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CARDINAL INVARIANTS OF BITOPOLOGICAL SPACES

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Summary. A non-symmetric distance function (see [7]) or quasiuniformity (see [2]) on a set X gives rise to two topologies on X; spaces with two topologies, called bitopological spaces, were introduced in [6] and [9]. We seek to extend the theory of cardinal functions to bitopological spaces, and obtain bounds on several of these functions involving the quasiuniform weight, a non-symmetric analogue of Juhasz's uniform weight (see [4]).

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1. SOME CARDINAL FUNCTIONS ON BITOPOLOGICAL SPACES

If φ is a cardinal function on a topological space (X, T), we often abbreviate $\varphi(X, T)$ to $\varphi(X)$ or $\varphi(T)$ when no confusion can arise. We are primarily interested in the following (standard) functions (from [5] unless otherwise indicated): $w(T)$ (weight of T) = least cardinality of a base for T, $d(T)$ (density of T) = least cardinality of a T-dense subset of X, $c(T)$ (cellularity of T) = sup of the cardinalities of pairwise disjoint sets of open sets in T, $s(T)$ (spread), $L(T)$ (Lindelof degree), $h(T)$ (height), $n(T)$ (net weight), $z(T)$ (width), and $e(T)$ (extent – see [1]) $= \sup \{|C|: C \subset X$ closed and discrete in $T\}$. Those not defined in the previous sentence are mentioned only in passing. If φ is any cardinal function on a topological space, then her $\varphi(X, T) = \sup \{\varphi(Y, T \mid Y): Y \subset X\}$ is called the *hereditary* φ of (X, T).

In addition to the above, we are interested in cardinal invariants related to metrization. Hodel's *metrization number* is $m(T)$ = smallest infinite cardinal m such that there is a base for T which is the union of m discrete collections (see [3]). Other invariants related to metrization have been defined externally to the topology: the *uniform weight* $u(T)$ of Juhasz [4] is the smallest possible cardinality of a base of a uniformity from which T arises.

The *Mrowka number* $M(T)$ (= the smallest cardinal of a set of pseudometrics from which T arises – see [10]) is essentially the same as the uniform weight (precisely, $u(T)\,\omega = M(T)\,\omega$). Another definition of the same cardinal may be made in terms of continuity spaces (see [7]) since $u(T)$ is the least cardinal of a base for the set of positives in a symmetric continuity space from which T arises. An internal characterization of this invariant is given in [8]; $m(T) \leqq u(T)$, but it is not presently known whether the two are always equal for completely regular spaces.

Theorem. *If T is an arbitrary topology and φ is any of the cardinal invariants in the first paragraph, then $c(T) \wedge e(T) \leqq \varphi(T) \leqq w(T)$. If T is metrizable, then $\varphi(T) = w(T)$ for each φ.*

Note that if $X_B = (X, T_1, T_2)$ is a bitopological space then the join $T_1 \vee T_2$ is a third topology on X determined by X_B. This topology is completely regular in the case we consider below and plays a central role in our work.

Given a cardinal function φ, the following considerations often lead to a definition of a bitopological analogue for φ, which we shall call $b\varphi$. φ is called *isotone* if $\varphi(T) \leqq \varphi(T')$ whenever T and T' are topologies on the same set and $T \subset T'$; *anti-isotone* is defined correspondingly. For the functions listed above, c, d, e, L, n, s, h, and z are isotone. If φ is isotone then so is her φ. If φ is isotone and T_1, T_2 are topologies on X then $\varphi(T_1)\,\varphi(T_2) \leqq \varphi(T_1 \vee T_2)$. We have found that for the cardinal functions listed above, $\varphi(T_1 \vee T_2)$ and $\varphi(T_1)\,\varphi(T_2)$ are both candidates for $b\varphi(X_B)$, and since our objective is to find an upper bound for these invariants, it is appropriate for our purposes to define: If φ is an isotone cardinal function and $X_B = (X, T_1, T_2)$ a bitopological space then $b\varphi(X_B) = \varphi(T_1 \vee T_2)$.

The weight is neither isotone nor anti-isotone, but satisfies the inequality $w(T_1)\,w(T_2) \geqq w(T_1 \vee T_2)$. The same is true for the net weight. For these cardinal invariants $b\varphi$ is defined by $b\varphi(X, T_1, T_2) = \varphi(T_1)\,\varphi(T_2)$. This corresponds to the fact that a *bibase* for a bitopological space is a base for each topology, thus its cardinality must be the sum of theirs.

We have now defined $b\varphi$ for all the cardinal functions φ considered so far, except for u, m, and M. The following properties are immediate for such φ:

(1) $b\varphi(X, T_1, T_2) = \varphi(X, T_1)\,\varphi(X, T_2)\,\varphi(X, T_1 \vee T_2)$.

(2) $b\varphi(X, T, T) = \varphi(X, T)$.

(3) if $\varphi_1 \leqq \varphi_2$ then $b\varphi_1 \leqq b\varphi_2$.

From (1) it follows that the known partial ordering of those topological cardinal functions and the results of the above theorem extend to the analogous bitopological invariants.

A *bisubspace* of a bitopological space is defined in the obvious way. For $X_B = (X, T_1, T_2)$, $S \subset X$, let $S_B = (S, T_1 \mid S, T_2 \mid S)$. For any bitopological invariant $b\varphi$ we define her $b\varphi$ by analogy with the usual topological definitions of hereditary cardinal invariants: her $b\varphi(X_B) = \sup\{b\varphi(S_B): S \subset X\}$. Here are some immediate consequences:

(4) her $b\varphi \geqq b\varphi$; if $\varphi \leqq \varphi'$ then her $b\varphi \leqq$ her $b\varphi'$,

(5) if φ satisfies (1) then her $b\varphi = b(\text{her } \varphi)$ (the proof uses $(T_1 \vee T_2) \mid S = (T_1 \mid S) \vee (T_2 \mid S)$).

We find it appropriate to define bitopological cardinal invariants related to M and u as follows: $bq(X_B)$ = least cardinality of a base for a quasiuniform space $(X, \mathscr{U})$ such that $T_1 = To(\mathscr{U})$ $(= \{P\colon$ if $x \in P$ then for some $U \in \mathscr{U}$, $\{Y\colon (x, y) \in U\} \subset P\})$ and $T_2 = To(\mathscr{U}^*)$ $(\mathscr{U}^* = \{U^{-1}\colon U \in \mathscr{U}\}$ – see [2] for more details). This is also the least cardinality of a base of a set of positives in a continuity space $X = (X, d, A, P)$ for which $T_1 = To(X)$ $(= \{P\colon$ if $x \in P$ then for some $r \in P$, $N_r(x) \subset P\}$, where $N_r(x) = \{y\colon d(x, y) \leqq r\})$ and $T_2 = To(X^*)$ $(X^* = (X, d^*, A, P)$ is the dual of X, where $d^*(x, y) = d(y, x)$; see [7]). $Q(X_B)$ = least cardinality of a set of quasimetrics D such that $To(D) = T_1$. Note that $\{S_{r,d}(x)\colon x \in X, r > 0, d \in D\}$ is an open subbase for $To(D)$, where $S_{r,d}(x) = \{y\colon d(x, y) < r\}$; thus if let $2^{-N} = \{2^{-k}\colon k = 1, 2, \dots\}$ and for $F \subset 2^{-N} \times D$ finite, $x \in X$ define $S_F(x) = \bigcap\{S_{r,d}(x)\colon (r, d) \in F\}$. then $\{S_F(x)\colon F \subset 2^{-N} \times D$ finite, $x \in X\}$ is a base for $To(D)$. We also use the notation, $\{d^*\colon d \in D\} = D^*$ (again $d^*(x, y) = d(y, x)$), and notice that $To(D^*) = T_2$. It is simple to show that these numbers are related in the same way as their topological (symmetric) analogues.

As is well known, $T_1 \vee T_2$ is a completely regular topology when X_B is obtained from a quasiuniformity, continuity space, or set of quasimetrics, due to the following considerations: With notation as in the last paragraph, let $\mathscr{U}^s = \{V \subset X \times X\colon$ for some $U \in \mathscr{U}$, $U \cap U^{-1} \subset V\}$, (a uniformity) and $d^s = d + d^*$ for quasimetrics or continuity functions, $X^s = (X, d^s, A, P)$ (a symmetric continuity space), $D^s = \{d^s\colon d \in D\}$ (a set of pseudometrics). If K denotes U, X, or D, then $To(K^s) = To(K) \vee To(K^*)$ (for continuity spaces simply note that for each $r \in P$, $N_r^s(x) \subset N_r(x) \cap N_r^*(x) \subset N_{2r}^s(x)$; the other arguments are similar). Thus $T_1 \vee T_2$ is completely regular since it arises from a uniformity, symmetric continuity space, or set of pseudometrics.

2. THE MAIN RESULTS

Recall that if $f\colon X \to Y$, $X_B = (X, T_1, T_2)$, $Y_B = (X, T_1', T_2')$, then f is *pairwise continuous* iff for each $i \in \{1, 2\}$, f is continuous from T_i to T_i'. A bitopological space (X, T_1, T_2) is *pairwise completely regular* if whenever $x \in P \in T_i$ there is an $f\colon X \to [0, 1]$ such that $f(x) = 1$, $f = 0$ off P, and f is pairwise continuous from (X, T_i, T_{3-i}) to $[0, 1]_B = ([0, 1], LO, UP)$, where $LO = \{(a, \infty)\colon a \in [-\infty, \infty]\}$, and $UP = \{(-\infty, a)\colon a \in [-\infty, \infty]\}$ (see [6] or [9]). A bitopological space is *pairwise* T_0 if for each pair x_1, x_2 of distinct points in X there is an $i \in \{1, 2\}$ and a $P \in T_1 \cup T_2$ such that $x_i \in P$ and $x_{3-i} \notin P$. Thus if X_B is pairwise completely regular then $T_1 \vee T_2$ is completely regular, and if X_B is also pairwise T_0 then $T_1 \vee T_2$ is Tychonoff. We now state the result mentioned in the introduction and give an example which shows that it cannot be sharpened. Proof of the result is given at the end of the paper.

1. Theorem. *If φ is any of the listed functions except M, m, u:*

(a) *$b\varphi + bq = bw$ for pairwise T_0, pairwise completely regular bitopological spaces;*

(b) *$\varphi + u = w$ for Tychonoff topological spaces.*

Example. The Sorgenfrey line (the reals, $\mathbb{R}$, topologized with base $\{(a, b]: a, b \in \mathbb{R}\}$) has $w = u = n = |\mathbb{R}| > \omega$, $c = d = e = L = h = s = z = \omega$; this topology arises from the quasimetric $So(x, y) = x - y$ if $x \geqq y$, $= 1$ if $x < y$. From the dual quasimetric So^* comes the other Sorgenfrey topology with base $\{[a, b): a, b \in \mathbb{R}\}$. Since this is a quasimetric space, $bq = \omega$ as well. The join of these topologies is the discrete topology, so bc is the cardinality of the continuum. Thus for the Sorgenfrey line, $bc + bq = c + u$, but $bc > c$ and $bq < u$. The latter inequality shows that the Sorgenfrey topology, while normal and quasimetrizable, is not metrizable.

The following lemma helps us to establish some properties of bq:

2. Lemma. (a) *If $\mathscr{B}$ is any set of subsets of X, T_1 is the topology generated by $\mathscr{B}$, and T_2 the topology generated by $\mathscr{B}^c = \{X - B: B \in \mathscr{B}\}$, then $X_{\mathscr{B}}$ is pairwise completely regular. In particular, if $\mathscr{B}$ is a base for a topology T_1 on X there is a topology T_2 on X such that X_B is pairwise completely regular. If the topology T_1 satisfies the T_1-separation property, then T_2 can be taken to be the discrete topology on X.*

(b) *The following are equivalent:*

(i) *(X, T) is completely regular,*

(ii) *(X, T, T) is pairwise completely regular.*

(c) *If $To(X) = To(X^*)$ then $To(X) = To(X^s)$.*

Proof. (a) For any finite $F \subset B$ define f_F, g_F by $f_F(y) = 1$ if $y \in \bigcap F$, $= 0$ otherwise, $g_F(y) = 0$ if $y \in \bigcup F$, $= 1$ otherwise. Then $f_F^{-1}[(a, +\infty)] = X$ if $a < 0$, $= \bigcap F$ if $0 \leqq a < 1$, $= 0$ if $a \geqq 1$, so f_F is continuous from T_1 to LO; similarly f_F is continuous from T_2 to UP, g_F from T_2 to LO and from T_1 to UP. If $x \in P \in T_1$ then for some finite subset F of B, $x \in \bigcap F \subset P$, so $f_F: X \to [0, 1]$ is continuous from T_1 to LO and from T_2 to UP, $f_F(x) = 1$ and $f_F = 0$ off $\bigcap F$, thus off P; g_F works similarly for T_2.

The proof of (b) is straightforward, and (c) is immediate from our observation that $To(X^s) = To(X) \vee To(X^*)$.

Let χ denote the neighborhood character ($= \sup\{\inf\{|B_x|: B_x$ a neighborhood base about $x\}: x \in X\}$; see [5]). The proof of 3(a) was suggested by a proof due to Engelking ([1], p. 115). Also we define $q(X, T)$ to be the least cardinality of a base of a quasiuniformity $\mathscr{U}$ such that $T = To(\mathscr{U})$.

3. Theorem. (a) *$bq \leqq bw$ for pairwise completely regular bitopological spaces.*

(b) *For any topology T, $bq(T, T) = u(T)$.*

(c) $\chi \leqq q \leqq w$ *for arbitrary topological spaces.*

(d) $\chi \leqq q \leqq u \leqq w$ *holds for completely regular topologies.*

Proof. (a) We give our proof essentially in terms of quasimetrics; it could easily be reformulated in terms of quasiuniformities or continuity spaces. Thus we find a set D of quasimetrics such that $To(D) = T_1$, $To(D^*) = T_2$, and $|D| \leqq w(T_1)\, w(T_2)$.

For $i = 1, 2$, let B_i be a base for T_i of minimal cardinality. Let $E = \{(R, S, i)$: $i = 1, 2$, $R, S \in B_i$, and for some pairwise continuous $f: (X, T_i, T_{3-i}) \to [0, 1]_B$, $f[R] \subset (.5,1]$ and $f[X - S] = \{0\}\}$. For each $k \in E$ choose such an f_k, and define $d_k(x, y) = \max\{f_k(x) - f_k(y), 0\}$ if $i = 1$, $= \max\{f_k(y) - f_k(x), 0\}$ if $i = 2$. Finally, let $D = \{d_k\colon k \in E\}$; this choice of D requires $|D| \leqq |B_1|\,|B_2| = w(T_1)\, w(T_2)$.

We now check that $T_1 = To(D)$, $T_2 = To(D^*)$: if $x \in Q \in T_1$ let $S \in B_1$, $x \in S \subset Q$, and by pairwise complete regularity find $f: X_B \to [0, 1]_B$ pairwise continuous and such that $f(x) = 1$ and $f = 0$ off S. Since $x \in f^{-1}[(0, 1]] \in T_1$ let $R \in B_1$, $x \in R \subset \subset f^{-1}[(.5, 1]]$; clearly $f[R] \subset (.5, 1]$, $f[X - S] = \{0\}$, so $e' = (R, S, 1) \in E$, $d = d_{e'} \in D$. Thus $S_{.5,d}(x) = \{y\colon d(x, y) < .5\} = \{y\colon f_{e'}(y) > .5\} \subset S \subset Q$. This shows $T_1 \subset To(D)$, and a very similar proof shows $T_2 \subset To(D^*)$. Next suppose $x \in Q \in To(D)$; we find $Q_x \in T_1$ such that $x \in Q_x \subset Q$, showing Q to be a T_1-neighborhood of each of its points, thus open in T_1. By definition of $To(D)$ we have some finite $F \subset 2^{-N} \times D$ such that $S_F(x) \subset Q$. Since this $S_F(x)$ satisfies our conditions, $To(D) \subset T_1$, and again $To(D^*) \subset T_2$ is shown similarly.

Proof of the converse, and proofs of (b), (c), (d) are clear.

Proof of 1: It suffices to prove (a) since (b) then follows from 3 (b). For (a) it suffices to show that $bw \leqq bc + bq$ and $bw \leqq be + bq$. In the following lemma we show that $bd \leqq bc + bq$ and $bd \leqq be + bq$, and we now show that $bw \leqq bd + bq$, completing the proof. It remains to note that $w(T_i) \leqq bq(X_B)\, bd(X_B)$. Let Y be *bidense* in X_B (dense in X with respect to the topology $T_1 \vee T_2$), and let $T_1 = = To(D)$, $T_2 = To(D^*)$, $|Y| = bd(X_B)$, D of minimal possible cardinality. Let $B_1 = \{S_F(y)\colon\ F \subset 2^{-N} \times D$ finite, $y \in Y\}$, $B_2 = \{S_F(y)\colon\ F \subset 2^{-N} \times D^*$ finite, $y \in Y\}$, where $S_F(y)$ is defined as in the next-to-last paragraph of Section 1. Then $|B_i| \leqq \omega|D|\,|Y| \leqq bd(X_B)\, bq(X_B)$, so it will do to show that B_i is a base for T_i, i.e., for each $x \in X$, F finite as above, there is a $y \in Y$ such that $x \in S_{F/2}(y) \subset S_F(x)$, where let $F/2 = \{(r/2, d)\colon (r, d) \in F\}$. Since Y is bidense and $To(D^s) = T_1 \vee T_2$, given $x \in X$ there is a $y \in Y$ such that for each $(r, d) \in F$, $d^s(x, y) \leqq r/2$, thus $x \in \in S_{F/2}(y)$ and if $z \in S_{F/2}(y)$, $(r, d) \in F$ then $d(x, z) \leqq d^s(x, y) + d^s(y, z) < r$, so $S_{F/2}(y) \subset S_F(x)$.

4. Lemma. *For any pairwose T_0 bitopological space X_B we have*: $bd(X_B) \leqq \leqq bq(X_B) + bc(X_B)$, *and* $bd(X_B) \leqq bq(X_B) + be(X_B)$.

Proof. For both inequalities, let D be a set of quasimetrics of minimal cardinality such that $X_B = (X, To(D), To(D^*))$ (if there is no such D then $bq(X_B) = \infty$, so there is nothing to show), and for each finite $F \subset 2^{-N} \times D^s$ let $Y_F \subset X$ be maximal for which $S(F) = \{S_F(y): y \in Y_F\}$ is a set of disjoint sets and if $y, y' \in Y$ and $S_F(y) = S_F(y')$ then $y = y'$. Next let $S = \bigcup\{S(F): F \subset 2^{-N} \times D^s \text{ finite}\}$, $Y = \bigcup\{Y_F: F \subset 2^{-N} \times D^s \text{ finite}\}$.

The first inequality now follows from the following two facts: (1) choice requires that $|Y| \leqq |S| \leqq |D| \, \omega c(To(D) \vee To(D^*)) \leqq bq(X_B) + bc(X_B)$. (2) Y is bidense: if $Q \in To(D) \vee To(D^*)$ is non-empty, we must show that $Y \cap Q$ is non-empty. Choose $x \in Q$; then for some finite $F \subset 2^{-N} \times D^s$, $S_F(x) \subset Q$. If $y \in Y$ then $S_{F/2}(x) \cap S_{F/2}(y) = \emptyset$ since if z is in their intersection and $(r, d) \in F$ then since $d \in D^s$, $d(x, y) \leqq d(x, z) + d(z, y) \leqq r/2 + r/2 = r$. But this contradicts the maximality of $S(r)$ or our choice of Y.

The second inequality follows from these facts: (1) each Y_F is discrete in $To(D) \vee To(D^*)$. (2) Y_F is closed in $To(D) \vee To(D^*)$; first notice that if $z \in X$ then $S_F(z)$ contains at most one of the $y \in Y_F$: by symmetry if $y, y' \in S_F(z)$ then $z \in S_F(y) \cap S_F(y')$, contradicting the discreteness of S_F and choice of Y_F. Since X_B is pairwise T_0, $To(D) \vee To(D^*)$ is Tychonoff, so there is a join-open set Q containing z but not this y, and $Q \cap S_F(z)$ does not intersect Y_F. (3) Thus $|Y_F| \leqq e(To(D) \vee To(D^*)) \leqq be(X_B)$ for each finite $F \subset 2^{-N} \times D^s$. But Y is bidense so $bd(X_B) = d(To(D) \vee To(D^*)) \leqq |Y| \leqq |D| \, \omega \sup \{|Y_F|: F \subset 2^{-N} \times D^s \text{ finite}\} \leqq bq(X_B) + be(X_B)$.

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Souhrn

KARDINÁLNÍ INVARIANTY BITOPOLOGICKÝCH PROSTORŮ

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Nesymetrická vzdálenost (viz [7]) nebo kvaziuniformita (viz [2]) na množině X generuje dvě topologie na X. Prostory se dvěma topologiemi, tzv. bitopologické prostory, byly zavedeny v [6] a [9]. Cílem autorů je rozšířit teorii kardinálních funkcí na tyto prostory a stanovit meze pro několik takových funkcí pomocí kvaziuniformní váhy, jež je nesymetrickou analogií Juhaszovy uniformní váhy (viz [4]).

Резюме

КАРДИНАЛЬНЫЕ ИНВАРИАНТЫ БИТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ

R. D. KOPPERMAN, P. R. MEYER

Несимметричное расстояние (см. [7]) или квазиравномерная структура (см. [2]) на множестве X порождает две топологии на X. Пространства с двумья топологиями, т.н. битопологические пространства были введены в [6] и [9]. Целью авторов является распространение теории кардинальных функций на эти пространства и установление оценок для нескольких таких функций при помощи квазиравномерного веса, который является несимметрическим аналогом равномерного веса Юхаса (см. [4]).

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