Časopis pro pěstování matematiky

Radko Mesiar

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Časopis pro pěstování matematiky, Vol. 114 (1989), No. 4, 337--342

Persistent URL: http://dml.cz/dmlcz/118384

ON CONTRACTIONS IN L_1

RADKO MESIAR, Bratislava

(Received December 8, 1986)

Summary. Let $(X, \mathscr{S}, m)$ be a σ-finite measure space, T a contraction on $L_1(X)$, $f \in L_1(X)$. For a given nondecreasing sequence $\{a_n\}$ of positive reals we study the pointwise convergence of $T^n f/a_n$. If the series $\Sigma\, 1/a_n$ is convergent, then $T^n f/a_n \to 0$, a.e. For a divergent series $\Sigma\, 1/a_n$ we establish a condition which enables us to construct a contraction P on $L_1(\langle 0, 1))$ and $f \in L_1(\langle 0, 1))$ such that $\limsup P^n f/a_n = \infty$, a.e.

Keywords: Pointwise convergence, contraction.

AMS Classification: 60F99.

Let $(X, \mathscr{S}, m)$ be a σ-finite measure space, $L_1(X)$ the Banach space of all classes of integrable real functions on $(X, \mathscr{S}, m)$ with the usual norm $\|\cdot\|$. A necessary condition of the pointwise convergence of Cesaro or other general means of a linear operator T acting on $L_1(X)$ is

(&) $m(\limsup T^n f/a_n = \infty) = 0$ for every $f \in L_1(X)$.

Here $T^n f/a_n$ are the last therms of the means involved (e.g. in the case of Cesaro means we have $a_n = n$). Of course, in the case $T^n f/a_n \to 0$, a.e., the condition (&) is fulfilled.

The nonfulfillment of the condition (&) is usual tool for the construction of examples in which the pointwise convergence of the means considered does not hold, see e.g. [2] for the Cesaro means or [3, 4] for more general means. The validity of the condition (&) for mean bounded operators is studied in [1].

In our paper we take into account only the linear contractions of the space $L_1(X)$. We propose to study the following problem: for a given nondecreasing sequence $\{a_n\}$ of positive reals, how do the properties of the sequence $\{a_n\}$ determine the fulfilling of the condition (&) (for every linear contraction)?

Theorem 1. *Let $\Sigma\, 1/a_n < \infty$, i.e. $\{1/a_n\} \in l_1$. Then for every linear contraction T on $L_1(X)$ and every $f \in L_1(X)$ we have*

$$T^n f/a_n \to 0\,, \quad a.e.$$

Proof. It suffices to define a linear operator $U = \Sigma T^n/a_n$. Then $\|U\| \leqq \Sigma \|T^n\|/a_n \leqq \Sigma\, 1/a_n < \infty$, so that $Uf \in L_1(X)$, which implies the theorem.

Chacón in [2] has given an example of a positive contraction $\boldsymbol{P}$ of the space $L_1(\langle 0, 1))$ and an integrable function f such that

$$\liminf \boldsymbol{M}_n f = 0\,, \quad \text{a.e.}$$

$$\limsup \boldsymbol{M}_n f = \infty\,, \quad \text{a.e.}$$

Here $\boldsymbol{M}_n = 1/n(\boldsymbol{I} + \boldsymbol{P} + \ldots + \boldsymbol{P}^{n-1})$ are the Cesaro means. Of course then $\liminf \boldsymbol{P}^n f/n = 0$, a.e. In Chacón's example the divergence of $\boldsymbol{M}_n$ is caused by the divergence of the last terms, when

$$\lambda(\limsup \boldsymbol{P}^n f/n = \infty) = 1 \quad (\lambda \text{ is the Lebesque measure})\,.$$

It is easy to see that for any unbounded sequence $\{a_n\}$ we get $\liminf \boldsymbol{T}^n f/a_n = 0$, a.e., for any contraction $\boldsymbol{T}$ and $f \in L_1(X)$. For convergent series $\Sigma\, 1/a_n$ Theorem 1 implies also $\limsup \boldsymbol{T}^n f/a_n = 0$, a.e. Let $\Sigma\, 1/a_n$ be a dense divergent series, i.e. $\inf n/a_n = B > 0$, see e.g. [5]. Then $\limsup \boldsymbol{P}^n f/a_n \geqq B.\ \limsup \boldsymbol{P}^n f/n$, so that Chacón's example implies the existence of a contraction $\boldsymbol{P}$ on $L_1(\langle 0, 1))$ and $f \in$ $\in L_1(\langle 0, 1))$ such that $\limsup \boldsymbol{P}^n f/a_n = \infty$, a.e. So it is sufficient to study the case of divergent series $\Sigma\, 1/a_n$ with $\inf n/a_n = 0$. The next example shows that even in this case the condition (&) need not hold.

Example 1. Let $a_n = n \ln n$, $n = 2, 3, \ldots$. We construct a positive contraction $\boldsymbol{P}$ on $L_1(\langle 0, 1))$ so that for a convenient integrable function $f \in L_1(\langle 0, 1))$ we get

$$\limsup \frac{\boldsymbol{P}^n f}{n\,.\ln n} = \infty\,, \quad \text{a.e.}$$

We modify Chacón's construction, see [2]. $\boldsymbol{P}$ will be induced by an invertible non-singular transformation t of $\langle 0, 1)$ (with the Lebesque measure λ),

$$\boldsymbol{P} f(x) = f(t^{-1}x)\frac{\mathrm{d}(\lambda \circ t^{-1})}{\mathrm{d}\lambda} \quad \text{for} \quad x \in \langle 0, 1)\,.$$

We define the transformation t step by step. It always maps the intervals into intervals. Let e.g. $\langle a, b) \to \langle c, d)$. Then we define

$$tx = c + \frac{d - c}{b - a}(x - a)\,.$$

Let $y \in \langle c, d)$. Then

$$\boldsymbol{P} f(y) = \frac{b - a}{d - c} f\left(a + \frac{b - a}{d - c}(y - c)\right).$$

It is obvious that $\boldsymbol{P}$ is a positive linear contraction of $L_1(\langle c, d))$ into $L_1(\langle a, b))$.

Denote $\mathsf{I}_1^1 = \langle 0, \frac{1}{2})$, $\mathsf{I}_2^1 = \langle \frac{1}{2}, 1)$. By induction we construct a disjoint partition $\mathsf{I}_1^n, \mathsf{I}_2^n, \ldots, \mathsf{I}_{N_n}^n$ of the unit interval $\langle 0, 1)$. The transformation t should map the interval I_i^n

into the interval I^n_{i+1}. Hence in the n-th step t should be well defined on the unit interval $\langle 0, 1)$ except the last interval $I^n_{N_n}$. Similarly the inverse transformation should be well defined except on the first interval I^n_1. Our construction implies $I^n_1 \searrow, \lambda(I^n_1) \to 0$, and similarly $I^n_{N_n} \searrow, \lambda(I^n_{N_n}) \to 0$, so that both t and t^{-1} should be defined on the whole interval $\langle 0, 1)$ except a set of measure zero.

Let the n-th partition $I^n_1, I^n_2, \ldots, I^n_{N_n}$ be defined. To get the $(n + 1)$-st partition we cut the intervals I^n_i into K_n disjoint subintervals $I^n_{i,j}$ such that

$$\bigcup_{j=0}^{K_n-1} I^n_{i,j} = I^n_i \,, \quad \text{for which} \quad x \in I^n_{i,j} \,, \quad y \in I^n_{i,j+1} \quad \text{always implies} \quad x < y \,.$$

We suppose $\lambda(I^n_{i,j}) = b_{n,j}\, \lambda(I^n_i)$ for all $i \in \{1, 2, \ldots, N_n\}$, $j \in \{0, 1, \ldots, K_n - 1\}$. Then

$$N_{n+1} = N_n K_n \,.$$

Here $b_{n,j}$ are positive constants independent of i, $\sum_j b_{n,j} = 1$.

The divergence of the series $\sum_{i=2} (1/i \,.\, \ln i)$ implies for each natural n, N_n the existence of a real positive constant c_n and a natural $K_n > 1$ such that

1) $c_n < (2^n \,.\, 4n \,.\, N_n \,.\, \ln N_n)^{-1}$,

2) $0 < 1 - (2^{-n} + c_n \sum_{i=2}^{K_n-1} (1/i \,.\, \ln i) < 2^{-n}$.

Note that the constant 4 in the first inequality may be replaced by an arbitrary constant A satisfying

$$A \geqq \sup_{\substack{p \geqq 2 \\ r \geqq 2}} \frac{\ln pr}{(\ln p) \,.\, \ln r} \,.$$

Now, we are able to define the constants $b_{n,j}$:

$$b_{n,0} = 2^{-n} \,,$$

$$b_{n,j} = \frac{c_n}{(j + 1) \,.\, \ln (j + 1)} \,, \quad j = 1, 2, K_n - 2,$$

$$b_{n,K_n-1} = 1 - \sum_{j=0}^{K_n-2} b_{n,j}.$$

It is obvious that $\sum_j b_{n,j} = 1$, $b_{n,0} > b_{n,j} > 0$ for $j > 0$. The $(n + 1)$-st partition should consist of the interval $I^n_{i,j}$. We order them to preserve the transformation t,

$$I^{n+1}_0 = I^n_{1,0} \,, \quad I^{n+1}_1 = I^n_{2,0} \,, \quad \text{etc.}$$

In general $I^{n+1}_{j.N_n+i} = I^n_{i,j}$, $i = 1, \ldots, N_n$, $j = 0, \ldots, K_{n-1}$. We extend the preserved definition of t on the n-th partition in a natural way, i.e., t should map $I^{n+1}_{j.N_n} = I^n_{N_n,j-1}$ into $I^{n+1}_{j.N_n+1} = I^n_{1,j}$.

Recurrency implies $\lambda(I^{n+1}_1) \geqq \lambda(I^{n+1}_i)$ for all $i \in \{1, \ldots, N_{n+1}\}$.

Let $f(x) = 1$ on $\langle 0, 1)$ and $x \in \mathsf{I}_{k+1}^{n+1}$, where $N_n \leqq k \leqq (K_n - 1) N_n - 1$, i.e. $k = rN_n + s$, $r \in \{1, 2, \ldots, K_n - 2\}$, $s \in \{0, 1, \ldots, N_n - 1\}$. Then

$$\boldsymbol{P}^k f(x) = \frac{\lambda(\mathsf{I}_k^{n+1})}{\lambda(\mathsf{I}_{k+1}^{n+1})} \boldsymbol{P}^{k-1} f(t^{-1}x) = \ldots = \frac{\lambda(\mathsf{I}_1^{n+1})}{\lambda(\mathsf{I}_{k+1}^{n+1})} f(t^{-k}x) = \frac{\lambda(\mathsf{I}_1^{n+1})}{\lambda(\mathsf{I}_{k+1}^{n+1})},$$

so that

$$\frac{\boldsymbol{P}^k f(x)}{k \,.\, \ln k} = \frac{\lambda(\mathsf{I}_{1,0}^n)}{\lambda(\mathsf{I}_{s+1,r}^n) \,.\, k \,.\, \ln k} = \frac{2^{-n} \,.\, \lambda(\mathsf{I}_1^n)}{\dfrac{c_n}{(r+1) \,.\, \ln(r+1)} \,.\, \lambda(\mathsf{I}_{s+1}^n) \,.\, .k \,.\, \ln k} >$$

$$> \frac{(4n \,.\, N_n \,.\, \ln N_n) \,.\, (r+1) \,.\, \ln(r+1)}{(r+1) \,.\, N_n \,.\, \ln((r+1) \,.\, N_n)} > n \,.$$

Denote

$$A_n = \Big(\bigcup_{k \leqq N_n} \mathsf{I}_k^{n+1}\Big) \cup \Big(\bigcup_{k > (K_n - 1) . N_n} \mathsf{I}_k^{n+1}\Big) \quad (\text{for } x \in A_n$$

we have not introduced the last inequality). Then

$$\lambda(A_n) = b_{n,0}\lambda(\langle 0, 1)) + b_{n,K_n-1}\lambda(\langle 0, 1)) < 2 \,.\, 2^{-n}, \quad \text{so that} \quad \lambda(\bigcup_n A_n) < \infty \,.$$

From the Borel-Cantelli lemma we get that for a.e. x and for infinite number of n the inequality

$$\frac{\boldsymbol{P}^k f(x)}{k \,.\, \ln k} > n \quad \text{for} \quad k \in \{N_n, \ldots, (K_n - 1) N_n - 1\} \quad \text{holds} \,.$$

Hence

$$\limsup_k \frac{\boldsymbol{P}^k f}{k \,.\, \ln k} = \infty \,, \quad \text{a.e.}$$

The construction of Example 1 is a modification of that of Chacón. It can be used with advantage for sequences $\{a_n\}$, $\sum 1/a_n = \infty$, with no "big jumps", which is expressed by the condition

(&&) *there exists infinite numbers of natural m such that for each m there exists a positive constant* d_m *for which*

$$B_m = \sup_n \frac{an_m}{a_n a_m (\sum_{i=1}^{n-1} 1/a_i)^{1-d_m}} < \infty \,.$$

Note that if the condition (&&) is fulfilled, then for all natural k there exists a positive constant d_k for which the supremum B_k is finite.

Theorem 2. *Let $\sum 1/a_n = \infty$ and let $\{a_n\}$ satisfy the condition (&&). Then there exist a positive linear contraction P on $L_1(\langle 0, 1))$ and an integrable function $f \in L_1(\langle 0, 1))$ such that*

$$\limsup_n P^n f / a_n = \infty\,, \quad \text{a.e.}$$

Proof. We repeat the construction of Example 1, only we take such constants c_n, K_n for which the following inequalities hold:

1) $c_n < (2^n n a_{N_n} (\sum_{i=1}^{K_n - 1} 1/a_i)^{1-d} \alpha_{N_n})^{-1} B_{Nn}^{-1}$

2) $0 < 1 - (2^{-n} + c_n \sum_{i=2}^{K_n - 1} 1/a_i) < 2^{-n}$.

The existence of c_n, K_n follows from the divergence of $\sum 1/a_n$.

Example 2. Let $a_1 = 1$, $a_n = 2^{2^i}$ for $2^{2^{i-1}} \leqq n < 2^{2^i}$, $i = 1, 2, \ldots$. Then

$$\sum_n 1/a_n = 1 + \sum_i \frac{2^{2^i} - 2^{2^{i-1}}}{2^{2^i}} = \infty\,.$$

Let $m \in \{2, 3, \ldots\}$. Then for $n = 2^{2^i} - 1$ we have

$$\sum_{j=1}^{n-1} 1/a_j < i + 1\,.$$

For an arbitrary positive constant d_m we get

$$\frac{a_{nm}}{a_n a_m (\sum_{j<n} 1/a_j)^{1-d_m}} > \frac{2^{2^{i+1}}}{2^{2^i} a_m (i+1)} = \frac{2^{2^i}}{a_m (i+1)}\,.$$

It follows that the sequence $\{a_n\}$ does not satisfy the condition (&&).

Example 2 shows that our problem for general divergent series $\sum 1/a_n$ remains open.

A similar problem for mean bounded linear operators is solved in [1]. There the convergence of the series $\sum 1/a_n$ is the best possible condition to ensure the pointwise convergence of $T^n f/a_n$ to zero. If the series $\sum 1/a_n$ is divergent, then we can construct (on a convenient measure space) a mean bounded linear operator T and an integrable function f such that $m(\limsup T^n f/a_n = \infty) > 0$, so that the condition (&) is not fulfilled.

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Súhrn

O KONTRAKCIÁCH V L_1

RADKO MESIAR

Nech $(X, \mathscr{S}, m)$ je σ-konečný priestor s mierou m, T kontrakcia na $L_1(X)$ a f integrovateľná funkcia z $L_1(X)$. Pre danú postupnosť $\{a_n\}$ kladných neklesajúcich reálnych čísiel rozoberáme bodovú konvergenciu výrazov $T^n f/a_n$. Ak rad $\Sigma\, 1/a_n$ konverguje, potom $T^n f/a_n \to 0$, s.v. V prípade divergentných radov $\Sigma\, 1/a_n$ sme našli podmienku, ktorej splnenie zaručuje existenciu takej kontrakcie P na $L_1(\langle 0, 1))$ a $f \in L_1(\langle 0, 1))$, že $\lim\sup P^n f/a_n = \infty$, s.v.

Резюме

О СЖИМАЮЩИХ ОПЕРАТОРАХ В L_1

RADKO MESIAR

Пусть $(X, \mathscr{S}, m)$ — пространство с σ-конечной мерой, T — сжимающий оператор в $L_1(X)$ и $f \in L_1(X)$. В статье изучается поточечная сходимость для $T^n f/a_n$, где $\{a_n\}$ — неубывающая последовательность положительных вещественных чисел.

Author's address: Katedra matematiky SvF SVŠT, Radlinského 11, 613 68 Bratislava.