear that G is not a tree. Further, it is obvious that the brink of every cyclic block B belonging to $\mathscr{A}_1$ contains either two vertices of type K_2 (case α) or one vertex of type K_2 and two vertices of type $\bar{K}_2$ (case β) or four vertices of type $\bar{K}_2$ (case γ). Thus the graph G must have a cut vertex. For B the case γ cannot occur if B is a cyclic endblock of G. By considering the cases α and β we conclude that G may contain only one cyclic endblock B. Thus the second endblock of G is acyclic.

The graph B can have at most one pair of vertices of the same type. Thus this pair of isomorphic neighborhoods are either K_2, K_2 or $\bar{K}_2, \bar{K}_2$ and the corresponding vertices are contained in a brink of B. If we go through all blocks of order at most eight belonging to $\mathscr{A}_1$ we find that in this area exactly four cyclic blocks B_i ($1 \leq$ $\leq i \leq 4$) have one pair of vertices of the same type each. The block B_1 of order six and the block B_2 of order seven are pictured as cyclic blocks of four graphs in Fig. 3 while B_3 of order seven and B_4 of order eight are shown in Fig. 4.

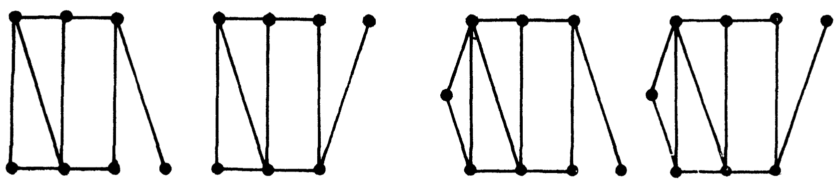

Fig. 3

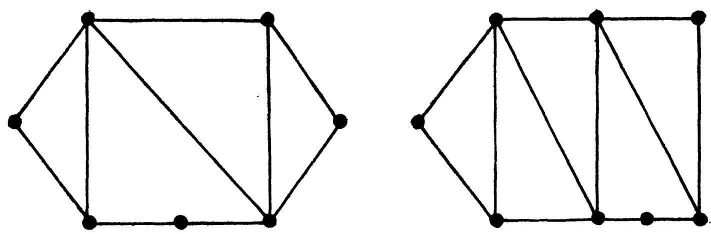

Fig. 4

If B had at least nine vertices then it would contain at most two outer diagonals. Let $d(B)$ be the subgraph of B induced on the set of all diagonals of B. Clearly the components of $d(B)$ are trees with at least two edges each.

Let v_i ($i = 1, 2, \ldots$) be the vertices of degree 1 in $d(B)$. In B the neighborhood $N_1(v_i)$ cannot be isomorphic with K_3, thus $N_1(v_i)$ is of type either $\bar{K}_3$ or $K_{1,2}$ or $\bar{K}_{1,2}$. Obviously $d(B)$ must be connected, thus $d(B)$ is a tree with two or three vertices v_i.

Let us show that the following does not hold:

$$N_1(v_i) \cong \bar{K}_3 . \qquad (2)$$

If $v_i u$ were an outer diagonal of B and if (2) held then there would exist at least three vertices of type $\bar{K}_2$. If $v_i u$ were not outer then we would obtain two vertices of type $\bar{K}_2$ not belonging to any brink and also a pair of vertices on a brink having isomorphic vertices.

The two remaining types of neighborhoods $N_1(v_i)$ are $K_{1,2}$ and $\bar{K}_{1,2}$. Thus $d(B)$ must be a path P_m with vertices v_1 and v_2 of degree one. Let us put

$$N_1(v_1) \cong K_{1,2}\,, \quad N_1(v_2) \cong \bar{K}_{1,2}\,.$$

Since B has at least nine vertices it must have at least four vertices of degree 4. The neighborhoods of these vertices are either P_3 or $K_2 \cup K_2$ or $K_{1,2} \cup K_1$, thus at least two of them are of the same type. For this reason an endblock with at least nine vertices cannot exist.

If B^* were not an endblock of G then it would contain exactly two cut vertices of G, say x and y. Let B^* be cyclic. Since both K_2 and $\bar{K}_2$ occur as vertex neighborhoods of each potential cyclic endblock B_i of G, the block B^* must have exactly two vertices of degree 2. These vertices are x and y. Let us choose an outer diagonal d in B^*. Clearly d cannot be isolated. The neighborhood of one endvertex of d would be of type $K_{1,2}$ but $K_{1,2}$ also occurs in B_i. Hence B^* cannot be cyclic.

The assumption that B^* is acyclic also leads to a contradiction.

Thus G has exactly two blocks. To obtain G we add an acyclic block to B_1 or B_2 as shown in Fig. 3. Neither B_3 nor B_4 (Fig. 4) lead to a graph G belonging to $\mathscr{A}_1 \cap \mathscr{C}_1$ as can be verified by the reader. □

Let v be a vertex of G. In [9] we also defined $N_2(v)$, the *neighborhood of the second type*, as the subgraph of G induced by the set of all edges at distance 1 from v. The neighborhoods of the second type were already discussed in [7] and [12].

In [10] the study of the neighborhoods $N_2(v)$ motivates the following generalization of outerplanar graphs: A *generalized outerplanar* graph G is a planar graph which can be embedded in the plane in such a way that at least one endvertex of each edge lies on the boundary of the same face Ω_2. The class of all generalized outerplanar graphs was denoted by $\mathscr{A}_2$.[1]) A necessary and sufficient condition for a graph to belong to $\mathscr{A}_2$ was also found in [10]. Let $\mathscr{A}_3$ stand again for the class of all planar graphs.

In connection with Theorem 3 the following question occurs: how large is the intersection $\mathscr{A}_2 \cap \mathscr{C}_1$? This is the second question we leave here open.

One of known results on planar graphs says that every planar graph with at least nine vertices has a nonplanar complement and nine is the smallest such number (see

[1]) A survey of some generalizations of outerplanar graphs can be found in Sysło [13].

[4], Theorem 11.11). In [4] we find a similar result on graphs of the class $\mathscr{A}_1$ which reads as follows: Every outerplanar graph with at least seven vertices has a non-outerplanar complement and seven is the smallest such number ([4], Theorem 11.12). The class $\mathscr{A}_2$ enables us to formulate a statement which is stronger than the above one. For proving this statement we need an auxiliary concept. A graph $G \in \mathscr{A}_i$ $(i = 1, 2, 3)$ is said to be *saturated* if the following holds: By adding an edge e, $e \notin E(G)$, we obtain a graph not belonging to $\mathscr{A}_i$.

Now we present the above mentioned improvement of Theorem 11.12 from [4].

Theorem 4. *Let G be an outerplanar graph with at least seven vertices. Then* $\bar{G}$ *does not belong to* $\mathscr{A}_2$.

Proof. It is sufficient to modify the proof of Theorem 11.12 from [4] and realize that each of four saturated outerplanar graphs on seven vertices has a complement not belonging to $\mathscr{A}_2$. Here one can use the characterization of graphs in $\mathscr{A}_2$ mentioned above. □

A natural analogue is the following

Theorem 5. *Let G be a generalized outerplanar graph with at least eight vertices. Then* $\bar{G}$ *does not belong to* $\mathscr{A}_3$. *There exists a graph* G_7 *of order seven so that* $G_7 \in \mathscr{A}_2$, $\bar{G}_7 \in \mathscr{A}_2$.

Proof. There exist exactly 19 saturated graphs on eight vertices belonging to $\mathscr{A}_2$ and one can easily verify that each of them has a nonplanar complement. This yields the first part of our statement. If we choose the graph from Fig. 5 for graph G_7 then we see that both G_7 and $\bar{G}_7$ lie in $\mathscr{A}_2$. □

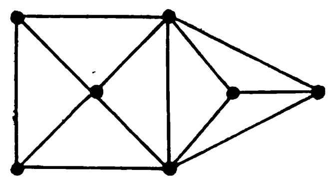

Fig. 5

IV. A REMARK ON FORD CIRCLES

Our final remark concerns the following known result: There exists an infinite graph G_∞ in which each $N_1(v)$ is isomorphic with the two-way infinite path. This statement is presented without proof in [5], and in [9] we described a construction of G_∞. Now we show a connection between this problem and the so called Ford circles.

In 1938 L. R. Ford proposed the following ingenious and elementary representation of fractions (see [3]). Let h/k be a reduced fraction, i.e. $(h, k) = 1$ with $k > 0$. Let us plot h/k on a line, say the x-axis as in analytic geometry, and let us draw a circle with radius $1/(2k^2)$ centred at the point $(h/k, 1/(2k^2))$. Such a circle tangent to the x-axis at h/k is called the *Ford circle* corresponding to the fraction h/k. Let F stand for the set of all Ford circles in the plane. It is known that the Ford circles representing any two different (reduced) fractions cannot intersect. In the extreme case they may be tangent. For a given Ford circle representing a fraction H/K there exists an infinite number of circles from F which are tangent to it. These adjacent circles can be arranged in two infinite sequences $c_1, c_2, c_3, \ldots$ and $c_{-1}, c_{-2}, c_{-3}, \ldots$ in such a way that c_i and c_j are tangent if and only if $|i - j| = 1$. If $K > 1$ then c_1 and c_{-1} are tangent, too.

If in addition we put

$$T = \{(x, 1) \mid x \in \mathbf{R}\},$$

then $V(G_\infty)$ can be defined by

$$V(G_\infty) = F \cup \{T\}$$

and two vertices are joint by an edge if and only if the corresponding two elements from $F \cup \{T\}$, i.e. two Ford circles or one Ford circle and the line T, have an unempty intersection. It is easy to see that each neighborhood $N_1(v)$ in G_∞ is isomorphic with a two-way infinite path.

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Souhrn

JEŠTĚ O LOKÁLNÍCH VLASTNOSTECH GRAFŮ

JIŘÍ SEDLÁČEK

V předcházejících pracích jsme se zabývali dvěma typy uzlových okolí, ale v tomto článku se jedná hlavně o prvním z nich. Je-li v uzel grafu G, pak $N_1(v)$, okolí prvního typu, je podgraf grafu G indukovaný na množině všech uzlů sousedících s uzlem v. Nechť $\mathscr{C}_1$ je třída všech grafů G řádu aspoň 2 takových, že žádné dva uzly z G, řekněme u a v, nemají $N_1(u)$ a $N_1(v)$ izomorfní. Nejprve dokazujeme, že libovolný graf lze pokládat za uzlové okolí vhodného grafu ze třídy $\mathscr{C}_1$. Další věta říká, že pro každé celé $n \geqq 7$ existuje lokálně souvislý graf řádu n patřící do $\mathscr{C}_1$. Je známo, že počet rovinných grafů patřících do $\mathscr{C}_1$ je konečný. Zde ukazujeme, že v $\mathscr{C}_1$ existují právě čtyři vnějškově rovinné grafy. Studium uzlových okolí druhého typu nás v [10] vedlo ke zobecnění vnějškově rovinných grafů. Nyní dokazujeme, že každý zobecněný vnějškově rovinný graf řádu aspoň osm má nerovinný komplement, přičemž osm je nejmenší takové číslo. Tento výsledek zapadá mezi věty 11.11 a 11.12 z knihy [4], přičemž samo tvrzení věty 11.12 se dá zesílit užitím zobecněné vnějškové rovinnosti. V závěru článku ukazujeme vztah mezi tzv. Fordovými kružnicemi a lokálními vlastnostmi grafů.

Резюме

ЕЩЕ РАЗ О ЛОКАЛЬНЫХ СВОЙСТВАХ ГРАФОВ

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В статье рассматриваются главным образом окрестности вершин в графах, принадлежащие к первому из двух типов окрестностей введенных в предыдущих работах автора. Если v — вершина графа G, то $N_1(v)$, окрестность первого типа, определяется как подграф, индуцированный на множестве всех вершин смежных с вершиной v. Пусть $\mathscr{C}_1$ — класс всех графов G порядка $\geqq 2$ таких, что $N_1(u)$ и $N_1(v)$ не изоморфны для никаких двух вершин $u \neq v$ графа G. В статье сначала доказывается, что произвольный граф изоморфен окрестности $N_1(u)$ некоторой вершины некоторого графа из класса $\mathscr{C}_1$. Следующий результат утверждает, что класс $\mathscr{C}_1$ содержит локально связные графы любого порядка $n \geqq 7$. Известно, что класс $\mathscr{C}_1$ содержит лишь конечное число попарно неизоморфных плоских графов. В статье доказано, что с точностью до изоморфизма $\mathscr{C}_1$ содержит точно четыре внешнепланарные графы. В связи с изучением окрестностей второго типа в работе [10] были введены обобщенные внешнепланарные графы. В настоящей статье доказано, что каждый обобщенный внешнепланарный граф порядка $\geqq 8$ обладает неплоским дополнением, причем 8 — наименьшее такое число. Этот результат дополняет теоремы 11.11 и 11.12 книги [4], причем утверждение теоремы 11.12 можно усилить при помощи обобщенной внешнепланарности. В заключении статьи рассматривается связь между т.н. окружностями Форда и локальными свойствами графов.

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