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# A NOTE ON THE GP-INTEGRAL

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*Summary.* Jiří Jarník, Jaroslav Kurzweil, Štefan Schwabik in [1] gave an example which showed that the GP-integral generally failed to depend continuously on the domain of integration. However, in this note, we show that the GP-integral depends continuously on the domain of integration except on the boundaries.

*Keywords:* Generalized Perron integral, continuous dependence on the integration domain.

Let $I \subset [a, b]$, $a, b \in R^n$, be a Cartesian product of compact intervals $[a_i, b_i] = R$ with $a_i < b_i$, $i = 1, 2, \ldots, n$. A $p$-partition of the interval $I$ is (cf. [2]) a finite family

$$\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$$

where $I^j$ are intervals such that $\{I^1, \ldots, I^m\}$ is a partition of $I$ and

$$x^j \in I^j \quad (j = 1, 2, \ldots, m).$$

With Mawhin [2], let us call the irregularity $\Sigma(\Pi)$ of $\Pi$ the positive number defined by

$$\Sigma(\Pi) = [\max_{1 \leq j \leq m} \sigma(I^j)]/\sigma(I)$$

where the rate of stretching $\sigma(I)$ of the interval $I$ is defined by

$$\sigma(I) = [\max_{1 \leq i \leq n} (b_i - a_i)]/[\min_{1 \leq i \leq n} (b_i - a_i)].$$

**Theorem.** *Let $f$ be* GP-*integrable on $[a, b]$, $a, b \in R^n$, let $x \in (a, b)$. Let $\{I^k\}_{k=1}^{\infty}$ be a sequence of intervals with $x \in I^k$ and $\lim_{x \to \infty} \operatorname{diag} I^k = 0$; then $\lim_{k \to \infty} (\mathrm{GP}) \int_{I^k} f = 0$.*

Proof. We prove the theorem for $n = 2$; when $n > 2$, the proof is similar. Suppose $x$ is in $(a, b)$, and the theorem is false at $x$. Then there exists a sequence $\{I^k\}_{k=1}^{\infty}$ of intervals such that $x \in I^k$, $k = 1, 2, \ldots$, $\lim_{k \to \infty} \operatorname{diag} I^k = 0$, but $(\mathrm{GP}) \int_{I^k} f \nrightarrow 0$. We may suppose that $\int_{I^k} f > \alpha > 0$; Otherwise, we would consider $-f$, where $\int_{I^k} f < < \alpha < 0$.

Without loss of generality, we suppose that $\int_I f = 0$. Given $\varepsilon > 0$, $\eta \geqq 2$, let $\delta$ be such a gauge on $I$ that

$$\left\|S(I,f,\Pi) - (\mathrm{GP})\textstyle\int_I f\right\| < \varepsilon$$

for every $\delta$-fine $p$-partition $\Pi$ of $I$ with $\Sigma(\Pi) \leqq \eta$.

We notice that $\sigma(I^k)/\sigma(I)$ may be larger than $\eta$. Next, we show that we can always construct a sequence $\{J^k\}_{k=1}^{\infty}$ of intervals such that $\lim_{k\to\infty} \operatorname{diag} J^k = 0$, $\sigma(J^k)/\sigma(I) \leqq \eta$ and $\left|\int_{J^k} f\right| > \alpha/3$. Since $x \in (a, b)$, we suppose that each $I^k$, $1 \leqq k$, is contained in a fixed square, centered at $x$ and contained in $(a, b)$.

If $\sigma(I^k)/\sigma(I) \leqq \eta$, put $J^k = I^k$. Suppose $\sigma(I^k)/\sigma(I) > \eta$, $I^k = [a_1^k, b_1^k] \times [a_2^k, b_2^k]$ and suppose $b_2^k - a_2^k > b_1^k - a_1^k$. Put

$$I_1^k = [b_1^k, c^k] \times [a_2^k, b_2^k]\,,$$

$$I_2^k = [d^k, a_1^k] \times [a_2^k, b_2^k]\,,$$

where $c^k$, $d^k$ are chosen such that

$$\sigma(I_1^k \cup I^k)/\sigma(I) \leqq \eta\,,$$

$$\sigma(I_2^k \cup I^k)/\sigma(I) \leqq \eta\,,$$

$$\sigma(I_1^k \cup I_2^k \cup I^k)/\sigma(I) \leqq \eta\,,$$

and $\operatorname{diag}(I_1^k \cup I^k)$, $\operatorname{diag}(I_2^k \cup I^k)$ and $\operatorname{diag}(I_1^k \cup I_2^k \cup I^k)$ are all less than $\sqrt{2}\operatorname{diag} I^k$. We claim that one of $I_1^k \cup I^k$, $I_2^k \cup I^k$, $I_1^k \cup I_2^k \cup I^k$ can be chosen as $J^k$.

$$\text{If} \quad \int_{I_1^k \cup I^k} f > \frac{\alpha}{3}\,, \quad \text{put} \quad J^k = I^k \cup I_1^k\,.$$

$$\text{If} \quad \int_{I_2^k \cup I^k} f > \frac{\alpha}{3} \quad \text{put} \quad J^k = I^k \cup I_2^k\,.$$

$$\text{Otherwise,} \int_{I_1^k \cup I^k} f \leqq \frac{\alpha}{3} \text{ and} \int_{I_2^k \cup I^k} f \leqq \frac{\alpha}{3}\text{; this implies}$$

$$\int_{I_1^k} f = \int_{I_1^k \cup I^k} f - \int_{I^k} f \leqq -\frac{2\alpha}{3} \quad \text{and}$$

$$\int_{I_2^k} f = \int_{I_1^k \cup I^k} f - \int_{I^k} f \leqq -\frac{2}{3}\alpha\,,$$

so that

$$\int_{I_1^k \cup I_2^k \cup I^k} f = \int_{I^k \cup I_1^k} f + \int_{I_2^k} f \leqq -\frac{\alpha}{3}\,.$$

Now we put $J^k = I_1^k \cup I_2^k \cup I^k$.

Since $f(x)$ is GP-integrable on $I \setminus J^k$, given $\varepsilon > 0$, $\eta \geqq 2$, let $\delta_1$ be such a gauge on $I \setminus J^k$ that

$$\|S(I \setminus J^k, f, \Pi_1) - (\mathrm{GP}) \int_{I \setminus J^k} f\| < \varepsilon$$

for every $\delta_1$-fine $p$-partition $\Pi_1$ of $I \setminus J^k$ with $\Sigma(\Pi_1) < \eta$.

Let $\delta_2(x) = \min(\delta(x), \delta_1(x))$ and let $\Pi_2$ be a $\delta_2$-fine $p$-partition of $I \setminus J^k$ with $\Sigma(\Pi_2) < \eta$. Then

$$\|S(I \setminus J^k, f, \Pi_2) - (\mathrm{GP}) \int_{I \setminus J^k} f\| < \varepsilon .$$

Let $\Pi = (x, J^k) \cup \Pi_2$, then $\Pi$ is a $\delta$-fine $p$-partition with $\Sigma(\Pi) < \eta$, so

$$\|S(I, f, \Pi)\| < \varepsilon , \tag{1}$$

but

$$\|S(I, f, \Pi)\| \geqq \|(\mathrm{GP}) \int_{I \setminus J^k} f\| - \|f(x)\, \mu I^k\| - \|S(I \setminus J^k, f, \Pi_2) - (\mathrm{GP}) \int_{I \setminus J^k} f\| \geqq$$
$$\geqq \tfrac{1}{3}\alpha - \varepsilon - \|f(x)\, \mu I^k\| .$$

When $k$ is large enough, we have

$$\|S(I, f, \Pi)\| \geqq \tfrac{1}{3}\alpha - 2\varepsilon ; \tag{2}$$

(1) and (2) form a contradiction if we choose $\varepsilon < \frac{1}{9}\alpha$. The proof is complete.

*References*


[1] *J. Jarník, J. Kurzweil, S. Schwabik:* On Mawhin's approach to multiple nonabsolutely convergent integral. Časopis pěst. Mat. *108* (1983), 356–380.

[2] *J. Mawhin:* Generalized multiple Perron integrals and the Green-Goursat theorem for differentiable vector fields. Czechoslovak Math. J. *31* (*106*) (1981), 614–632.



Souhrn

POZNÁMKA KE GP-INTEGRÁLU

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J. Jarník, J. Kurzweil a Š. Schwabik podali příklad, že zobecněný Perronův integrál obecně nezávisí spojitě na integračním oboru. V poznámce je ukázáno, že nespojitá závislost může nastat pouze v hraničních bodech integračního oboru.



Резюме

ЗАМЕЧАНИЕ О GP-ИНТЕГРАЛЕ

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И. Ярник, Я. Курцвейл и Ш. Швабик показали на примере, что обобщенный интеграл Перрона независит в общем случае непрерывно от области интегрирования. В заметке показано, что это может случиться только в граничных точках области интегрирования.



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