Časopis pro pěstování matematiky

Jan Chrastina

On formal theory of differential equations. II.

Časopis pro pěstování matematiky, Vol. 114 (1989), No. 1, 60--105

Persistent URL: http://dml.cz/dmlcz/118369

ON FORMAL THEORY OF DIFFERENTIAL EQUATIONS II

JAN CHRASTINA, Brno

Dedicated to my teacher Professor Otakar Borůvka on the occasion of his ninetieth birthday

(Received June 29, 1987)

Summary. As a very particular application of the theory of infinite prolongations of partial differential equations (diffieties), a method for solving the ancient Monge problem (~ 1780) is proposed. The original problem is whether the general solution of an underdetermined system of ordinary differential equations (the number of unknown functions exceeds the number of equations) can be expressed by explicit formulae containing some arbitrary "parametric" functions. The proposed method is even more powerful and yields a deep insight into the structure of the considered equations.

Keywords: underdetermined system, infinite prolongation, Pfaffian system, Monge problem, explicit solvability.

AMS classification: 34A05, 35A30, 58A15, 58A17.

Continuing the previous part [1], we should like to establish some interrelations between our approach and the common one. However, the variety of actual topics proves to be so immense that it is not possible to cope with this imposition at the first attempt. So we shall deal only with the particular case $n = 1$, the case of ordinary differential equations, through the main body of the present article.

In classical terms, our intentions are as follows: we shall deal with local theory of quite arbitrary smooth systems of ordinary differential equations near the generic points. If the number of unknown functions is equal to the number of equations, the system can be transformed into $\mathrm{d}y^j/\mathrm{d}x = 0$ $(j = 1, \ldots, m)$ which is a trivial object, of course. Similar conclusions can be drawn if the number of equations exceeds the number of unknown functions. However, the remaining *underdetermined case* deserves more attention. For instance, the famous *Monge equation* $f(x, y, z, \mathrm{d}y/\mathrm{d}x, \mathrm{d}z/\mathrm{d}x) = 0$ for two unknown functions $y = y(x)$, $z = z(x)$ has been thoroughly studied for a long time. Already Euler (in a particular case) and Monge found the interesting general solution of the type

$$x = \bar{x}(\&)\,, \quad y = \bar{y}(\&)\,, \quad z = \bar{z}(\&) \quad (\& = (t, u(t), \mathrm{d}u(t)/\mathrm{d}t, \mathrm{d}^2u(t)/\mathrm{d}t^2))\,,$$

with certain fixed functions $\bar{x}, \bar{y}, \bar{z}$ of the argument & and an arbitrary $u(t)$. Since then, the problem of resolving an underdetermined system by means of analogous formulae with a generalized argument & possibly involving some arbitrary constants and higher order derivatives of several arbitrary functions is called the *Monge* *problem*. We can mention only two achievements here and refer to [2] for more information. It was Hilbert who first succeded in finding a negative result, namely that $dz/dx = (d^2y/dx^2)^2$ cannot be resolved in this manner. Shortly after, E. Cartan gave a complete discussion of the particular case of $m - 1$ equations with m unknown functions but with a pessimistic remark (rather unusual in his scientific work) concerning the possible development of these result, cf. [3]. In fact, to the best of our knowledge, no essential progress followed although the underdetermined systems became very popular owing to the optimal control theory; see also [4] and the references therein.

We would like to have more space (and patience) to present the solution of the Monge problem in full generality here but only the cases of $m - 1$ and $m - 2$ equations for m unknown functions can be analyzed in more detail. We believe, however, that these particular cases are sufficient to demonstrate the universality of the method. Instead, for compensation, we shall look at some modifications of the problem if certain quadratures in the argument & are permitted. It is to be noted that the approach proposed yields a deep insight into the structure of the differential equations under consideration not fully exploited here. For instance, all explicitly solvable systems admit a large group of symmetries so that some links to the *differential Galois theory* surely exist. Nevertheless, the necessary tool, an intrinsical and easily manageable pseudogroup theory, is still lacking.

The present paper is made independent (to a large extent) of the preceding part [1]. (Added in proof: see the errata to [1] and some comments at the end of the present article.)

COMPLEMENTS TO THE GENERAL THEORY

1. Review of fundamental concepts from [1, Sections 1—3, 5, 12]. For the convenience of exposition, we recall some needful facts slightly modifying the notation and terminology. The *underlying spaces* for all considerations will be the inverse limits $J = \lim\operatorname{inv} J^l$, where J^l $(l = 0, 1, \ldots)$ are certain (smooth, Hausdorff, with countable basis, real) manifolds of finite dimensions n^l (but $n^l \to \infty$ in all interesting cases) and the resulting limit arises from certain given surjective submersions $j^l_k \colon J^l \to J^k$ $(l \geqq k)$ satisfying $j^l_i \circ j^i_k = j^l_k$. Only the final result J is important for us, not the specific determination by the data J^l and j^l_k. Since the pullbacks j^{l*}_k identify every space $\Psi^{[k]}_s$ of exterior differential s-forms on J^k with a subspace of the analogous space $\Psi^{[l]}_s$, the space $\Psi_s = \cup \Psi^k_s$ of *differential s-forms on J* can be easily introduced. In particular, $\Psi_0 = \cup \Psi^k_0 = C^\infty(J)$ are functions and $\Psi = \oplus \Psi_s$ are all differential forms on J. The concept of the *exterior derivative* d is familiar and the *vector fields* $X, Y, \ldots$ on the space J, the *inner products* $X \urcorner^* \psi \in \Psi$ (where $\psi \in \Psi$), the

*) For typographical reasons the author's symbol $\lrcorner$ was replaced by $\urcorner$ throughout the paper.

Lie derivatives $\mathscr{L}_X = \mathrm{d}X \urcorner + X \urcorner \mathrm{d}$, and the *Lie brackets* $[X, Y] = XY - YX$ can be employed and satisfy the common rules.

According to the definition of the inverse limit, a point $p \in J$ is represented by an array $p^0, p^1, \ldots,$, where $p^l \in J^l$, $j_k^l(p^l) \equiv p^k$. We denote $j_k = \lim j_k^l \colon J \to J^k$, hence $p^l = j_l(p)$. A *coordinate system* at a point $p \in J$ consists of a sequence $f^1, f^2, \ldots \in \Psi_0$ such that every piece $f^1, \ldots, f^{n^l}$ $(l = 0, 1, \ldots)$ belongs already to $\Psi_0^{[l]} = C^\infty(J^l)$ and provides a coordinate system on J^l near the point p^l. A differential form can be expressed by a finite number of coordinates but a vector field X may be only identified with a formal sum $X = \Sigma g^j \, \partial/\partial f^j$, where $g^j \equiv Xf^j$. Values at a point $p \in J$ of various objects will be often denoted by subscripts, e.g., $f(p) = f_p \in \mathbb{R}$ $(f \in \Psi_0)$, $X_p \in T_pJ$ is a tangent vector at p, $(\Psi_1)_p = T_p^*J$ is the cotangent space. We shall often deal with various Ψ_0-modules Ξ and the relevant $\mathbb{R}$-linear spaces Ξ_p. The dimension of the space Ξ_p will be denoted by $\ell(\Xi_p)$. We shall prefer the *regular case* when $\ell(\Xi_q)$ is a (finite) constant for all $q \in J$ lying near the given point p under consideration. Then the Ψ_0-module Ξ is locally free near the point p and $\ell(\Xi_p)$ is the number of (local) generators; we shall abbreviate it (a little inaccurately) by $\ell(\Xi)$ omitting the point p. If $\Xi \subset \Psi_1$ is an arbitrary submodule of the Ψ_0-module Ψ_1, then $\Xi^\perp$ denotes the Ψ_0-module of all vector fields X on J satisfying $X \urcorner \xi \equiv 0$ $(\xi \in \Xi)$. If $\Xi^\perp$ is regular, then $\xi \in \Xi$ is characterized by the property $X \urcorner \xi \equiv 0$ $(X \in \Xi^\perp)$.

The main object of our investigations will be the so called *diffieties*. The last term was invented to denote a special type of submodules $\Omega \subset \Psi_1$ satisfying certain axioms $\mathscr{L}oc$, $\mathscr{D}im$, $\mathscr{C}los$ and $\mathscr{F}in$ specified below. (The diffieties represent the infinite prolongations of general systems of partial differential equations in a very concise, intrinsical and self-contained manner eliminating all accidental and misleading features, as we shall soon see.) The axiom $\mathscr{L}oc$ means that a form $\psi \in \Psi_1$ belongs to a given diffiety Ω if ψ belongs to Ω locally (that is, if for every $p \in J$ there exists $f \in \Psi_0$ with $f(p) \neq 0, f\psi \in \Omega$). The axiom $\mathscr{D}im$ requires the Ψ_0-module Ψ_1/Ω to be locally free with dimensions $\ell((\Psi_1/\Omega)_p) = n(\Omega)$ independent of $p \in J$. (More explicitly, for every $p \in J$ there exist forms $\psi^1, \ldots, \psi^n \in \Psi_1$ $(n = n(\Omega))$ such that there are unique decompositions $\psi = \Sigma f^i \psi^i + \omega$ $(f^i \in \Psi_0,\ \omega \in \Omega)$ of every form $\psi \in \Psi_1$ near the point p. One can then see that $\psi_p^1, \ldots, \psi_p^n$ is a basis of $(\Psi_1/\Omega)_p$.) Let us denote $\mathscr{H} = \mathscr{H}(\Omega) = \Omega^\perp$; clearly $\mathscr{H}$ is a regular module and $\ell(\mathscr{H}) = n$. The axiom $\mathscr{C}los$ is expressed by $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$ or, equivalently, by $\mathscr{L}_{\mathscr{H}}\Omega \subset \Omega$ (that is, $\mathscr{L}_X\omega \in \Omega$ for any $X \in \mathscr{H}$ and $\omega \in \Omega$). The axiom $\mathscr{F}in$ postulates the existence of filtrations

$$(1) \qquad \Omega^*\colon \Omega^0 \subset \Omega^1 \subset \ldots \subset \Omega^l \subset \ldots \subset \Omega = \cup \Omega^l$$

(for technical reasons we occasionally put $\Omega^l = \{0\}$ if $l < 0$) by finitely generated submodules $\Omega^l \subset \Omega$ satisfying

$$(2)_{1,2} \qquad \Omega^{l+1} \supset \Omega^l + \mathscr{L}_{\mathscr{H}}\Omega^l \ (\text{all } l), \quad \Omega^{l+1} = \Omega^l + \mathscr{L}_{\mathscr{H}}\Omega^l \ (l \text{ large enough});$$

these are the *good filtrations* of Ω.

We shall mention one simple result of the theory. Let $\odot\mathscr{H} = \Psi_0 \oplus \mathscr{H} \oplus \oplus (\mathscr{H} \odot \mathscr{H}) \oplus (\mathscr{H} \odot \mathscr{H} \odot \mathscr{H}) \oplus \ldots$ be the free commutative Ψ_0-algebra over the Ψ_0-module $\mathscr{H}$. Then the graded Ψ_0-module $\mathscr{G} = \oplus \mathscr{G}^l$ $(\mathscr{G}^l = \Omega^l/\Omega^{l-1})$ turns into a $\odot\mathscr{H}$-module with the multiplication

$$X\hat{\omega} = (\mathscr{L}_X\omega)^\wedge = (X \urcorner \mathrm{d}\omega)^\wedge \in \mathscr{G}^{l+1} \quad (X \in \mathscr{H},\ \omega \in \Omega^l,\ \hat{\omega} \in \mathscr{G}^l),$$

where the hats indicate the relevant classes in $\mathscr{G}$. Suppose regularity of all modules $\mathscr{G}^l$ at a point $p \in J$. One can then see that the relation $\hat{\omega}_p = 0$ implies $(X\hat{\omega})_p = 0$ and it follows that the rule

$$X_p \,.\, \hat{\omega}_p = (X\hat{\omega})_p \quad (X_p \in \mathscr{H}_p,\ \hat{\omega} \in \mathscr{G})$$

defines a well-founded $\odot\mathscr{H}_p$-module structure on the graded $\mathbb{R}$-linear space $\mathscr{G}_p = \oplus \mathscr{G}^l_p$. Owing to the $\mathscr{D}im$ axiom, we have a Noetherian module. So we find ourselves in the realm of the classical commutative algebra and in particular, we may consider the familiar *Hilbert polynomial*

$$\ell(\mathscr{G}^0_p \oplus \ldots \oplus \mathscr{G}^l_p) = \ell(\Omega^l) \cong \chi(\mathscr{G}_p, l) = \mu\, l^\nu/\nu! + (\ldots)$$

(the equality is true for large l and the dots denote some lower degree terms), where $\mu = \mu(\Omega, p) > 0$, $\nu = \nu(\Omega, p) \geqq 0$ are certain integers independent of the choice of the filtration (1). (The trivial case $\Omega = \{0\}$ is omitted to ensure $\mu > 0$.)

2. Morphisms of diffieties. Turning to new concepts, we shall consider a quite another diffiety beside the given Ω, and then the notation will be as follows: the underlying space for the other diffiety Θ will be $I = \lim\operatorname{inv} I^l$ arising from certain surjective submersions $i^l_k\colon I^l \to I^k$ between manifolds. The spaces of differential forms on I^l or I will be denoted $\Phi^{[l]}_s$ or Φ_s, and $\Phi = \cup\Phi_s$. The diffiety Θ is a submodule of the Φ_0-module Φ_1 satisfying the relevant axioms, of course. In order not to fall into deep water, we shall always tacitly suppose the equality $n(\Omega) = n = n(\Theta)$, hence $\ell(\mathscr{H}(\Omega)) = n = \ell(\mathscr{H}(\Theta))$.

We begin with the concept of a mapping $\iota\colon I \to J$ between the underlying spaces. Such a mapping is represented by a limit $\iota = \lim \iota^{k(l)}_l$, where $\iota^{k(l)}_l\colon I^{k(l)} \to J^l$ $(0 \leqq \leqq k(0) \leqq k(1) \leqq \ldots,\ k(l) \to \infty)$ are certain mappings between the usual finite-dimensional manifolds satisfying the commutativity law $\iota^{k(l)}_l \circ i^{k(l+1)}_{k(l)} = j^{l+1}_l \circ \iota^{k(l+1)}_{l+1}$ $(l = 0, 1, \ldots)$. The mapping ι is also determined by the relations $j_l \circ \iota = \iota^{k(l)}_l \circ i_{k(l)}$ (recall that $j_l = \lim j^k_l\colon J \to J^l$ and $i_l = \lim j^k_l\colon I \to I^l$). However, the above concept is too wide for the common practice. Surjective submersions onto the submanifolds

(3) $\qquad \iota^{k(l)}_l\colon I^{k(l)} \to J^l \ \ (\iota^{k(l)}_l(I^{k(l)}) \subset J^l$ is an embedded submanifold)

are preferable. In particular, surjective submersions $\iota^{k(l)}_l$ onto the whole manifold J^l are tacitly employed if the resulting mapping ι is surjective. The point is that in this case the pullbacks $\iota^*\colon \Psi \to \Phi$, $\iota^*_p\colon \Psi_p \to \Phi_p$ are fair injections. (Undoubtedly, these measures are in reality excessively strong and can be weakened by Sard's theorem; however, the lure will be ignored.)

The above mapping ι is called a *morphism* between the diffieties Ω and Θ, if the conditions

$$(4)_{1,2} \qquad \iota^*\Omega \subset \Theta\,, \quad \iota^*\Psi_1 + \Theta = \Phi_1$$

are satisfied. (Equivalently and more geometrically, every space $\mathscr{H}(\Theta)_p$ is bijectively projected onto $\mathscr{H}(\Omega)_{\iota(p)}$ by the tangent mapping $(\iota^*)_p\colon T_pI \to T_{\iota(p)}J$.) If moreover ι is injective, then Θ is called a *subdiffiety* of Ω, and if ι is a surjection, then Ω is called a *factordiffiety* of Θ. An *isomorphism* or *automorphism* is obtained for bijective ι, of course.

A large supply of deep problems is coming on: decide whether given diffieties are isomorphic, find whether a given diffiety is a factordiffiety or subdiffiety of another diffiety, find whether there exists a common factordiffiety of two given diffieties, investigate automorphisms and infinitesimal automorphisms of a given diffiety, and so on. Connections to some actual topics and to several traditional areas (cf. the Bäcklund transformations, the equivalence problem, the Monge problem, the Lie and Lie-Bäcklund infinitesimal symmetries, and so on) are obvious and some of them will be discussed in what follows.

3. From differential equations to diffieties, see [5, 6]. We intend to outline this way as briefly as possible so that only the local theory is presented without any specification of the geometric contents. Let B be a manifold with coordinates $t^1, \ldots, t^n$ and E another manifold with coordinates $t^1, \ldots, t^n, u^1, \ldots, u^m$ and the obvious surjection $\pi\colon E \to B$ identifying some of the coordinates: $\pi^* t^i = t^i$ $(i = 1, \ldots, n)$. We introduce the jet space $J(\pi)$ where the coordinates are

$$(5) \qquad t^i, u^j_{i_1 \ldots i_s} \quad (i, i_1, .., i_s = 1, \ldots, n;\ i_1 \leqq \ldots \leqq i_s;\ j = 1, \ldots, m;\ s = 0, 1, \ldots).$$

Clearly $J(\pi) = \lim\operatorname{inv} J^l(\pi)$, where $J^l(\pi)$ $(l = 0, 1, \ldots)$ is parametrized by the coordinates (5) with s constrained only to the values $0, \ldots, l$. In particular $J^0(\pi) = E$. The inverse limit arises from the obvious surjections $j^k_l(\pi)\colon J^l(\pi) \to J^l(\pi)$ $(k \geqq l)$ omitting the coordinates (5) with $s = k + 1, \ldots, l$. The relevant spaces of differential forms on the spaces $J^l(\pi)$ or J will be denoted by $\Psi^{[l]}_s(\pi)$ or $\Psi_s(\pi)$, and $\Psi(\pi) = \bigcup \Psi_s(\pi)$. The notation can be simplified by the multi-index abbreviation $\mathscr{I} = i_1 \ldots i_s$, $|\mathscr{I}| = s$, and by the convention $u^j_{\mathscr{I}} = u^j_{\mathscr{I}'}$, if the nondecreasing multiindex $\mathscr{I} = i_1 \ldots i_s$ $(i_1 \leqq \ldots \leqq i_s)$ is only a permutation of $\mathscr{I}' = i'_1 \ldots i'_s$. Owing to this convention, we have the familiar *contact forms* $\omega^j_{\mathscr{I}}(\pi) = \mathrm{d}u^j_{\mathscr{I}} - \Sigma u^j_{\mathscr{I}i}\,\mathrm{d}t^i \in \Psi^{[l+1]}_1(\pi) \subset \Psi_1(\pi)$. Then the *Cartan diffiety* $\Omega(\pi)$ is defined as the $\Psi_0(\pi)$-module consisting of all forms $\Sigma f^j_{\mathscr{I}}\, \omega^j_{\mathscr{I}}(\pi)$ with arbitrary $f^j_{\mathscr{I}} \in \Psi_0(\pi)$. On can easily verify all axioms $\mathscr{L}oc, \ldots, \mathscr{F}in$. In particular, the relevant module $\mathscr{H}(\pi) = \mathscr{H}(\Omega(\pi))$ is freely generated by the well-known *formal derivatives*

$$\partial_i = \partial/\partial t^i + \Sigma u^j_{\mathscr{I}i}\, \partial/\partial u^j_{\mathscr{I}} \quad (i = 1, \ldots, n)$$

and we have the so called *standard filtration*

$$(6)\qquad \Omega(\pi)^*: \Omega(\pi)^0 \subset \Omega(\pi)^1 \subset \ldots \subset \Omega(\pi)^l \subset \ldots \subset \Omega(\pi) = \cup\, \Omega(\pi)^l\,,$$

where $\Omega(\pi)^l$ is the submodule of $\Omega(\pi)$ generated by all contact forms $\omega^j_{\mathscr{I}}(\pi)$ with $|\mathscr{I}| \leqq l$. (From the common point of view, the above spaces $J^l(\pi)$ and $\Omega(\pi)^l$ are considered for intrinsical objects preserved by all reasonable automorphisms. However, we take only the resulting $J(\pi)$ and $\Omega(\pi)$ to be meaningful and quite other filtrations than (6) are permitted as well.)

The diffiety $\Omega(\pi)$ corresponds to the very special system of nm equations $\partial u^j/\partial t^i \equiv \equiv u^j_i$ containing $(n + 1)\, m$ unknown functions u^j, u^j_i or better, to the infinite prolongation $\partial u^j_{\mathscr{I}}/\partial t^i \equiv u^j_{\mathscr{I}i}$ thereof. An arbitrary system of partial differential equations appears if certain additional relations among the variables (5) are taken into account.

Having this in mind, let $\{f^\kappa;\ \kappa \in K\} \subset \Psi_0(\pi)$ be a set of functions labelled by κ varying in an (in general infinite) index set K. Let $J \subset J(\pi)$ be the subset consisting of all points $p \in J$ that satisfy $f^\kappa(p) \equiv 0$ $(\kappa \in K)$. We wish to restrict the diffiety $\Omega(\pi)$ to the subset J to obtain a new diffiety corresponding to the system

$$(7)\qquad f^\kappa(\ldots, t^i, \ldots, \partial^s u^j/\partial t^{\mathscr{I}}, \ldots) \equiv 0 \quad (\kappa \in K)\,;$$

the derivatives stand at the places of the variables $u^j_{\mathscr{I}}$. In order to derive this result without much effort, the functions f^κ will be submitted to strong restriction $\mathscr{F}i\ell$ (*filtration*) and $\mathscr{P}as$ (*passivity*).

The first assumption $\mathscr{F}i\ell$ ensures the existence of a filtration $K^0 \subset K^1 \subset \ldots \ldots \subset K = \cup K^l$ by finite subsets with the property that every set of functions $\{f^\kappa;\ \kappa \in K^l\}$ is a subset of $\Psi_0^{[l]}(\pi)$ and may be completed to a coordinate system on the space $J^l(\pi)$. It follows that $J^l = j_l(\pi)\,(J)$ (where $j_l(\pi) = \lim j^k_l(\pi)$) are embedded submanifolds of the space $J^l(\pi)$, and $J = \lim\operatorname{inv} J^l$ with respect to the obvious mappings $j^k_l: J^k \to J^l$ induced by $j^k_l(\pi)$. If we denote by $\iota(\pi) = \lim j_l(\pi): J \to J(\pi)$ the natural inclusion, then $\Psi_s = \iota(\pi)^*\, \Psi_s(\pi)$ are the spaces of exterior s-forms on J.

The second assumption $\mathscr{P}as$ requires for every fixed $f \in \Psi_0(\pi)$ that if the identity $f(p) \equiv 0$ $(p \in J)$ is satisfied, then $\partial_i f(p) \equiv 0$ $(p \in J;\ i = 1, \ldots, n)$, hence $X f(p) \equiv 0$ $(p \in J,\ X \in \mathscr{H})$. Consequently, every vector field $X \in \mathscr{H}(\pi)$ induces a well-defined operator X' on the space Ψ_0 by the rule $X'g = \iota(\pi)^*\, Xf$ $(g = \iota(\pi)^* f,\ f \in \Psi_0(\pi))$. Clearly X' is a vector field on J and we shall abbreviate the notation by identifying $X = X'$, in particular $\partial'_i \equiv \partial_i$. We usually denote $\iota(\pi)^*\, t^i \equiv x^i$, $\iota(\pi)^*\, u^j_{\mathscr{I}} \equiv y^j_{\mathscr{I}}$ and $\iota(\pi)^*\, \omega^j_{\mathscr{I}}(\pi) \equiv \omega^j_{\mathscr{I}}$ for more clarity.

At this stage, one can easily verify that $\Omega = \iota(\pi)^*\, \Omega(\pi)$ is a diffiety on the underlying space J. For instance, $\mathscr{D}im$ follows from

$$\mathrm{d}f = \Sigma\, \partial_i f\, \mathrm{d}t^i + \Sigma\, \partial f/\partial u^j_{\mathscr{I}}\, \omega^j_{\mathscr{I}}(\pi) \quad (f \in \Psi_0(\pi))$$

after applying $\iota(\pi)^*$, $\mathscr{C}\ell os$ is trivial since (roughly speaking) $\mathscr{H}(\Omega) = \mathscr{H}(\Omega(\pi))$

along the subspace $J \subset J(\pi)$, and $\mathscr{F}in$ follows from the (so called *standard*) filtration of Ω consisting of the terms $\Omega^l = \iota(\pi)^* \Omega(\pi)^l$, by definition.

(A few words about the realization of the above assumptions $\mathscr{F}il$ and $\mathscr{P}as$ should be useful. One usually starts with a finite set $f^1, \ldots, f^c \in \Psi_0(\pi)$ which is successively completed by adding the derivatives $\partial_i f^j, \partial_i \partial_{i'} f^j, \ldots$ that are functionally independent of the previously calculated functions. The resulting set is filtered according to the norms $|\mathscr{I}|$ of the variables $u^j_{\mathscr{I}}$ effectively appearing in the functions. If necessary, some exceptional points of the space $J(\pi)$ must be left out to ensure the functional independence. If the final space J consisting of all points $p \in J(\pi)$ which satisfy $f^j(p) = \partial_i f^j(p) = \partial_i \partial_{i'} f^j(p) = \ldots \equiv 0$ is nonempty (the *compatibility condition*), we are done. Careful analysis of the procedure proves to be rather tedious (but relatively easy in all current cases) and since we are interested only in the final result, it is not important to us. If necessary, the reader is referred to extensive literature [5–10] with other references and many examples.)

4. Module structure related to the standard filtrations can be made very explicit. Since the $\Psi_0(\pi)$-modules $\mathscr{G}(\pi)^l = \Omega(\pi)^l / \Omega(\pi)^{l-1}$ are freely generated by the classes $\hat{\omega}^j_{\mathscr{I}}(\pi)$ of contact forms, and the localizations $(\hat{\omega}^j_{\mathscr{I}}(\pi))_p$ ($|\mathscr{I}| = l$) yield a basis of the space $\mathscr{G}(\pi)^l_p$, we obtain the $\odot\, \mathscr{H}(\pi)_p$-module structure on $\mathscr{G}(\pi)_p$ by taking the values at p:

$$X_p\, \hat{\omega}^j_{\mathscr{I}}(\pi)_p = (X \urcorner \mathrm{d}\omega^j_{\mathscr{I}}(\pi))^{\wedge}_p = \Sigma\, g^i(p)\, \hat{\omega}^j_{\mathscr{I}i}(\pi)_p \quad (X = \Sigma\, g^i \partial_i)\,.$$

As the subdiffiety $\Omega = \iota(\pi)^* \Omega(\pi)$ is concerned, let $\omega \in \Omega(\pi)^l$ and assume $(\iota(\pi)^* \omega)^{\wedge}_p = 0$ for the corresponding class $(\iota(\pi)^* \omega)^{\wedge} \in \mathscr{G}^l$. It follows that

$$\omega = h\varphi + \psi + \Sigma g^\kappa\, \mathrm{d}f^\kappa \quad (h, g^\kappa \in \Psi_0(\pi);\ \varphi \in \Omega(\pi)^l,\ \psi \in \Omega(\pi)^{l-1})$$

with $h(p) = 0$. But $Xf^\kappa \equiv 0$ and $\iota(\pi)^*\, \mathrm{d}f^\kappa \equiv 0$ so that

$$(X(\iota(\pi)^* \omega)^{\wedge})_p = (\iota(\pi)^* X\hat{\omega})_p = (\iota(\pi)^* (X \urcorner \mathrm{d}\omega)^{\wedge})_p = (\iota(\pi)^* (X \urcorner h\, \mathrm{d}\varphi)^{\wedge}))_p = 0$$

and the $\odot\, \mathscr{H}_p$-module structure on $\mathscr{G}_p$ is well-founded. Moreover, as follows from the above formulae, the $\odot\, \mathscr{H}_p$-module $\mathscr{G}_p$ is a factormodule of the $\odot\, \mathscr{H}(\pi)_{\iota(p)}$-module $\mathscr{G}(\pi)_{\iota(p)}$ with respect to the submodule of $\mathscr{G}(\pi)_{\iota(p)}$ generated by the family of the classes of differentials $\mathrm{d}f^\kappa$ ($\kappa \in K$), for all points $p \in J$.

5. From diffieties to differential equations. Analogously as in Section 3, we should like to present here only the most essential information employing some local arguments. So, let Ω be an arbitrary diffiety, let (1) be good filtration satisfying the equality $(2)_2$ for $l \geqq c$. Since the Ψ_0-module Ω^c is finitely generated and the generators can be expressed by a finite number of functions, there exist a certain m and functions $y^1, \ldots, y^m \in \Psi_0$ such that every form lying in Ω^c can be expressed by a sum $\Sigma g^j\, \mathrm{d}y^j$ with appropriate functions $g^1, \ldots, g^m \in \Psi_0$. Now, according to $\mathscr{D}im$, there are local formulae of the type

$$\mathrm{d}y^j = \Sigma z^j_i \psi^i + \omega^j \quad (y^j_i \in \Psi_0,\ \omega^j \in \Omega) \tag{8}$$

near the point $p \in J$ under consideration, where $\psi^1, \ldots, \psi^n$ $(n = n(\Omega))$ are certain fixed forms. But one can easily see that these forms can be chosen to be of special type $\psi^i \equiv \mathrm{d}x^i$, where $x^1, \ldots, x^n$ are appropriate *sufficiently general* functions (i.e., satisfying $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \neq 0$ on the subspace $\mathscr{H}(\Omega)_p$).

Passing to the crucial point of the construction, let $\bar{\Omega}^0$ be the submodule of Ω generated by the above forms $\omega^1, \ldots, \omega^m$, and let us put inductively $\bar{\Omega}^{l+1} = \bar{\Omega}^l + \mathscr{L}_{\mathscr{H}}\bar{\Omega}^l$ $(l = 0, 1, \ldots)$. According to (8), every form of the type $\Sigma g^j \,\mathrm{d}y^j$ lying in Ω necessarily is a linear combination of $\omega^1, \ldots, \omega^m$, that is, it lies in $\bar{\Omega}^0$. Consequently $\Omega^c \subset \bar{\Omega}^0$, hence $\Omega^{c+1} = \Omega^c + \mathscr{L}_{\mathscr{H}}\Omega^c \subset \bar{\Omega}^0 + \mathscr{L}_{\mathscr{H}}\bar{\Omega}^0 = \bar{\Omega}^1$, and in general $\Omega^{c+k} \subset \bar{\Omega}^k$ for every $k = 0, 1, \ldots$, by a similar argument. It follows that

$$(9) \qquad \bar{\Omega}^*\colon \bar{\Omega}^0 \subset \bar{\Omega}^1 \subset \ldots \subset \bar{\Omega}^l \subset \ldots \subset \Omega = \cup \bar{\Omega}^l$$

is a good filtration.

Let the vector fields $X_1, \ldots, X_n \in \mathscr{H}$ be (locally) defined by the requirements $X_i x^j \equiv 0$ $(i \neq j)$, $X_i x^i \equiv 1$. Then the Ψ_0-module $\bar{\Omega}^l$ is clearly generated by the forms of the type

$$\omega^j_{\mathscr{I}} = \mathscr{L}_{X_{i_1}} \ldots \mathscr{L}_{X_{i_s}} \omega^j = \mathrm{d}y^j_{\mathscr{I}} - \Sigma y^j_{\mathscr{I}i}\,\mathrm{d}x^i \quad (y^j_{\mathscr{I}} = X_{i_1} \ldots X_{i_s} y^j;\ s = 0, \ldots, l)\,.$$

Now, let us recall the diffiety $\Omega(\pi)$ of Section 3 on the underlying space with the coordinates $t^i, u^j_{\mathscr{I}}$. Then the formulae $\iota^* t^i \equiv x^i$, $\iota^* u^j_{\mathscr{I}} \equiv y^j_{\mathscr{I}}$ determine a (local) injection $\iota\colon J \to J(\pi)$, and Ω turns into a subdiffiety of $\Omega(\pi)$ since clearly $\iota^* \omega^j_{\mathscr{I}}(\pi) \equiv \omega^j_{\mathscr{I}}$. Speaking more expressively, if $f^\kappa(\ldots, t^i, \ldots, u^j_{\mathscr{I}}, \ldots) = 0$ $(\kappa \in K)$ are all interrelations among the functions $t^i, u^j_{\mathscr{I}}$, then the diffiety Ω corresponds to the system (7) of partial differential equations.

ORDINARY DIFFERENTIAL EQUATIONS

6. Introduction. From now on we shall restrict ourselves to the particular case $n = 1 = \dim \mathscr{H}_p$ without further explicit warning. Then the notation can be simplified: the variables t^i, x^i (where $i = 1$) and $u^j_{\mathscr{I}}, y^j_{\mathscr{I}}$ (where $\mathscr{I} = 1 \ldots 1$ with s equal terms) will be abbreviated to t, x and u^j_s, y^j_s, respectively. (We occasionally write $u^j_0 \equiv u^j$, $y^j_0 \equiv y^j$.) Quite analogously, we have the contact forms $\omega^j_s(\pi) = \mathrm{d}u^j_s - u^j_{s+1}\,\mathrm{d}t$ which generate the diffiety $\Omega(\pi)$ and the vector field $\partial = \partial/\partial t + \Sigma u^j_{s+1}\,\partial/\partial u^j_s$ which generates the module $\mathscr{H}(\pi)$.

Let us mention the correspondence between (ordinary) differential equations and diffieties in more detail. Clearly, the diffiety $\Omega(\pi)$ corresponds to the underdetermined system $\mathrm{d}u^j_0/\mathrm{d}t \equiv u^j_1$ of m differential equations involving $2m$ unknown functions $u^1_0, \ldots, u^m_0, y^1_1, \ldots, u^m_1$ and one independent variable t. According to Section 3, an arbitrary system of ordinary differential equations may be represented as a subdiffiety Ω of $\Omega(\pi)$ with appropriate m. This may be achieved in various manners, however, all subdiffieties corresponding to the same system are mutually isomorphic.

If one follows the approach of Section 3 rather closely, the given system is filtered according to the order of the derivatives involved. First, there may be equations of order zero:

$$(10)_0 \qquad f^{\kappa}(t, u_0^1, \ldots, u_0^m) = 0 \qquad (\kappa \in K^0),$$

where the *regularity* (i.e., the maximum possible rank of the Jacobian $(\partial f^{\kappa}/\partial u_0^j)$) is supposed. Secondly, there are equations of order one:

$$(10)_1 \qquad f^{\kappa}(t, u_0^1, \ldots, u_0^m, u_1^1, \ldots, u_1^m) = 0 \quad (\kappa \in K^1 - K^0,\ u_1^j = \mathrm{d}u^j/\mathrm{d}t),$$

again with the *regularity* (i.e., the maximum rank of $(\partial f^{\kappa}/\partial u_1^j)$). Thirtly, there are equations of the second order:

$$(10)_2 \qquad f^{\kappa}(t, u_0^1, \ldots, u_2^m) = 0 \quad (\kappa \in K^2 - K^1,\ u_1^j = \mathrm{d}u^j/\mathrm{d}t,\ u_2^j = \mathrm{d}^2u^j/\mathrm{d}t^2)$$

with the regularity, and so on. Then the condition $\mathcal{F}il$ is a consequence of the presumed regularity, and $\mathcal{P}ass$ is satisfied if, for instance, every relation

$$\mathrm{d}f^{\kappa}(t, \ldots, \mathrm{d}^s u^j/\mathrm{d}t^s, \ldots)/\mathrm{d}t = \partial f^{\kappa}(t, \ldots, u_s^j, \ldots) = 0$$
$$(\kappa \in K^l - K^{l-1},\ u_s^j \equiv \mathrm{d}^s u^j/\mathrm{d}t^s)$$

of equations of order l is included into the next group $(10)_{l+1}$.

In practise, however, the matters turn to be much simpler. One usually starts with a given finite system $f^k(t, \ldots, \mathrm{d}^s u^j/\mathrm{d}t^s, \ldots) = 0$ $(k = 1, \ldots, c)$ which may be replaced by an equivalent first order system by introducing new variables for the higher order derivatives involved. Then, in the regular case, the arising first order system can be resolved with respect to some derivatives, e.g.,

$$(11) \qquad \mathrm{d}u^k/\mathrm{d}t = g^k(t, u^1, \ldots, u^m, \mathrm{d}u^{c+1}/\mathrm{d}t, \ldots, \mathrm{d}u^m/\mathrm{d}t) \quad (k = 1, \ldots, c).$$

If new variables $u^{m+1} = \mathrm{d}u^{c+1}/\mathrm{d}t, \ldots, u^{c+m} = \mathrm{d}u^m/\mathrm{d}t$ are introduced, an equivalent but formally simpler system

$$(12) \qquad \mathrm{d}u^k/\mathrm{d}t = g^k(t, u^1, \ldots, u^m) \quad (k = 1, \ldots, c)$$

is obtained (the meaning of c, m was changed) without reducing the generality of results. The system (12) corresponds to the system $(10)_1$ above, $(10)_0$ is now empty. In order to fulfil $\mathcal{P}ass$, we take

$$\mathrm{d}^2u^k/\mathrm{d}t^2 = \partial g^k/\partial t + \Sigma\, \mathrm{d}u^j/\mathrm{d}t.\ \partial g^k/\partial u^j (= \partial g^k) \quad (k = 1, \ldots, c)$$

for $(10)_2$, then $\mathrm{d}^3u^k/\mathrm{d}t^3 = \partial^2 g^k$ for $(10)_3$, and so on. So we have functions

$$(13) \qquad f^{\kappa} = u_s^k - \partial^{s-1} g^k \quad (\kappa \in K = \cup K^l),$$

where the index set K^l consists of all pairs $\kappa = (k, s)$ with $k = 1, \ldots, s$ and $s = 1, \ldots, l$. Let $J \subset J(\pi)$ be the set of all $p \in J(\pi)$ such that $f^{\kappa}(p) \equiv 0$, as usual. According to (13), the functions

$$(14) \qquad x, y_0^1, \ldots, y_0^m, y_1^{c+1}, \ldots, y_1^m, y_2^{c+1}, \ldots, y_2^m, \ldots$$

(where $x = \iota(\pi)^* t$, $y_s^j = \iota(\pi)^* u_s^j$) may serve as coordinates on J, and the diffiety $\Omega = \iota(\pi)^* \Omega(\pi)$ is freely generated by the forms

$$\begin{aligned} &\mathrm{d}y_0^k - g^k \,\mathrm{d}x \quad (k = 1, \ldots, c)\,, \\ &\mathrm{d}y_s^j - y_{s+1}^j \,\mathrm{d}x \quad (j = c + 1, \ldots, m;\ s = 0, 1, \ldots)\,. \end{aligned} \tag{15}$$

7. Involutiveness. Passing to the general theory, let Ω be a diffiety with a good filtration (1). Since $\odot \mathscr{H}_p$ are polynomials of one variable, the degree ν of the Hilbert polynomial $\chi(\mathscr{G}_p, l)$ is at most 1. The case $\nu = 0$ clearly implies $\Omega = \Omega^l$ for all l large enough so that the space J is of finite dimension and we deal with a determined system of ordinary differential equations, a system locally expressible by $\mathrm{d}u^j/\mathrm{d}t = 0$ ($j = 1, \ldots, m$), and therefore uninteresting from our point of view. For this reason, we shall assume $\nu = 1$, $\chi(\mathscr{G}_p, l) = \mu l + \text{const.}$ ($\mu > 0$) unless otherwise stated.

Since $\mathscr{G}_p$ is a finitely generated module, the multiplication $X_p\colon \mathscr{G}_p^l \to \mathscr{G}_p^{l+1}$ ($X_p \in \mathscr{H}_p$, $X_p \neq 0$) is surjective for l large enough. But the dimension $\ell(\mathscr{G}_p^l) = \mu$ is constant for l large; then the multiplication is even a bijection. We speak of an *involutive case* if the above multiplication is surjective whenever $l \geqq 0$ and bijective for $l \geqq 1$ (compare with [1, Section 34]). We shall also need the so called *semi-involutive case* (a new concept); then the above multiplication is supposed bijective for $l \geqq 1$.

The involutiveness can be easily achieved by a simple modification of the original filtration (1), if necessary. For instance, one can use the so called *c-normal prolongation of* (1). This is the filtration

$$\bar{\Omega}^* = \Omega^{*+c}\colon \bar{\Omega}^0 = \Omega^c \subset \bar{\Omega}^1 = \Omega^{c+1} \subset \ldots \subset \bar{\Omega}^l = \Omega^{l+c} \subset \ldots \subset \Omega = \cup \bar{\Omega}^l\,,$$

cf. [1, Section 18]. As follows from the above reasoning, given an arbittary good filtration (1), Ω^{*+c} is involutive for all c large enough. But even the involutive filtrations are rather unpleasant and of little use in subtler investigations.

8. Examples. (i) Looking at the standard filtration (6) in the particular case $n = 1$, one can check that $\mu(\Omega(\pi)) = m$. The diffiety $\Omega(\pi)$ corresponds to the system $\mathrm{d}u_0^j/\mathrm{d}t \equiv u_1^j$ of m differential equations involving $2m$ unknown functions and the difference $2m - m$ (the number of "*arbitrary functions*") is exactly $m = \mu(\Omega(\pi))$. On the other hand, every term $\Omega(\pi)^l$ ($l \geqq 0$) is freely generated by certain $m(l + 1)$ differential forms expressible just by $m(l + 2) + 1$ variables. The difference between these numbers diminished by 1 (due to the presence of the independent variable t) is $m(l + 2) + 1 - m(l + 1) = m = \mu(\Omega(\pi))$ as before.

(ii) The diffiety $\Omega = \iota(\pi)^* \Omega(\pi)$ of Section 6 is filtered by the modules $\Omega^l = \iota(\pi)^* \Omega^l(\pi)$ and the existence of generators (15) implies $\mu(\Omega) = m - c$. It corresponds to the system (12) of c equations involving m unknown functions, the difference $m - c = \mu$ agrees. The modules Ω^l are freely generated by the forms (15) with $s = 0, \ldots, l$ in the total number of $m + (m - c)\,l$, and expressible just by

$l + m + (m - c)(l + 1)$ variables. The difference of these numbers diminished by 1 equals $m - c = \mu(\Omega)$.

(iii) Let us consider the space J with coordinates $t, u_0, v_0, u_1, v_1, \ldots$ and the vector field

$$\partial/\partial t + (u_1 + (v_2)^2)\,\partial/\partial u_1 + v_1\,\partial/\partial v_0 + u_2\,\partial/\partial u_1 + v_2\,\partial/\partial v_1 + \ldots .$$

This field determines a module $\mathscr{H}(\Omega)$ of a diffiety Ω corresponding to the system $du_0/dt = u_1 + (v_2)^2$, $dv_0/dt = v_1$, $dv_1/dt = v_2$ of three differential equations involving five unknown functions, hence $5 - 3 = 2 = \mu(\Omega)$. The modules Ω^l ($l \geqq 0$) freely generated by the forms

$$du_0 - u_1\,dt - v_2\,dv_1\,, \quad du_s - u_{s+1}\,dt + v_{s+2}\,dv_1 - v_2\,dv_{s+1} \quad (s = 1, \ldots, l)\,,$$
$$dv_s - v_{s+1}\,dt \quad (s = 0, \ldots, l)$$

determine an involutive filtration of Ω. There are $2l$ generators of Ω^l expressible just by $2l + 4$ variables (cf. the next section for the proof) and the difference diminished by 1 equals $2l + 4 - 2l - 1 = 3 > \mu(\Omega)$. The discrepancy may be interpreted as the presence of certain "parasite variables" in the generators of Ω^l.

9. Adjoint variables. Let us made a digression to some general concepts of the general theory of exterior systems (cf. [11]) suitably adapted for our needs. Dealing as usual with the space J, let $\Xi \subset \Psi_1$ be a regular submodule and $\operatorname{Adj} \Xi \subset \Psi_1$ the submodule consisting of all forms of the type $X \urcorner d\xi$ with $X \in \Xi^\perp$ and $\xi \in \Xi$. One can successively verify that the inclusion $Y \in (\operatorname{Adj} \Xi)^\perp$ is equivalent to any of the folloving conditions:

(i) $Y \urcorner X \urcorner d\xi = -X \urcorner Y \urcorner d\xi \equiv 0$ for all $X \in \Xi^\perp$, $\xi \in \Xi$,
(ii) $Y \urcorner d\xi \in \Xi$ for all $\xi \in \Xi$,
(iii) $Y \urcorner d(f\xi) = Yf \,.\, \xi - Y \urcorner \xi \,.\, df + fY \urcorner d\xi \in \Xi$ for all $f \in \Psi_0$, $\xi \in \Xi$,
(iv) $Y \in \Xi^\perp$ and $Y \urcorner d\xi \in \Xi$ for all $\xi \in \Xi$,
(v) $Y \in \Xi^\perp$ and $\mathscr{L}_Y \xi \in \Xi$ for all $\xi \in \Xi$,
(vi) $\mathscr{L}_{fY}\xi = f\mathscr{L}_Y\xi + Y \urcorner (df \wedge \xi) \in \Xi$ for all $f \in \Psi_0$, $\xi \in \Xi$,
(vii) $X \urcorner \mathscr{L}_{fY}\xi \equiv 0$ for all $X \in \Xi^\perp$, $f \in \Psi_0$, $\xi \in \Xi$.

In virtue of the rule $0 = \mathscr{L}_{fY}(X \urcorner \xi) = [fY, X] \urcorner \xi + X \urcorner \mathscr{L}_{fY}\xi$, we may continue with

(viii) $[fY, X] \in \Xi^\perp$ for all $f \in \Psi_0$, $X \in \Xi^\perp$,
(ix) $Y \in \Xi^\perp$ and $[Y, X] \in \Xi^\perp$ for all $\Phi \in \Xi^\perp$.

Always $\Xi \subset \operatorname{Adj} \Xi$ and (according to (ix)) the case $\ell(\operatorname{Adj} \Xi) - \ell(\Xi) = 1$ cannot occur. Moreover, $\operatorname{Adj}(\operatorname{Adj} \Xi) = \operatorname{Adj} \Xi$ or, equivalently, $[X, Y] \in \operatorname{Adj} \Xi^\perp$ for every $X, Y \in \operatorname{Adj} \Xi^\perp$ (use (ix) and the Jacobi identity, or (iv) and the rule $\mathscr{L}_{[X,Y]} = \mathscr{L}_X\mathscr{L}_Y - \mathscr{L}_Y\mathscr{L}_X$). Note that a submodule $\Xi \subset \Psi_1$ is called *completely integrable* if $[\Xi^\perp, \Xi^\perp] \subset \Xi^\perp$; it follows that $\operatorname{Adj} \Xi$ is completely integrable. In general, complete integrability of a regular submodule $\Xi \subset \Psi_1$ is equivalent to any of the conditions

$\operatorname{Adj} \Xi = \Xi$, $X \neg \mathrm{d}\xi \in \Xi$ $(X \in \Xi^{\perp},\ \xi \in \Xi)$, $\mathscr{L}_X \Xi \subset \Xi$ $(X \in \Xi^{\perp})$ as follows from the points (ix), (ii) and (v) above.

$\operatorname{Adj} \Xi$ is clearly a finitely generated module. Assume moreover $\operatorname{Adj} \Xi$ to be regular with $\ell(\operatorname{Adj} \Xi) = a$. Then, applying the Frobenius theorem near a fixed point $p \in J$, one can find a coordinate system $f^1, f^2, \ldots$ at the point p such that the differentials $\mathrm{d}f^1, \ldots, \mathrm{d}f^a$ may serve as (local) free generators of the module $\operatorname{Adj} \Xi$. Moreover, *there exist local free generators of the Ψ_0-module Ξ expressible only in terms of the functions $f^1, \ldots, f^a$.* (Proof: Since $\operatorname{Adj} \Xi \subset \Xi$, every form lying in Ξ can be represented by a sum $\Sigma g^i \, \mathrm{d}f^i$ with appropriate $g^1, \ldots, g^a \in \Psi_0$. We conclude (using the Gauss elimination method and some redesignation of variables) that there exist local free generators of Ξ of the type

(16) $$\mathrm{d}f^k - \Sigma h_j^k \, \mathrm{d}f^j \quad (k = 1, \ldots, c;\ \text{sum over } j = c + 1, \ldots, a)\,.$$

Then point (iii) gives $Y \neg \mathrm{d}h_j^k \equiv 0$ $(Y \in \operatorname{Adj} \Xi^{\perp})$, that is, $\mathrm{d}h_j^k \in \operatorname{Adj} \Xi$ and h_j^k are functions of $f^1, \ldots, f^a$.)

The functions $f^1, \ldots, f^a$ will be called *adjoint variables* to the module Ξ at the point p. (More accurately, every composed function $f(f^1, \ldots, f^a)$ may be called *an adjoint variable* which may be adapted to a global concept. Nonetheless, we prefer the former (ancient) terminology.) We already know that appropriate generators of Ξ can be expressed in terms of them. Conversely, if some generators of Ξ can be expressed by certain functions $g^1, \ldots, g^b \in \Psi_0$, then already the adjoint variables $f^1, \ldots, f^a$ are functions of $g^1, \ldots, g^b$. (Hint: If $Xg^j \equiv 0$, then $X \in \operatorname{Adj} \Xi^{\perp}$, hence $Xf^j \equiv 0$.) It follows that *the adjoint variables are the most economical family in this respect.*

10. Cartan filtrations. A semi-involutive filtration (1) is called a *Cartan filtration* if

(17) $$\Omega^1 = \Omega \cap \operatorname{Adj} \Omega^0\,.$$

Assume $\operatorname{Adj} \Omega^0 \subset \Omega$ for a moment. Then $\operatorname{Adj} \Omega^0 = \Omega^1$ by (17), hence $\mathscr{L}_{\mathscr{H}} \Omega^1 \subset \Omega^1$ (we use $\mathscr{H} = \Omega^{\perp} \subset \Omega^{1\perp}$ and the complete integrability of $\Omega^1 = \operatorname{Adj} \Omega^0$), hence $\Omega^2 = \Omega^1 + \mathscr{L}_{\mathscr{H}} \Omega^1 = \Omega^1$ (we use the semi-involutiveness). Quite analogously, $\Omega^2 = \Omega^3 = \ldots = \cup \Omega^l = \Omega$ and, consequently, J is of finite dimension. Omitting this trivial case, we have $\operatorname{Adj} \Omega^0 \neq \Omega^1$, hence $\ell(\operatorname{Adj} \Omega^0) = \ell(\Omega^1) + 1$ according to $\mathscr{Dim}$ and (17).

Our next aim is to find a fair family of local free generators to the terms Ω^l of a Cartan filtration. Let $f^1, \ldots, f^a$ be the adjoint variables to the module Ω^0, where $\mathrm{d}f^1 \notin \Omega$. We may assume the existence of local free generators of the type (16) to the module Ω^0. And, introducing (a little artificial) notation of variables $x = f^1$, $y_1^1 = f^2, \ldots, y_1^m = f^a$ (hence $m = a - 1$), the formulae (16) can be rewritten as

$(18)^0$ $$\mathrm{d}y_1^k - \Sigma h_j^k \, \mathrm{d}y_1^j - h_{m+1}^k \, \mathrm{d}x \quad (k = 1, \ldots, c;\ \text{sum over } j = c + 1, \ldots, m)\,,$$

where h_j^k, h_{m+1}^k are functions of the variables $x, y_1^1, \ldots, y_1^m$. Now, the sense of the notation is clarified by looking at Ω^1. Indeed, according to (17), there are certain free generators of the type

$$(18)^1 \qquad \mathrm{d}y_1^1 - y_2^1\,\mathrm{d}x, \ldots, \mathrm{d}y_1^m - y_2^m\,\mathrm{d}x$$

to the module Ω^1, where $y_2^j \in \Psi_0$ are appropriate functions. Let $X \in \mathscr{H}$ be normalized by $Xx = 1$. Then, if we employ the semi-involutiveness (in particular, the equality $(2)_2$), the forms $(18)^1$ together with

$$(18)^s \qquad \mathrm{d}y_s^j - y_{s+1}^j\,\mathrm{d}x \quad (j = 1, \ldots, m;\ y_{s+1}^j = Xy_s^j),$$

where $s = 2, \ldots, l$, generate the module Ω^l ($l \geqq 2$). But we do not have free generators since there are only $\mu(\Omega) = \ell(\Omega^1/\Omega^0) = m - c$ forms in every family $(18)^s$ that are independent of the previous $(18)^1 \cup \ldots \cup (18)^{s-1}$. However, the free generators can be easily found. In virtue of $(18)^0$ and $(18)^1$ the relations $y_2^k \equiv g^k$ hold ($g^k = \Sigma h_j^k y_2^j + h_m^{k+1}$) and it follows that

$$(19) \qquad \mathrm{d}y_1^k - g^k\,\mathrm{d}x \quad (k = 1, \ldots, c),$$

$$\mathrm{d}y_s^j - y_{s+1}^j\,\mathrm{d}x \quad (j = c + 1, \ldots, m;\ s = 1, \ldots, l)$$

are the desired generators to Ω^l ($l \geqq 1$). Note that the functions

$$(20) \qquad x, y_1^1, \ldots, y_1^m, y_2^{c+1}, \ldots, y_2^m, y_3^{c+1}, \ldots, y_3^m, y_4^{c+1}, \ldots$$

may serve for local coordinates on the space J.

Using the generator $\partial/\partial x + \Sigma g^k\, \partial/\partial y_1^k + \Sigma y_{s+1}^j\, \partial/\partial y_s^j$ of the module $\mathscr{H}(\Omega)$, one can easily verify that normal 1-prolongations of our Cartan filtrations are involutive. Moreover, as follows from the explicit expressions of the generators given above, $\Omega \cap \operatorname{Adj}\Omega^l = \Omega^{l+1}$ (so that the c-prolongations are Cartan filtrations, too) and $\ell(\operatorname{Adj}\Omega^l) - \ell(\Omega^l) = \ell(\Omega^{l+1}/\Omega^l) + 1 = m - c + 1 = \mu(\Omega) + 1$ (so that the "parasite variables" in the sense of the last example of Section 8 do not appear).

11. Theorem. *The first term Ω^0 of a Cartan filtration* (1) *permits to reconstruct the whole filtration* (1), *hence the diffiety Ω. For any regular submodule $\Xi \subset \Phi_0$ there exists a diffiety Ω on an appropriate underlying space J and a Cartan filtration* (1) *such that $\Omega^0 = \Xi$.*

Proof. Using the notation of Section 10, let $x = f^1$, $y_1^1 = f^2, \ldots, y_1^m = f^a$ be the adjoint variables to the module Ω^0, where $\mathrm{d}f^1 \notin \Omega$. Moreover, let $(18)^0$ be the local generators to Ω^0. Then the relevant functions g^k needful for the reconstruction of the remaining generators (19) are known; this concludes the proof of the first assertion.

To prove the second assertion, we begin with another underlying space I for technical reasons. Then the proof runs as follows: Let $(18)^0$ be local generators of Ξ, where $x, y_1^1, \ldots, y_1^m \in \Phi_0$ are the adjoint variables to the module Ξ. Introducing a new underlying space J with coordinates denoted by $x, y_1^1, \ldots, y_1^m$ (and lying in Φ_0) and additional coordinates

$$(21) \qquad y_2^{c+1}, \ldots, y_2^m, y_3^{c+1}, \ldots, y_3^m, y_4^{c+1}, \ldots, \ldots$$

(not lying in Φ_0), the desired diffiety is generated by the forms (19).

12. Existence of a Cartan filtration for an arbitrary diffiety was already established in Section 5 if one chooses $n = 1$. Let us recall the construction: Starting with a filtration (1), let $f^1, \ldots, f^a$ be the adjoint variables to the module Ω^c. We may assume $f^1 \notin \Omega$. Then, if we denote $x = f^1, y^1 = f^2, \ldots, y^m = f^a$ $(m = a - 1)$, there are forms $\omega_0^j = \mathrm{d}y^j - y_1^j \, \mathrm{d}x \in \Omega$ and the filtration (9) with terms $\bar{\Omega}^l$ generated by the forms

$$\omega_0^j = \mathrm{d}y^j - y_1^j \, \mathrm{d}x \,, \quad \omega_s^j = \mathrm{d}y_s^j - y_{s+1}^j \, \mathrm{d}x \quad \left(y_{s+1}^j = Xy_s^j, \ s = 1, \ldots, l\right),$$

where the vector field $X \in \mathscr{H}$ normalized by $Xx = 1$ is used. If the arising filtration is involutive, we are done. If not, we turn to an appropriate normal prolongation. This concludes the construction.

THE MONGE PROBLEM AND ITS GENERALIZATIONS

13. Cartan's case. We ask whether the "general solution" of a system of certain $m - 1$ ordinary differential equations for m unknown functions $y^1, \ldots, y^m$ of an independent variable x can be represented by formulae of the type

$$(22) \qquad x = \bar{x}(\&) \,, \quad y^j = \bar{y}^j(\&) \quad \left(j = 1, \ldots, m; \ \& = \left(t, u(t), \ldots, \mathrm{d}^a u(t)/\mathrm{d}t^a\right)\right),$$

where $\bar{x}, \bar{y}^j$ are certain fixed functions of the argument & involving the "arbitrary" function $u(t)$. (One can then easily derive analogous formulae for the derivatives $\mathrm{d}^s y^j/\mathrm{d}x^s$.)

Without discussing the rather vague concepts of a "general solution" and "arbitrary function $u(t)$", we shall substitute the following more precise setting for the above mentioned classical problem: Instead of the original system (not specified above), we introduce its infinite prolongation, that is, the relevant diffiety Ω on an underlying space J with the coordinates $x, y_0^j = y^j, y_s^j$ $(j = 1, \ldots, m; \ s = 1, 2, \ldots)$, where y_s^j stand for the derivatives $\mathrm{d}^s y^j/\mathrm{d}x^s$. Let moreover $\Theta = \Omega(\pi)$ be the diffiety generated by all contact forms $\mathrm{d}u_i - u_{i+1} \, \mathrm{d}t$ $(i = 0, 1, \ldots)$ on the underlying space $I = J(\pi)$ of the coordinates $t, u_0, u_1, \ldots$. The problem is whether Ω can be represented as a factordiffiety of Θ. If this is the case, then the relevant surjection $\iota: I \to J$ yields the formulae (22) provided we put $\bar{x} = \iota^* x, \bar{y}^j \equiv \iota^* y^j$.

In fact, E. Cartan investigated the more general case when the argument & may also depend on some arbitrary constants $c_1, \ldots, c_b$. In terms of diffieties, the problem consists in representing a given diffiety Ω as a factordiffiety of another diffiety Θ

generated by the preceding contact forms $\mathrm{d}u_i - u_{i+1}\,\mathrm{d}t$ together with forms $\mathrm{d}v^1, \ldots$ $\ldots, \mathrm{d}v^b$, differentials of certain additional functions. The relevant underlying space I has the coordinates $t, v^1, \ldots, v^b, u_0, u_1, \ldots$. We shall see, however, that this generalization does not bring too many difficulties.

14. True generalizations are easy to formulate. For instance, one might be interested in the case of the argument

$$\& = (t, u(t), v(t), \ldots, \mathrm{d}^a u(t)/\mathrm{d}t^a\,, \quad \mathrm{d}^a v(t)/\mathrm{d}t^a)$$

involving two arbitrary functions $u(t), v(t)$. Then the given diffiety Ω is to be expressed as a factordiffiety of the diffiety Θ generated by two series of contact forms $\mathrm{d}u_i - u_{i+1}\,\mathrm{d}t$, $\mathrm{d}v_i - v_{i+1}\,\mathrm{d}t$ $(i = 0, 1, \ldots)$ on the underlying space I with the coordinates $t, u_0, v_0, u_1, v_1, \ldots$. A little more generally, some constants may be permitted in &, too, but we shall not deal with this problem. Instead, we shall consider the case when a quarature is present:

$$\& = (t, u(t), \ldots, \mathrm{d}^a u(t)/\mathrm{d}t^a, \textstyle\int h(t, u(t), \ldots, \mathrm{d}^a u(t)/\mathrm{d}t^a)\,\mathrm{d}t)\,.$$

Then the relevant diffiety Θ involves the form $\mathrm{d}v - h(t, u_0, \ldots, u_a)\,\mathrm{d}t$ and the underlying space involves the additional coordinate function v.

15. Determination of factordiffieties. Let Ω be a factordiffiety of a diffiety Θ, let $\iota: I \to J$ be the relevant surjection of the underlying spaces. Let (1) be a good filtration. Recalling the definition of morphisms of diffieties, one can verify the commutative diagram

$$\text{(23)}\quad \begin{array}{lllllll} \mathscr{L}_X\colon & \iota^*\Omega^0 & \to & \iota^*\Omega^1 & \to & \iota^*\Omega^2 & \to \ldots \quad (X \in \mathscr{H}(\Theta)) \\ & \uparrow & & \uparrow & & \uparrow & \\ \mathscr{L}_Y\colon & \Omega^0 & \to & \Omega^1 & \to & \Omega^2 & \to \ldots \quad (Y = \iota^* X \in \mathscr{H}(\Omega)) \end{array}$$

with ι-related vector fields X, Y. Since ι^* is injective, the inequality $\mu(\Omega) \leqq \mu(\Theta)$ follows by a simple argument employing the multiplication by X and Y in the modules $\oplus\, \Theta^l/\Theta^{l-1}$ and $\mathscr{G}^l = \oplus\, \Omega^l/\Omega^{l-1}$, respectively, and the relevant Hilbert polynomials. If (1) is involutive, then the equality $(2)_2$ is valid for every $l \geqq 0$ and the mapping $\iota^*\colon \Omega \to \Theta$ can be reconstructed already from the left vertical arrow $\iota^*\colon \Omega^0 \to \Theta$ of the diagram (23). If moreover (1) is a Cartan filtration, then the diffiety Ω can be reconstructed from the first term Ω^0 (cf. Theorem 11), hence (up to an isomorphism) from the module $\iota^*\Omega^0$ (since ι^* is injective). It follows that the knowledge of the submodule $\iota^*\Omega^0 \subset \Theta$ permits to determine all the other data.

In more detail, let Θ be a diffiety and let a regular submodule $\Xi \subset \Theta$ be given. Our aim is to identify Ξ with the above mentioned submodule of the kind $\iota^*\Omega^0$ for a certain factordiffiety Ω of Θ not yet known. The method proposed is quite clear: according to Theorem 11, there exists a diffiety Ω with $\Omega^0 = \Xi$ the first term of a Cartan filtration. Then, recalling the notation from the proof, we observe that the generators $(18)^0$ are expressed by the functions $x, y_1^1, \ldots, y_1^m$ lying in both spaces Φ_0 and Ψ_0 so that they must be identified by the sought morphism $\iota: I \to J$. As the remaining variables (22) are concerned, we may employ the diagram (23) which gives

$$(24) \qquad \iota^* x = x\,, \quad \iota^* y_1^j = y_1^j \quad (j = 1, \ldots, m)\,, \quad \iota^* y_{s+1}^k = \iota^* \mathscr{L}_Y y_s^k = \mathscr{L}_X y_s^k$$
$$(k = c+1, \ldots, m;\ s = 1, 2, \ldots;\ X \in \mathscr{H}(\Theta),\ Y \in \mathscr{H}(\Omega),\ Xx = Yx = 1)\,.$$

Clearly $\iota^* \Omega^0 = \Xi \subset \Theta$ and in general $\iota^* \Omega^l = (\mathscr{L}_X)^l\, \Xi \subset \Theta$, by induction. Consequently, $(4)_1$ is satisfied. The condition $(4)_2$ is satisfied if, e.g., ι^* Adj $\Omega^0 =$ Adj $\Xi \not\subset \Theta$, which will be assumed. So we have a morphism of diffieties.

However, we are interested in a surjective morphis , that is, in the case of the injective pull-back ι^* when the images in (3) are not proper submanifolds. This is ensured, at least locally, if the family of forms

$$\mathrm{d}x = \iota^* \mathrm{d}x\,, \quad \mathrm{d}y_1^j = \iota^* \mathrm{d}y_1^j\,, \quad \iota^* \mathrm{d}y_s^k$$
$$(j = 1, \ldots, m;\ k = c+1, \ldots, m;\ s = 2, 3, \ldots)$$

is linearly independent at every point or, equivalently, if the forms

$$(25) \qquad \mathrm{d}y_1^j - \iota^* y_2^j\, \mathrm{d}x\,, \quad \iota^*(\mathrm{d}y_s^k - y_{s+1}^k\, \mathrm{d}x)$$
$$(j = 1, \ldots, m;\ k = c+1, \ldots, m;\ s = 2, 3, \ldots)$$

lying in Θ are linearly independent. (The differential $\mathrm{d}x$ is omitted since we assume Adj $\Xi \not\subset \Theta$, hence we may also assume $\mathrm{d}x = \mathrm{d}f^1 \notin \Theta$.) But

$$\iota^*(\mathrm{d}y_s^k - y_{s+1}^k\, \mathrm{d}x) = (\mathscr{L}_X)^s\, (\mathrm{d}y_1^j - \iota^* y_2^j\, \mathrm{d}x)$$

according to (24), so that the forms (25) with s restricted to $2, 3, \ldots, l$ generate the module $\Xi + \mathscr{L}_X \Xi + \ldots + (\mathscr{L}_X)^l\, \Xi$. It follows that the last independence condition may be expressed without any use of coordinates by

$$(26) \qquad \ell(\Xi + \mathscr{L}_{\mathscr{H}} \Xi + \ldots + (\mathscr{L}_{\mathscr{H}})^l\, \Xi) = c + (m - c)\, l \quad (l = 0, 1, \ldots)\,.$$

(Besides, note that this seemingly complicated condition will be trivially satisfied in all examples below.) In view of $m - c = \mu(\Omega) \leqq \mu(\Theta)$, the above results may be expressed as follows:

16. Theorem. *Let $\Xi \subset \Theta$ be a regular submodule freely generated by certain c differential forms expressible by just $m + 1$ variable and satisfying* Adj $\Xi \not\subset \Theta$ *and* (26). *Let* (1) *be a Cartan filtration of a diffiety Ω with $\Omega^0 = \Xi$. Then Ω is a factordiffiety of Θ and $m - c = \mu(\Omega) \leqq \mu(\Theta)$.*

17. Solution of the Monge problem to be examined in the present paper is based on Theorem 16. We ask whether a given diffiety Ω may be represented as a factordiffiety of another given diffiety Θ. Then the proposed way is as follows: All submodules $\Xi \subset \Theta$ satisfying the conditions of Theorem 16 are to be determined (which seems to be the most difficult step). Then, a Cartan filtration (1) to the given diffiety Ω can be (as a rule) easily found and, since the relevant morphism $\iota: I \to J$ is determined already by the (bijective) portion $\iota^*: \Omega^0 \to \Xi$ of the pullback $\iota^*: \Psi_1 \to \Phi_1$, the concluding step consists in identifying the first term Ω^0 of the filtration (1) with one of the found submodules Ξ. This is already the common classical equivalence problem (which may lead, however, to lengthy discussions).

We continue this paper by applying the method proposed to particular problems already mentioned in Sections 13 and 14. We begin with the instructive and relatively easy primary problem of Section 13 for then the procedure can be developed without any auxiliary technical rearrangements. In addition to the results known already to Cartan, some new (and promising) details can be picked up. Then the other problems are analyzed following the same schema of reasoning. Nonetheless, some modifications (specific to every problem) clarifying some further generalizations and shortening the exposition (but unfortunately, obscuring a little the basic idea) prove to be necessary and are employed.

EXPLICIT SOLVABILITY WITH ONE FUNCTION

18. Preliminaries. We begin with the primary problem of Section 13. In the course of exposition, some useful abbreviations and conventiones will be introduced. We shall (tacitly) use only regular finite generated modules which means that some nowhere dense subsets must be left out from the spaces under consideration. Moreover, following the experience of old masters of analysis, we will omit some "degenerate" subcases in the concluding Sections 28 and 29 in order not do enlarge the exposition.

So we are going to study factordiffieties of the diffiety $\Theta = \{du_i - u_{i+1}\,dt;\ i = 0, 1, \ldots\}$, i.e., the diffiety generated by the above mentioned forms $\vartheta_i = du_i - u_{i+1}\,dt$ on the underlying space I with the coordinates $t, u_0, u_1, \ldots$. Clearly $\mu(\Theta) = 1$ and $\mathscr{H}(\Theta)$ is generated by $\partial = \partial/\partial t + \Sigma u_{i+1}\,\partial/\partial u_i$. We temporarily introduce the notation $\delta f = \Sigma\, \partial f/\partial u_i . \vartheta_i$ ($f \in \Phi_0$), hence $df = \partial f . dt + \delta f$ and

$$d\xi = dt \wedge \Sigma(f^{i-1} + \partial f^i)\,\vartheta_i + \Sigma \delta f^i \wedge \vartheta_i \quad (\xi = \Sigma f^i \vartheta_i)\,. \tag{27}$$

According to Theorem 16, we wish to determine submodules $\Xi \subset \Theta$ with $\ell(\Xi) = c$ and $\ell(\mathrm{Adj}\,\Xi) = m + 1$, where $m - c \leqq \mu(\Theta) = 1$, hence $m \leqq c + 1$ and $\ell(\mathrm{Adj}\,\Xi) \leqq c + 2$. The other requirements of Theorem 16 will be automatically satisfied and need not be taken into account.

19. The hierarchy of generators. Let $\xi_0 \in \Xi$ be a nonvanishing form with the lowest ϑ_i-order. In more explicit terms, we assume

$$(28)_0 \qquad \xi_0 = \sum_{i=0}^{n} f_0^i \vartheta_i \in \Xi \quad (f_0^n(p) \neq 0)$$

near a point $p \in J$ under consideration and suppose that the integer n appearing in $(28)_0$ is the lowest possible one. According to (27), we have

$$\xi_1 = \partial \neg \mathrm{d}\xi_0 = \Sigma(f_i^{i-1} + \partial f_0^i)\,\vartheta_i = \sum_{i=0}^{n+1} f_1^i \vartheta_i \in \mathrm{Adj}\,\Xi$$

and if even $\xi_1 \in \Xi$, we may continue with $\xi_2 = \partial \neg \mathrm{d}\xi_1 \in \mathrm{Adj}\,\Xi$, and so on. As a final result, there appears a certain chain of forms

$$(28)_j \qquad \xi_j = \partial \neg \mathrm{d}\xi_{j-1} = \Sigma(f_{j-1}^{i-1} + \partial f_{j-1}^i)\,\vartheta_i = \sum_{i=0}^{n+j} f_j^i \vartheta_i \in \Xi \quad (j = 1, \ldots, N-1)$$

with $\xi_N = \partial \neg \mathrm{d}\xi_{N-1} \in \mathrm{Adj}\,\Xi$, but $\xi_N \notin \Xi$. Obviously $f_0^n = \ldots = f_N^{n+N}$ and we can normalize $f_n^0 = \ldots = f_N^{n+N} = 1$, at least locally. Let us now look at the formula

$$(29) \qquad \mathrm{d}\xi_j = \mathrm{d}t \wedge \xi_{j+1} + \sum_{i=0}^{n+j} \delta f_j^i \wedge \vartheta_i \quad (j = 0, \ldots, N-1)$$

for the case $j = N - 1$. Since $\xi_N \notin \Xi$, there exists $X \in \Xi^\perp$ with $X \neg \xi_N = 1$. It follows that

$$\tau = X \neg \mathrm{d}\xi_{N-1} = \mathrm{d}t + (\text{linear combination of } \vartheta_0, \ldots, \vartheta_{n+N-1}) \in \mathrm{Adj}\,\Xi$$

but $\tau \notin \Xi$. So we have two forms $\xi_N, \tau \in \mathrm{Adj}\,\Xi$ not lying in Ξ, and another linearly independent form of this property cannot exist.

It follows easily from the last sentence that (31) are generators of the module Ξ (hence $N = c = \ell(\Xi)$) since another form independent of them cannot exist in Ξ. Indeed, the ϑ_i-order of such a form would be at least $n + N$ and by applying the above procedure, a third essentially new form lying in Adj Ξ but not in Ξ could be derived, which is impossible.

20. Structural formulae. We shall analyse the formula (29) using the new family of generators $\mathrm{d}t, \vartheta_0, \ldots, \vartheta_{n-1}, \xi_0, \ldots, \xi_N, \vartheta_{n+N+1}, \vartheta_{n+N+2}, \ldots$ of the module Φ_1. According to the common rules of exterior algebra, (29) can be rewritten as

$$(30) \qquad \mathrm{d}\xi_j = \tau_j \wedge \xi_{j+1} + \sum_{i=0}^{j} \alpha_j^i \wedge \xi_i + \sum_{i=0}^{n-1} \beta_j^i \wedge \vartheta_i \quad (j = 0, \ldots, N-1)\,,$$

where $\tau_j, \alpha_j^i, \beta_j^i$ are forms of certain special type:

$$(31)_{1,2,3} \qquad \tau_j \in \mathrm{d}t + \{\vartheta_{<n}\}\,, \quad \alpha_j^i \in \{\text{all except } \mathrm{d}t\}\,, \quad \beta_j^i \in \{\xi_{j+1<,<N}\}\,.$$

This (a little unusual but useful) notation means that τ_j is a sum of $\mathrm{d}t$ and a linear combination of $\vartheta_0, \ldots, \vartheta_{n-1}$, then α_j^i is a linear combination of all forms of the new family of generators except $\mathrm{d}t$, and β_j^i is a linear combination of $\xi_{j+2}, \ldots, \xi_{N-1}$. (More complicated situations will appear later.)

Existence of such a formula (30) can be established as follows: First, the second summand at the right hand side (29) is expressed by the new generators. Secondly, all terms of this summand which involve the factors $\xi_0, \ldots, \xi_j$ are separated, which determines α_j^i. Thirdly, terms with the factor ξ_{j+1} determine τ_j. At last, all the remaining terms of the second summand of (29) are already of the kind $\Sigma\beta_j^i \wedge \vartheta_i$ with $i = 0, \ldots, n-1$, but β_j^i can involve neither $\vartheta_0, \ldots, \vartheta_{n-1}, \xi_N, \xi_{n+N+1}, \xi_{n+N+2}, \ldots$ (look at Adj Ξ), nor $\xi_0, \ldots, \xi_{j+1}$ (due to the second and the third step above), so that $(31)_3$ is true.

21. Reduction procedure. According to $(31)_3$ we have $\beta_{N-1}^i \equiv 0$. Then, using the identities

$$
\begin{aligned}
(32) \qquad 0 = \mathrm{d}^2\xi_j = {} & \mathrm{d}\tau_j \wedge \xi_{j+1} + \\
& + \tau_j \wedge (\tau_{j+1} \wedge \xi_{j+2} + \Sigma\alpha_{j+1}^k \wedge \xi_k + \Sigma\beta_{j+1}^k \wedge \vartheta_k) + \\
& + \Sigma\, \mathrm{d}\alpha_k^i \wedge \xi_i + \Sigma\alpha_j^i \wedge (\tau_i \wedge \xi_{i+1} + \Sigma\alpha_i^k \wedge \xi_k + \Sigma\beta_i^k \wedge \vartheta_k) + \\
& + \Sigma\, \mathrm{d}\beta_j^i \wedge \vartheta_i + \Sigma\beta_j^i \wedge \mathrm{d}t \wedge \vartheta_{i+1}\,, \quad (j = 0, \ldots, N-2)
\end{aligned}
$$

and the congruences (31), we shall prove $\beta_j^i \equiv 0$ by a descending induction argument: Assume $\beta_{N-1}^i = \ldots = \beta_{j+1}^i = 0$ but $\beta_j^i \neq 0$. If $f\xi_J$ is a nontrivial summand of β_j^i with maximum J (necessarily $j + 1 < J < N$), then the term $\mathrm{d}\beta_j^i \wedge \vartheta_i$ appearing in the last line of (32) contains the summand $f\,\mathrm{d}t \wedge \xi_{J+1} \wedge \vartheta_i$ and, using (31) with the maximum property of J and the induction argument, one can check that it cannot be cancelled by any other summand of (32), which is a contradiction. So we conclude $\beta_j^i \equiv 0$, and looking at (32) again, one sees that the products $\tau_j \wedge \tau_{j+1}$ identically vanish. Hence $\tau_0 = \ldots = \tau_N = \tau \in$ Adj Ξ and

$$
(33)_j \qquad \mathrm{d}\xi_j = \tau \wedge \xi_{j+1} + \sum_{i=0}^{j} \alpha_j^i \wedge \xi_i \quad (j = 0, \ldots, N-1)
$$

is valid which simplifies considerably the primary structure equations.

22. The canonical formulae. If we consider the module $\{\xi_0\}$ with the single generator ξ_0, $(33)_0$ implies Adj $\{\xi_0\} = \{\tau, \xi_0, \xi_1\}$. It follows that there exists another generator $f\xi_0$ ($f \neq 0$) expressible by exactly three variables, $f\xi_0 = \mathrm{d}y_0 - y_1\,\mathrm{d}x$ (the Darboux theorem). Then, if the primary generator ξ_0 is replaced by the new one $f\xi_0$, the structural equations (33) are not disturbed provided $\xi_1, \ldots, \xi_{N-1}$ and α_j^i are slightly corrected. After the change, $(33)_0$ simplifies to $\mathrm{d}\xi_0 = \tau \wedge \xi_1 + \alpha_0^0 \wedge \xi_0 + \mathrm{d}x \wedge \mathrm{d}y_1$. Since $\mathrm{d}x \notin \Xi$, we have $\mathrm{d}y_1 - y_2\,\mathrm{d}x \in \Xi$ for an appropriate $y_2 \in \Phi_0$ (the $\mathcal{D}im$ axiom). Moreover, $\xi_1 \in$ Adj $\{\xi_0\} = \{\mathrm{d}x, \mathrm{d}y_0, \mathrm{d}y_1\} = \{\mathrm{d}x, \xi_0, \mathrm{d}y_1\}$ so that the next generator ξ_1 can be replaced by the form $\mathrm{d}y_1 - y_2\,\mathrm{d}x$. (The forms $\xi_2, \ldots, \xi_{N-1}$ and α_j^i must be slightly corrected.) After the change, $(33)_1$ turns into $\mathrm{d}\xi_1 = \tau \wedge \xi_2 + \alpha_1^0 \wedge \xi_0 + \alpha_1^1 \wedge \xi_1 = \mathrm{d}x \wedge \mathrm{d}y_2$, and continuing similarly as above, a certain form $\mathrm{d}y_2 - y_3\,\mathrm{d}x$ we may substitute for ξ_2, and so on. As the final result we obtain new generators

$$(34)_j \qquad \xi_j = \mathrm{d}y_j - y_{j+1}\,\mathrm{d}x \quad (j = 0, \dots, N-1)$$

of the module Ξ. In particular, we have

23. Theorem. *Every factordiffiety of the diffiety Θ from Section 18 is isomorphic to the diffiety Θ itself.*

24. The cross-section method by non-adjoint variables can be alternatively employed to derive the generators (34) without any use of the Darboux theorem. It will be also useful in other situations to appear later on. It is based on the formulae (28), (33) and runs as follows: Since the generators $\xi_0, \dots, \xi_{N-1}$ of Ξ can be expressed in terms of the adjoint variables to Ξ, we may consider (28), (33) under the family of constraints

$$u_0 = a_0, \dots, u_{n-1}, \; u_{N+1} = a_{N+1}, \; u_{N+2} = a_{N+2}, \; \dots$$

(where a_i are constants), which do not introduce any interrelations among the adjoint variables. Let us denote by tildes the results of constraining (whenever necessary for more clarity). Then $(28)_0$ turns into

$$\tilde{\xi}_0 = \mathrm{d}u_n - \left(u_{n+1} + f_0^{n-1}u_n + f_0^{n-2}a_{n-1} + \dots + f_0^0 a_1\right)\mathrm{d}t = \mathrm{d}y_0 - y_1\,\mathrm{d}x$$

with the notation $y_0 = u_n$, $y_1 = u_{n+1} + \dots + f_0^0 a_1$, $x = t$. This is exactly $(34)_0$. Clearly $\operatorname{Adj}\{\tilde{\xi}_0\} = \{\tilde{\tau}, \tilde{\xi}_0, \tilde{\xi}_1\} = \{\mathrm{d}t, \mathrm{d}u_n, \mathrm{d}u_{n+1}\}$, but at the same time $\operatorname{Adj}\{\tilde{\xi}_0\} = \{\mathrm{d}x, \mathrm{d}y_0, \mathrm{d}y_1\}$. It follows that u_{n+1} can be expressed by t, y_0, y_1. Then $(28)_1$ turns into the formula

$$\begin{aligned}\tilde{\xi}_1 &= \mathrm{d}u_{n+1} + f_1^n\,\mathrm{d}u_n - \left(u_{n+2} + f_1^{n+1}u_{n+1} + f_1^n u_n + f_1^{n-1}a_{n-1} + \dots + f_1^0 a_0\right)\mathrm{d}x = \\ &= f(\mathrm{d}y_1 - y_2\,\mathrm{d}t) + g(\mathrm{d}y_0 - y_1\,\mathrm{d}t)\end{aligned}$$

with certain functions f, g, y_2. Necessarily $f \neq 0$ so that the next form $(34)_1$ has appeared. Continuing in this way, one can obtain the whole family of generators (34).

25. The derived modules. All submodules $\Xi \subset \Theta$ satisfying the assumptions of Theorem 16 are determined and, following the instructions of Section 17, we have to find a criterion whether a given module Ω^0 can be identified with some of these modules Ξ. In our particular case, this equivalence problem can be easily resolved by a method due to Cartan, and the relevant background will be derived here. We shall use the notation from Section 9 for a moment since both topics are closely related.

So we are in the space J and a submodule $\Xi \subset \Psi_1$ is considered. Let us introduce the submodule $\operatorname{Der}\Xi \subset \Psi_1$ consisting of all forms ξ that satisfy any of the following equivalent conditions:

(i) $\xi \in \Xi$ and $X \urcorner \mathrm{d}\xi \in \Xi$ for all $X \in \Xi^\perp$,
(ii) $\mathscr{L}_X \xi \in \Xi$ for all $X \in \Xi^\perp$,
(iii) $Y \urcorner \mathscr{L}_X \xi \equiv 0$ for all $X, Y \in \Xi^\perp$,
(iv) $[X, Y] \urcorner \xi \equiv 0$ for all $X, Y \in \Xi^\perp$.

(Note that (ii) $\Rightarrow$ (i) follows from the rule $\mathscr{L}_{fX}\xi = f\mathscr{L}_X\xi + X \urcorner \xi . \mathrm{d}f$ under the regularity of Ξ, (iii) $\Leftrightarrow$ (iv) follows from $0 = \mathscr{L}_X(Y \urcorner \xi) = [X, Y] \urcorner \xi + Y \urcorner \mathscr{L}_X\xi$, and other implications are easy.) Clearly $\Xi \supset \operatorname{Der} \Xi \supset \operatorname{Der}^2 \Xi \supset \ldots$ and $\Xi' = \bigcap \operatorname{Der}^l \Xi$ is the greatest completely integrable submodule of Ξ. In particular, Ξ is completely integrable if $\Xi = \operatorname{Der} \Xi$, and in this case $\Xi = \operatorname{Adj} \Xi$. Using (ix) Section 9 and the above point (iv), one can also verify the inclusion $(\operatorname{Adj} \operatorname{Der} \Xi)^\perp \supset (\operatorname{Adj} \Xi)^\perp$, that is, $\operatorname{Adj} \operatorname{Der} \Xi \subset \operatorname{Adj} \Xi$.

We shall look at the "nearly completely integrable case" when

$$\ell(\Xi) = \ell(\operatorname{Der} \Xi) + 1 = \ldots = \ell(\operatorname{Der}^e \Xi) + e = \ell(\operatorname{Der}^l \Xi) + e \quad (l \geqq e)\,; \tag{35}$$

here $e = e(\Xi, p) \geqq 0$ is an integer independent of $p \in J$. We shall need only the particular case when $\Xi = \Omega^0$ is the initial term of a Cartan filtration of Ω with $\mu(\Omega) = 1$. Then $\ell(\Omega^1/\Omega^0) = \mu(\Omega) = 1$, hence

$$\ell(\operatorname{Adj} \Xi) = \ell(\operatorname{Adj} \Omega^0) = \ell(\Omega^1) + 1 = \ell(\Omega^0) + 2 = \ell(\Xi) + 2\,. \tag{36}$$

So we shall suppose $\ell(\operatorname{Adj} \Xi) = \ell(\Xi) + 2$ for the module Ξ.

Under this assumption, let us look at $\operatorname{Der} \Xi$. Either $e = 0$ and $\Xi = \operatorname{Der} \Xi$ is completely integrable, or $e = 1$. In the latter case clearly

$$\operatorname{Adj} \Xi^\perp \subset \Xi^\perp \subset \operatorname{Der} \Xi^\perp\,. \tag{37^0}$$

Let X, Y be vector fields generating together with $\operatorname{Adj} \Xi^\perp$ the module $\Xi^\perp$, that is, $\Xi^\perp = \{X, Y, \operatorname{Adj} \Xi^\perp\}$. Then $\operatorname{Der} \Xi^\perp = \{X, Y, [X, Y], \operatorname{Adj} \Xi^\perp\}$ and (as follows from $\ell(\operatorname{Der} \Xi) - \ell(\operatorname{Der}^2 \Xi) = 1$) there exists a vector field $Z = fX + gY$ such that $[Z, [X, Y]] \in \operatorname{Der} \Xi^\perp$. Consequently $Z \in \operatorname{Adj} \operatorname{Der} \Xi^\perp$, hence $\operatorname{Adj} \operatorname{Der} \Xi^\perp = \{Z, \operatorname{Adj} \Xi^\perp\}$ and we have the inclusions

$$\operatorname{Adj} \operatorname{Der} \Xi^\perp \subset \operatorname{Der} \Xi^\perp \subset \operatorname{Der}^2 \Xi^\perp \tag{37^1}$$

analogous to $(37)^0$. One can also easily verify $\ell(\operatorname{Adj} \operatorname{Der} \Xi) = \ell(\operatorname{Der} \Xi) + 2$.

So we may look at $\operatorname{Der}^2 \Xi$. Either $e = 1$ and $\operatorname{Der} \Xi = \operatorname{Der}^2 \Xi$ is completely integrable, or $e = 2$. In the latter case, the above procedure may be repeated with $(37)^1$ instead of $(37)^0$, and so on, up to the completely integrable module $\operatorname{Der}^e \Xi = \operatorname{Der}^{e+1} \Xi$. Let us also mention the relations

$$\ell(\operatorname{Adj} \operatorname{Der}^l \Xi) = \ell(\operatorname{Der}^l \Xi) + 2 \quad (l = 0, \ldots, e - 1)\,, \tag{38}$$

$$\operatorname{Adj} \operatorname{Der}^l \Xi \supset \operatorname{Der}^{l-1} \Xi \qquad (l = 1, \ldots, e)\,, \tag{39}$$

appearing in the course of the reasoning, cf. the role of the vector fields X, Y, Z above in the first step $l = 1$.

The relations (38) and (39) permit to determine some fair generators of the module Ξ.

Denoting $\ell(\Xi) = c$ as usual, (35) clearly implies $\ell(\mathrm{Der}^e\,\Xi) = c - e$. Since $\mathrm{Der}^e\,\Xi$ is completely integrable, the Frobenius theorem gives $\mathrm{Der}^e\,\Xi = \{\mathrm{d}v^1, \ldots, \mathrm{d}v^{c-e}\}$, at least locally, with appropriate functions $v^1, \ldots, v^{c-e} \in \Psi_0$. Since $\ell(\mathrm{Der}^l\,\Xi) = c - l$ $(l = 0, \ldots, e)$, we have

$$(40)_l \qquad \mathrm{Der}^l\,\Xi = \{\mathrm{d}v^1, \ldots, \mathrm{d}v^{c-e}, \eta_j, \ldots, \eta_{e-l-1}\} \quad (l = 0, \ldots, e)$$

with appropriate forms $\eta_0, \ldots, \eta_{e-1} \in \Psi_1$. These forms can be chosen to be of a certain very special type. We begin with the form η_0 (which is more advantageous in some examples, but an alternative way starting with η_{l-1} may be also useful, cf. Section 45). Relation (38) gives $\ell(\mathrm{Adj}\,\mathrm{Der}^{e-1}\,\Xi) = c - e + 3$ so that the form η_0 first appearing in $(40)_{e-1}$ is expressible by $v^1, \ldots, v^{c-e}$ and three additional variables. Applying the Darboux theorem, we may even assume $\eta_0 = \mathrm{d}y_0 - y_1\,\mathrm{d}x$ where x, y_0, y_1 are the additional variables, $\mathrm{d}x \notin \mathrm{Der}^{e-1}\,\Xi$. We continue with the form η_1 first appearing in $(40)_{e-2}$. Relation (38) gives $\ell(\mathrm{Adj}\,\mathrm{Der}^{e-2}\,\Xi) = c - e + 4$ so that η_1 depends on one additional variable $y_2 \in \Psi_0$. According to (39), we have $\eta_1 \in \mathrm{Adj}\,\mathrm{Der}^{e-1}\,\Xi$, hence η_1 may be assumed a linear combination of $\mathrm{d}x$ and $\mathrm{d}y_1$, hence $\eta_2 = \mathrm{d}y_1 - y_2\,\mathrm{d}x$ (the Darboux theorem is not needed!). A quite analogous reasoning gives the remaining generators $\eta_3, \ldots, \eta_{e-1}$ of the kind $\eta_j = \mathrm{d}y_j - y_{j+1}\,\mathrm{d}x$. Hence

$$(41) \qquad \Xi = \{\mathrm{d}v^1, \ldots, \mathrm{d}v^{c-e}, \mathrm{d}y_0 - y_1\,\mathrm{d}x, \ldots, \mathrm{d}y_{e-1} - y_e\,\mathrm{d}x\}\,.$$

Not only this final result, but also the method is of independent interest since it must be applied if we deal with applications, cf. Sections 28, 29 and the papers [3, 12].

26. The equivalence problem. Let us return to the Monge problem. We begin with assuming Ω to be a factordiffiety of Θ with $\iota\colon J \to I$ the relevant surjection. We know that the submodule $\Xi = \iota^*\Omega^0 \subset \Theta$ possesses the generators (34), that is,

$$(42) \qquad \Xi = \{\mathrm{d}y_0 - y_1\,\mathrm{d}x, \ldots, \mathrm{d}y_{N-1} - y_N\,\mathrm{d}x\}\,,$$

which is a particular case $c = e$ of (41). One can then explicitly verify the property (35) with $c = e = N = \ell(\Xi)$ and, by virtue of the obvious equalities

$$(43) \qquad \mathrm{Der}^l\,\Xi = \mathrm{Der}^l\,\iota^*\Omega^0 = \iota^*\,\mathrm{Der}^l\,\Omega^0$$

and the injectivity of ι^*, the same property

$$(44) \qquad \ell(\Omega^0) = \ell(\mathrm{Der}\,\Omega^0) + 1 = \ldots = \ell(\mathrm{Der}^c\,\Omega^0) + c \quad (c = \ell(\Omega^0))$$

holds for the module Ω^0.

Let conversely Ω be a diffiety admitting a Cartan filtration (1) with the first term Ω^0 satisfying (44), that is, (35) rewritten for Ω^0 instead of Ξ. It follows that there are generators (41) with lacking terms dv^k for the module Ω^0. According to Theorem 11, the diffiety Ω is even isomorphic to Θ (not a mere factordiffiety of Θ), which is in full agreement with Theorem 23. However, the main result of reasoning may be formulated as follows:

27. Theorem. *The initial term Ω^0 of a Cartan filtration* (1) *of a diffiety Ω fulfils* (44) *if and only if Ω is a factordiffiety of the diffiety Θ from Section* 18.

28. Example. In this and the next section we shall proceed in the spirit of the matematical analysis of the past century, and the notation and terminology will be slightly adapted to the geometrical contents. We commence with the Monge problem for the system of ordinary differential equations

$$\mathrm{d}x/\mathrm{d}t = x' , \quad \mathrm{d}y/\mathrm{d}t = y' , \quad \mathrm{d}z/\mathrm{d}t = z'$$

with two relations

$$f(t, x, y, z, x', y', z') = 0 , \quad g(t, x, y, z, x', y', z') = 0 \tag{45}$$

between the variables involved; the Jacobian of f, g with respect to x', y', z' is supposed to be of rank two. According to Section 6, diffieties can be called for help. So we introduce the diffiety Ω where the first term Ω^0 of the Cartan filtration is generated by the forms

$$\xi = \mathrm{d}x - x'\,\mathrm{d}t , \quad \eta = \mathrm{d}y - y'\,\mathrm{d}t , \quad \zeta = \mathrm{d}z - z'\,\mathrm{d}t$$

with relations (45) between the variables. Abbreviating the notation, the partial derivatives will be temporarily indicated by subscripts so that the differentials are related by

$$\mathrm{d}f = \partial' f\,\mathrm{d}t + f_x\xi + f_y\eta + f_z\zeta + f_{x'}\,\mathrm{d}x' + f_{y'}\,\mathrm{d}y' + f_{z'}\,\mathrm{d}z' = 0 ,$$
$$\mathrm{d}g = \partial' f\,\mathrm{d}t + g_x\xi + g_y\eta + g_z\zeta + g_{x'}\,\mathrm{d}x' + g_{y'}\,\mathrm{d}y' + g_{z'}\,\mathrm{d}z' = 0 ,$$

where $\partial' = \partial/\partial t + x'\,\partial/\partial x + y'\,\partial/\partial y + z'\,\partial/\partial z$ is the truncated operator ∂. Clearly $\ell(\Omega^0) = 3$, $\ell(\mathrm{Adj}\,\Omega^0) = 5$, and the forms

$$\chi = f_{x'}\xi + f_{y'}\eta + f_{z'}\zeta , \quad \pi = g_{x'}\xi + g_{y'}\eta + g_{z'}\zeta$$

generate $\mathrm{Der}\,\Omega^0$. Hence $\ell(\Omega^0) = \ell(\mathrm{Der}\,\Omega^0) + 1$ which is the first equality in (44).

It follows that the requirements of Theorem 27 are satisfied if $\ell(\mathrm{Der}^2\,\Omega^0) = 1$, that is, if the forms $\mathrm{d}\chi, \mathrm{d}\pi$ are proportional under the conditions $\mathrm{d}f = \mathrm{d}g = \chi = = \pi = 0$. (The degenerate case $\mathrm{d}\chi = \mathrm{d}\pi = 0$ fits well into the framework of Section 30 and is not analyzed here.) Since we are in a $7 - 4 = 3$-dimensional space, at most two conditions on the coefficients of the forms $\mathrm{d}\chi, \mathrm{d}\pi$ can arise. However, owing to

$$\mathrm{d}\chi = (\mathrm{d}f_{x'} - f_x\,\mathrm{d}t) \wedge \xi + (\mathrm{d}f_{y'} - f_y\,\mathrm{d}t) \wedge \eta + (\mathrm{d}f_{z'} - f_z\,\mathrm{d}t) \wedge \zeta\,,$$

an analogous expression for $\mathrm{d}\pi$ (easily following from $\mathrm{d}f = \mathrm{d}g = 0$) and a very particular type of the conditions $\chi = \pi = 0$ (namely, they interrelate only the forms ξ, η, ζ), the conditions reduce to exactly one.

Denoting $\varrho = u\,\mathrm{d}\chi + v\,\mathrm{d}\pi$ for a moment, where u/v is the undetermined proportionality factor, the condition is expressed by the requirement that the rank of the matrix

$$\begin{bmatrix} 0 & f_{x'} & f_{y'} & f_{z'} & 0 & 0 & 0 \\ 0 & g_{x'} & g_{y'} & g_{z'} & 0 & 0 & 0 \\ \partial' f & f_x & f_y & f_z & f_{x'} & f_{y'} & f_{z'} \\ \partial' g & g_x & g_y & g_z & g_{x'} & g_{y'} & g_{z'} \\ \partial' \urcorner \varrho & \partial/\partial x \urcorner \varrho & \partial/\partial y \urcorner \varrho & \partial/\partial z \urcorner \varrho & \partial/\partial x' \urcorner \varrho & \partial/\partial y' \urcorner \varrho & \partial/\partial z' \urcorner \varrho \end{bmatrix}$$

is five. (The vector fields $\partial', \partial/\partial x, \ldots, \partial/\partial z'$ used here are dual to the coframe $\mathrm{d}t, \xi, \eta, \zeta, \mathrm{d}x', \mathrm{d}y', \mathrm{d}z'$.) Under this condition, the form $u\chi + v\pi$ generates $\mathrm{Der}^2\,\Omega^0$ and the next module $\mathrm{Der}^3\,\Omega^0$ is trivial.

We shall not evaluate the rank condition explicitly but mention a few particular cases. At first, assume (45) of the following special type:

$$(f =)\ x' - h(t, x, y, z, z') = 0\,, \quad (g =)\ y' - k(t, x, z, z') = 0\,.$$

The above matrix becomes considerably simpler and after easy calculations, the rank condition is expressed by $h_{z'z'}\,\partial' \urcorner \mathrm{d}\pi = k_{z'z'}\,\partial' \urcorner \mathrm{d}\chi$ or, more explicitly, by

$$h_{z'z'}(k_x\xi + k_y\eta + (k_z - k_{z't})\,\zeta) = k_{z'z'}(h_x\xi + h_y\eta + (h_z - h_{z't})\,\zeta)\,.$$

But ξ, η, ζ are related by $\chi = \xi - h_{z'}\zeta = 0, \pi = \eta - k_{z'}\zeta = 0$, and the final result is

$$h_{z'z'}((k_x + k_y)\,h_{z'} + k_z + k_{z't}) = k_{z'z'}((h_x + h_y)\,k_{z'} + h_z + h_{z't})\,.$$

The particular case $k = x$ was mentioned in [12]. Then the relations (45) again reduce to $x = h(t, x, y, z, z')$, $y' = x$ and, after denoting $y'' = x'$, the original system turns into the second order equation $y'' = h(t, y', y, z, z')$ and the rank condition is $h_{z'z'} = 0$. It is not satisfied for the equation $y'' = (z')^{1/2}$ which is Hilbert's negative result mentioned above.

29. Continuation. We shall consider the same problem once more but in several dimensions and using some dual and (from our point of view) non-intrinsical concepts. The results are partly due to Goursat, cf. [2]. Altering the notation, we introduce the system

$$\mathrm{d}x^i/\mathrm{d}z = t^i \quad (i = 1, \ldots, n) \tag{46}$$

in the space of variables $z, x^1, \ldots, x^n, t^1, \ldots, t^n$ related by $n - 1$ conditions

$$f^j(z, x^1, \ldots, x^n, t^1, \ldots, t^n) = 0 \quad (j = 1, \ldots, n - 1)\,, \tag{47}$$

where the Jacobian $(\partial f^j/\partial t^i)$ is of rank $n - 1$. Introducing the relevant diffiety Ω with the initial module Ω^0 generated by the forms $\xi^i = \mathrm{d}x^i - t^i\,\mathrm{d}z$ on the subspace (47), we shall prove that *the property* (44) *is equivalent to the involutiveness* (in the classical sense) *of a certain system of* $n - 1$ *partial differential equations of the first order with one unknown function* $z(x^1, \ldots, x^n)$.

Let us first assume that (44) is fulfilled (with $c = n$). In virtue of (47), the variables $t^1, \ldots, t^n$ can be expressed in terms of the remaining functions $z, x^1, \ldots, x^n$ and one auxiliary variable t (say, $t = t^1$), $y^i = \bar{y}^i(z, x^1, \ldots, x^n, t)$. Then $\Omega^{0\perp}$ is generated by the vector fields

$$Z = \partial/\partial z + \Sigma \bar{y}^i\,\partial/\partial x^i\,, \quad T = \partial/\partial t$$

and by vector fields from $\mathrm{Adj}\,\Omega^{0\perp}$ (but the latter do not matter much). Hence, according to (44) and (iv) Section 25, $\mathrm{Der}^l\,\Omega^0$ is generated by the vector fields mentioned together with the Lie brackets

$$[T, Z] = \Sigma \bar{y}^i_t\,\partial/\partial x^i, \ldots, [T, \ldots, [T, Z] \ldots] = \Sigma \bar{y}^i_{t\ldots t}\,\partial/\partial x^i \quad (l \text{ terms } T \text{ and } t) \tag{48}$$

(we tacitly assume linear independence of these vectors, a more general case is covered by the theory of Sections 30–33). One can see that the form $\zeta = \mathrm{d}z - p^1\,\mathrm{d}x^1 - \ldots \ldots - p^n\,\mathrm{d}x^n$ (which is also a linear combination of $\xi_1, \ldots, \xi_n$) generating the module $\mathrm{Der}^{n-1}\,\Omega^0$ is determined by the conditions

$$Z \urcorner \zeta = [T, Z] \urcorner \zeta = \ldots = [T, \ldots, [T, Z] \ldots] \urcorner \zeta = 0 \tag{49}$$
$$(\text{up to } n - 1 \text{ terms } T)\,.$$

According to (48), these conditions may be interpreted as osculating requirements for the hyperplane $\zeta = 0$ and the cone consisting of all the above vectors Z at a fixed point $z, x^1, \ldots, x^n$ with varying parameter t. It follows that the functions $p^1, \ldots, p^n$ can be expressed in terms of $z, x^1, \ldots, x^n$ and the auxiliary variable t so that there exist exactly $n - 1$ relations of the type

$$g^j(z, x^1, \ldots, x^n, p^1, \ldots, p^n) = 0 \quad (j = 1, \ldots, n - 1)\,, \tag{50}$$

if t is eliminated. The equation $\zeta = \mathrm{d}z - \Sigma p^i\,\mathrm{d}x^i = 0$ together with (50) may be interpreted as a system of partial differential equations for one unknown function z, the so called *associated system* to (47). Now, an appropriate multiple of ζ is equal to the form η_0 of Section 25:

$$w(\mathrm{d}z - \Sigma p^i\,\mathrm{d}x^i) = \mathrm{d}y_0 - y_1\,\mathrm{d}x\,. \tag{51}$$

Here y_0 is a function of $z, x^1, \ldots, x^n, t$, but it may be also expressed by $z, x^1, \ldots, x^n, x$. Denoting $y_0 = \bar{y}(z, x^1, \ldots, x^n, x)$ for more clarity, the relation (51) expresses the fact that the functions $\bar{z}(x^1, \ldots, x^n, a, b)$ implicitly defined by $\bar{y}(z, x^1, \ldots, x^n, a) = b$ ($a, b \in \mathbb{R}$ are constants) satisfy the adjoint system. In other words, we have a complete integral, hence *the associated system is involutive.*

Before passing to the proof of the converse result, let us look at the particular case $n = 3$ of the preceding section. The associated system (we use the form $\zeta = \mathrm{d}t - p\,\mathrm{d}x - q\,\mathrm{d}y - r\,\mathrm{d}z$ here)

$$F(t, x, y, z, p, q, r) = 0\,, \quad G(t, x, y, z, p, q, r) = 0$$

to (45) can be derived from the osculating conditions (49) or, more easily, by substituting $x' = \bar{x}(\lambda)$, $y' = \bar{y}(\lambda)$, $z' = \bar{z}(\lambda)$ into (45) and then by eliminating the derivatives $\mathrm{d}\bar{x}/\mathrm{d}\lambda, \ldots, \mathrm{d}^2\bar{z}/\mathrm{d}\lambda^2$ from the equations

$$f = g = \mathrm{d}f/\mathrm{d}\lambda = \mathrm{d}g/\mathrm{d}\lambda = \mathrm{d}^2 f/\mathrm{d}\lambda^2 = \mathrm{d}^2 g/\mathrm{d}\lambda^2 = p\bar{x} + q\bar{y} + r\bar{z} - 1 =$$
$$= \mathrm{d}(p\bar{x} + q\bar{y} + r\bar{z})/\mathrm{d}\lambda = \mathrm{d}^2(p\bar{x} + q\bar{y} + r\bar{z})/\mathrm{d}\lambda^2 = 0\,.$$

According to our results, the well-known involutiveness condition

$$(G_x + pG_t)\,F_p + (G_y + qG_t)\,F_q + (G_z + rG_t)\,F_r =$$
$$= (F_x + pF_t)\,G_p + (F_y + qF_t)\,G_q + (F_z + rF_t)\,G_r$$

is equivalent to the rank condition mentioned (but not explicitly stated) in Section 28.

We turn to the converse assertion. Let (47) be given with the involutive associated system (50). Our aim is to verify (44). Because of the involutiveness, there exists a complete integral $\bar{y}(z, x^1, \ldots, x^n, x) = y$ depending on two parameters x, y. This means that (51) is satisfied with $y_0 = \bar{y}$, $w = \bar{y}_z$, $y_1 = \bar{y}_x$ so that the first differential form $\eta_0 = \mathrm{d}y_0 - y_1\,\mathrm{d}x$ of Section 25 results. In order to find the following forms $\eta_1, \ldots, \eta_{n-1}$ we must look at the osculating conditions from the dual (and hence equivalent) point of view. At every fixed point $z, x^1, \ldots, x^n$, the hyperplane $\zeta = \mathrm{d}z - \Sigma p^i\,\mathrm{d}x^i = 0$ of the cotangent space is enveloping to the cone

$$\text{(52)} \qquad \mathrm{d}z - \Sigma\bar{p}^j\,\mathrm{d}x^j = 0\,, \quad -\Sigma \mathrm{d}^l\bar{p}^i/\mathrm{d}t^l\,\mathrm{d}x^i = 0 \quad (l = 1, \ldots, n-1)$$

where the functions $\bar{p}^j$ are implicitly defined by (50). The relations (52) are in fact equivalent to the original system $\xi^1 = \ldots = \xi^n = 0$ with variables related by (47), of course. Keeping this in mind, we can rewrite (52) in new coordinates $y_0, \ldots, y_{n-1}, x$ introduced (a little artificially) by the equations

$$\text{(53)} \qquad y_0 = \bar{y}(z, x^1, \ldots, x^n, x)\,,$$
$$y_1 = y_x(z, x^1, \ldots, x^n, x), \ldots, y_n = \bar{y}_{x\ldots x}(z, x^1, \ldots, x^n, x)$$

and any one of the (equivalent) relations

$$\bar{y}_{x^i}(z, x^1, \ldots, x^n, x)/\bar{y}_z(z, x^1, \ldots, x^n, x) = -\bar{p}^i(z, x^1, \ldots, t)\,.$$

Then, owing to (51), the first equation (52) multiplied by u_z may be rewritten as

$$\bar{y}_z\,\mathrm{d}x + \Sigma\bar{y}_{x^i}\,\mathrm{d}x^i + (\bar{y}_x\,\mathrm{d}x - \bar{y}_x\,\mathrm{d}x) = \mathrm{d}\bar{y} - \bar{y}_x\,\mathrm{d}x = 0$$

which is equivalent to $\mathrm{d}y_0 - y_1\,\mathrm{d}x = 0$. (This is already known.) But (52) with

$l = 1$ may be replaced by the equivalent relation $\mathrm{d}(\bar{y}_z(\mathrm{d}z - \Sigma\bar{p}^j\,\mathrm{d}z^j))/\mathrm{d}x = 0$, i.e., by

$$\bar{y}_{zx}\,\mathrm{d}z + \Sigma\bar{y}_{x^i x}\,\mathrm{d}x^i + (\bar{y}_{xx}\,\mathrm{d}x - \bar{y}_{xx}\,\mathrm{d}x) = \mathrm{d}\bar{y}_x - \bar{y}_{xx}\,\mathrm{d}x = 0\,,$$

which is equivalent to $\mathrm{d}y_1 - y_2\,\mathrm{d}x = 0$. Continuing in this way, one can obtain the generators

$$\Omega^0 = \{\mathrm{d}y_0 - y_1\,\mathrm{d}x, \ldots, \mathrm{d}y_{n-1} - y_n\,\mathrm{d}x\}\,,$$

that is, (44) is satisfied (with the constant $c = n$) and this concludes the proof.

Since the original system (46) written in the coordinates $y_0, \ldots, y_n, x$ is of the simple form $\mathrm{d}y_i/\mathrm{d}x \equiv y_{i+1}$ $(i = 0, \ldots, n-1)$, the general solution is expressed by the formulae $y_0 = u(t)$, $y_1 = \mathrm{d}u(t)/\mathrm{d}t, \ldots, y_n = \mathrm{d}^n u(t)/\mathrm{d}t^n$ and the transformation (53) gives the solution in terms of the original variables

$$z = \bar{z}(\&)\,, \quad x^i = \bar{x}^i(\&) \quad (i = 1, \ldots, n;\ \& = (t, u(t), \ldots, \mathrm{d}^n u(t)/\mathrm{d}t^n))$$

by inversion. The particular case $n = 2$ of the system (47) consisting of one equation $f^1(z, x^1, x^2, \mathrm{d}x^1/\mathrm{d}z, \mathrm{d}x^2/\mathrm{d}z) = 0$ proves to be relatively easy since then the involutiveness condition is trivial. In this case, the formulae (53) give the solution

$$u(t) = y(z, x^1, x^2, x),\ \mathrm{d}u(t)/\mathrm{d}t = y_x(z, x^1, x^2, x)\,,\quad \mathrm{d}^2u(t)/\mathrm{d}t^2 = y_{xx}(z, x^1, x^2, x)\,,$$

discovered already by Monge.

EXPLICIT SOLVABILITY WITH CONSTANTS

30. Preliminaries. We shall deal with factordiffieties of the diffiety $\Theta = \{\mathrm{d}v^1, \ldots, \mathrm{d}v^b, \mathrm{d}u_0 - u_1\,\mathrm{d}t, \mathrm{d}u_1 - u_2\,\mathrm{d}t, \ldots\}$ on the underlyings space I with the coordinates $t, v^1, \ldots, v^b, u_0, u_1, \ldots$. Clearly $\mu(\Theta) = 1$ and $\mathscr{H}(\Theta)$ is generated by (formally) the same vector field $\partial = \partial/\partial t + \Sigma u_{i+1}\,\partial/\partial u_i$ as in Section 18. Nonetheless, (27) must be replaced by the more complicated relation

$$\begin{aligned}(54)\qquad \mathrm{d}\xi = {} & \mathrm{d}t \wedge (\Sigma(f^{i-1} + \partial f^i)\,\vartheta_i + \Sigma\,\partial g^k\,\mathrm{d}v^k) + \\ & + \Sigma(\delta f^i + \Sigma\,\partial f^i/\partial v^k\,.\,\mathrm{d}v^k) + \Sigma(\delta g^k + \Sigma\,\partial g^k/\partial v^l\,.\,\mathrm{d}v^l) \wedge \mathrm{d}v^k\,,\end{aligned}$$

where $\xi = \Sigma f^i \vartheta_i + \Sigma g^k\,\mathrm{d}v^k$. We wish to determine submodules $\Xi \subset \Theta$ with $\ell(\Xi) = c$, $\ell(\mathrm{Adj}\,\Xi) \leqq c + 2$.

31. The cross-section method. As far as the generators of Ξ are concerned, there may exist some forms in Ξ expressible only by the differentils $\mathrm{d}v^1, \ldots, \mathrm{d}v^b$. Let these forms be linear combinations of

$$(55)\qquad \pi^k = \mathrm{d}v^k + \sum_{l=a+1}^{b} h_l^k\,\mathrm{d}v^l \quad (k = 1, \ldots, a)\,.$$

Moreover, there are generators of Ξ involving some contact forms, especially one with the lowest ϑ_i-order:

$$(56)\qquad \xi_0 = \sum_{i=0}^{n} f_0^i \vartheta_i + \sum_{l=a+1}^{b} g_0^l \,\mathrm{d}v^l \in \vartheta_n + \{\vartheta_{<n}, \mathrm{d}v^{\mathrm{all}}\} \quad (f_0^n = 1)\,.$$

Proceeding with (56) quite analogously as in Section 19 with $(28)_0$, one can get some forms

$$\xi_j = \partial \urcorner \mathrm{d}\xi_{j-1} = \vartheta_{n+j} + \{\vartheta_{<n+j}, \mathrm{d}v^{\mathrm{all}}\} \in \Xi \quad (j = 1, \ldots, N-1)$$

but with $\xi_N = \partial \urcorner \mathrm{d}\xi_{N-1} = \vartheta_{n+j+1} + \{\ldots\} \in \mathrm{Adj}\,\Xi$ not lying in Ξ. Moreover, quite analogously as in Section 19, $\tau = X \urcorner \mathrm{d}\xi_N = \mathrm{d}t + (\ldots) \in \mathrm{Adj}\,\Xi$ is a second form not lying in Ξ. Hence

$$\Xi = \{\pi^{\mathrm{all}}, \xi_{<N}\}\,, \quad \mathrm{Adj}\,\Xi = \{\pi^{\mathrm{all}}, \xi_{\leq N}, \tau\}\,, \quad a + N = c\,.$$

Consequently, we may introduce the constraints

$$v^{a+1} = c^1, \ldots, v^b = c^{b-a}$$

with fixed constants $c^1, \ldots, c^{b-a}$ without disturbing essentially the module Ξ. But the above data become considerably simpler since the differentials $\mathrm{d}v^{a+1}, \ldots, \mathrm{d}v^b$ completely disappear. In particular,

$$\pi^k \equiv \mathrm{d}v^k\,, \quad \xi_j \equiv \sum_{i=0}^{n+j} f_j^i \vartheta_i \quad (k = 1, \ldots, a;\; j = 0, \ldots, N;\; f_j^{nj} \equiv 1)\,.$$

The coefficients f_j^i may depend on the variables $v^1, \ldots, v^a$ but this fact does not cause any troubles and the reasoning of Sections 19–24 can be carried over to our problem word by word. As a final result, some forms of the type (34) may replace the original generators ξ_j so that the final result is (41) with the constants $c - e = a$, $e = N$.

32. The equivalence problem. Given a diffiety Ω with a Cartan filtration (1), we ask whether there exists a surjective morphism $\iota\colon J \to I$ with $\iota^*\Omega^0 = \Xi$, Ξ being specified by (44). The solution is analogous as in Section 26. First, let such an ι exist. Then, the property (38) follows from (41) for the module Ξ by easy calculation. And, according to (43), the same property (35) is true for the module Ω^0. Then the reasoning of Section 25 applied to Ω^0 (instead of Ξ) yields certain generators of the type (41) for the module Ω^0 so that Ω is a factordiffiety of Θ, in fact isomorphic to Θ.

33. Theorem. *The property (35) rewritten for Ω^0 instead of Ξ is typical for the initial term Ω^0 of a Cartan filtration (1) of a diffiety Ω in order that Ω be a factor diffiety of a diffiety Θ from Section 30. Then Ω is even isomorphic to Θ with possibly another constant b, namely with $b = a$.*

34. Preliminaries. We wish to determine the factordiffieties of the diffiety $\Theta = \{du_i - u_{i+1}\,dt,\ dv_i - v_{i+1}\,dt;\ i = 0, 1, \ldots\}$ on the underlying space I with the coordinates $t, u_0, v_0, u_1, v_1, \ldots$. Clearly $\mu(\Theta) = 2$ and $\mathscr{H}(\Theta)$ is generated by the vector field $\partial = \partial/\partial x + \Sigma(u_{i+1}\,\partial/\partial u_i + v_{i+1}\,\partial/\partial v_i)$. We shall abbreviate $\vartheta_i = du_i - u_{i+1}\,dt$, $\eta_i = dv_i - v_{i+1}\,dt$, $\delta f = \Sigma(\partial f/\partial u_i\,.\,\vartheta_i + \partial f/\partial v_i\,.\,\eta_i)$, hence $df = \partial f\,dt + \delta f\ (f \in \Phi_0)$ and

$$d\xi = dt \wedge \Sigma((f^{i-1} + \partial f^i)\,\vartheta_i + (g^{i-1} + \partial g^i)\,\eta_i + \Sigma(\delta f^i \wedge \vartheta_i + \delta g^i \wedge \eta_i)\,, \tag{57}$$

where $\xi = \Sigma(f^i\vartheta_i + g^i\eta_i)$. According to Theorem 16, we are interested in submodules $\Xi \subset \Theta$ with $\ell(\Xi) = c$, $\ell(\mathrm{Adj}\,\Xi) \leqq c + \mu(\Theta) = c + 3$.

35. The hierarchy of generators. An arbitrary form $\xi \in \Xi$ may be written as a sum

$$\xi = \sum_{i=0}^{n} f^i\vartheta_i + \sum_{i=0}^{m} g^i\eta_i\,. \tag{58}$$

If the first (second) summand on the right hand side is not really present, we formally put $n = -1$ $(m = -1)$. In the other cases, we tacitly suppose $f^n \neq 0$, $g^m \neq 0$ and speak of n (m) as of the ϑ_i-order (η_i-order) of ξ. Now, if (58) is a nonvanishing form in Ξ with the lowest possible m, then either $m \geqq 0$ or $m = -1$.

Let us begin with the first case $m \geqq 0$. Then the relevant minimal form

$$\xi_0 = \sum_{i=0}^{n} f_0^i\vartheta_i + \sum_{i=0}^{m} g_0^i\eta_i \quad (m \text{ minimal, } m \geqq 0) \tag{59$_0$}$$

is determined up to a nonvanishing factor. Analoguously as in Section 19, there appears a chain of forms

$$\xi_j = \partial \urcorner d\xi_{j-1} = \sum_{i=0}^{n+j} f_j^i\vartheta_i + \sum_{i=0}^{m+j} g_j^i\eta_i \in \Xi \quad (j = 1, \ldots, M - 1) \tag{59$_j$}$$

with $\xi_M = \partial \urcorner d\xi_{M-1} \in \mathrm{Adj}\,\Xi$ but $\xi_M \notin \Xi$. These forms can be explicitly calculated by applying (57) and one can see that the forms $(59)_j$ are also unique in a certain weakened sense: all forms in Ξ of the η_i-order at most $m + j$ $(j = 0, \ldots, M - 1)$ are linear combinations of $\xi_0, \ldots, \xi_j$.

It may well happen that already the forms (59) generate Ξ. However, then we are just within the domain of Sections 18–29 of explicit solvability with one function.

Omitting this easy subcase, there surely exists a nonvanishing form

$$\bar{\xi}_0 = \sum_{i=0}^{\bar{n}} \bar{f}_0^i\vartheta_i + \sum_{i=0}^{\bar{m}} \bar{g}_0^i\eta_i \quad (\bar{m} \geqq m + M) \tag{60$_0$}$$

with the minimal possible $\bar{m}$. It is determined up to a factor and a linear combination of the preceding forms (59). Then, due to the existence of the form $\xi_M \notin \Xi$, the following steps are a little more complicated than before. According to (57), we have the form

$$\xi_1' = \partial \urcorner \mathrm{d}\bar{\xi}_0 = \Sigma(\bar{f}_0^{i-1} + \partial \bar{f}_0^i)\,\vartheta_i + \Sigma(\bar{g}_0^{i-1} + \partial \bar{g}_0^i)\,\eta_i \in \operatorname{Adj} \Xi\,.$$

If there exists a function $f_0 \in \Phi_0$ such that $\bar{\xi}_1 = \xi_1' - f_0\xi_M \in \Xi$, we may continue with $\xi_2' = \partial \urcorner \mathrm{d}\bar{\xi}_1 \in \operatorname{Adj} \Xi$, and so on. As a final result, there appears a certain chain of forms of the type

$$(60)_j \qquad \bar{\xi}_j = \partial \urcorner \mathrm{d}\bar{\xi}_{j-1} - f_j\xi_M = \sum_{i=0}^{\bar{n}+j} \bar{f}_j^i\vartheta_i + \sum_{i=0}^{\bar{m}+j} \bar{g}_j^i\eta_i \quad (j = 1, \ldots, N-1)$$

with $\bar{\xi}_N = \partial \urcorner \mathrm{d}\bar{\xi}_{N-1} \in \operatorname{Adj} \Xi$, where $\bar{\xi}_N - f\xi_M \notin \Xi$ for any choice of the function $f \in \Phi_0$. So we have two different forms in the module $\operatorname{Adj} \Xi$ not lying in Ξ, and the existence of a third form of the type $\tau \in \mathrm{d}t + \{\vartheta_{\mathrm{all}}, \eta_{\mathrm{all}}\}$ can be proved quite analogously as in Section 19. It follows that the forms (59), (60) generate the module Ξ.

Let us mention the second case $m = -1$. Let $(28)_0$ be a form in the module Ξ with the lowest possible n. Then there appear forms $(28)_j$ with $\xi_N = \partial \urcorner \mathrm{d}\xi_{N-1} \notin \Xi$. The subcase when these forms (28) generate Ξ may be omitted so that there exist other forms in Ξ. Let $(60)_0$ be such a form with $\bar{m}$ minimal (note that the inequality $\bar{m} \geqq m + M$ is ignored here). Then either $\bar{m} \geqq 0$ or $\bar{m} = -1$. In the first subcase we continue with the forms $(60)_j$. In the latter subcase, we take a form $(60)_0$ with the lowest possible $\bar{n}$, $\bar{n} \geqq n + N$ (the second sum in $(60)_0$ is not present here!) and continue with the forms $\partial \urcorner \mathrm{d}\bar{\xi}_0, \partial \urcorner \mathrm{d}\bar{\xi}_1, \ldots$ exactly as before. It follows that (28), (60) (without the second sums) are generators of the module Ξ. These generators are unique in a certain weakened sense which need not be specified here.

36. Technical rearrangements are useful in order to include all the above subcases into a unified and lucid schema (permitting further generalizations to the case of explicit solvability with several arbitrary functions). We should like to prove that the generators of the module Ξ can be assumed to be of the type

$$(61) \quad \xi_j = \sum_{i=0}^{n+j} (f_j^i\vartheta_i + g_j^i\eta_i) \quad (j = 0, \ldots, N-1;\ \xi_{j+1} = \partial \urcorner \mathrm{d}\xi_j)\,,$$

$$(62) \quad \bar{\xi}_k = \sum_{i=0}^{m+k} (\bar{f}_k^i\vartheta_i + \bar{g}_k^i\eta_i) \quad (k = 0, \ldots, M-1;\ \bar{\xi}_{k+1} = \partial \urcorner \mathrm{d}\bar{\xi}_k + f_k\xi_N;\ m \geqq n + N)$$

with a totally ordered family of (equal ϑ_i- and η_i-) orders which fill up certain non-overlapping intervals $n \leqq j \leqq n + N - 1$, $m \leqq k \leqq m + M - 1$.

Such a state can be achieved in both the above cases .The proof runs as follows: In the first case $m \geqq 0$ with the primary generators (59) and (60), we make a shift of all lower indices

$$\eta_i \mapsto \eta_{i+\mathrm{const.}} \quad \text{(const. is large enough)}$$

in the expressions for the generators thus ensuring the η_i-orders exceed the corresponding ϑ_i-orders, that is, ensuring $m \geqq n$ and $\bar{m} \geqq \bar{n}$ after the shift just made. (One can realize that the sought factordiffiety of Θ is not substantially changed. Moreover, the η_i-orders of the generators are again totally ordered and fill up two non-overlapping intervals exactly as before the shift.) Then, after a sufficiently general linear substitution with constant coefficients of the type

$$u_i \mapsto Au_i + Bv_i\,, \quad v_i \mapsto Cu_i + Dv_i$$

and the same substitution on the forms ϑ_i, η_i, the ϑ_i-order can be made equal to the η_i-order for every generator of Ξ. That is, we have formulae of the type (61), (62). The second case $m = -1$ with the primary generators (28) and (60) is quite analogous (and even easier since some summands with the forms η_i are missing) so that the above reasoning need not be repeated.

We conclude this section with the formulae

$$(63) \qquad \mathrm{d}\xi_j = \mathrm{d}t \wedge \xi_{j+1} + \sum_{i=0}^{n+j} (\delta f_j^i \wedge \vartheta_i + \delta g_j^i \wedge \eta_i) \quad (j = 0, \ldots, N-1)\,,$$

$$(64) \qquad \mathrm{d}\bar{\xi}_k = \mathrm{d}t \wedge (\bar{\xi}_{k+1} + f_k \xi_N) + \sum_{j=0}^{m+k} (\delta \bar{f}_k^i \wedge \vartheta_i + \delta g_k^i \wedge \eta_i) \quad (k = 0, \ldots, M-1)$$

playing the role of (29). They easily follow from (57), (61), (62).

37. Structural formulae. Instead of the original generators $\mathrm{d}t, \vartheta_0, \eta_0, \vartheta_1, \eta_1, \ldots$ of the module Φ_1, we shall use the family of forms which is obtained if every form ϑ_j $(j = n, \ldots, n+N)$ is replaced by ξ_j, and every form η_k $(k = m, \ldots, m+M)$ is replaced by $\bar{\xi}_k$. Then the formulae (63), (64) can be rewritten as follows:

$$\mathrm{d}\xi_j = \tau_j \wedge \xi_{j+1} + \sum_{i=0}^{j} \alpha_j^i \wedge \xi_i + \sum_{i=0}^{n-1} \beta_j^i \wedge \vartheta_i + \sum_{i=0}^{n+j} \gamma_j^i \wedge \eta_i \quad (j = 0, \ldots, N-1)\,,$$

$$\mathrm{d}\bar{\xi}_k = \bar{\tau}_k \wedge (\bar{\xi}_{k+1} + f_k \xi_N) + \sum_{i=0}^{N-1} \bar{\alpha}_k^i \wedge \xi_i + \bar{\alpha}_k \wedge \xi_N + \sum_{i=0}^{k} \bar{\bar{\alpha}}_k^i \wedge \bar{\xi}_i +$$
$$+ \Big(\sum_{i=0}^{n-1} + \sum_{i=n+N+1}^{m+k}\Big) \bar{\beta}_k^i \wedge \vartheta_i + \sum_{i=0}^{m-1} \bar{\gamma}_k^i \wedge \eta_i \quad (k = 0, \ldots, M-1)\,,$$

where the forms on the right hand sides are of the special type

$\tau_j \in \mathrm{d}t + \{\vartheta_{<n}, \eta_{\leqq n+j}\}$; $\alpha_j^i \in \{\text{all except } \mathrm{d}t\}$; $\beta_j^i, \gamma_j^i \in \{\xi_{j+1<,<N}, \bar{\xi}_{<M}\}$;
$\bar{\tau}_k \in \mathrm{d}t + \{\vartheta_{<n}, \vartheta_{N+1<,<m+k}, \eta_{<m}, \xi_N\}$; $\bar{\alpha}_k^i, \bar{\bar{\alpha}}_k^i \in \{\text{all except } \mathrm{d}t\}$;
$\bar{\alpha}_k \in \{\bar{\xi}_{k+1<}\}$; $\bar{\beta}_k^i, \bar{\gamma}_k^i \in \{\bar{\xi}_{k+1<,<M}\}$.

This can be derived quite analogously as in Section 20.

38. Reduction procedure of the structural equations is a little tedious but follows the same lines as in Section 21 and need not be repeated here in much detail. As fas as the first group of differentials $d\xi_j$ is concerned, clearly $\beta^i_{N-1}, \gamma^i_{N-1} \in \{\xi_{<N}\}$, and using the identity $d^2\xi_{N-1} = 0$ one can prove $\beta^i_{N-1} = \gamma^i_{N-1} \equiv 0$ by an induction argument on the lower index of the presumed summand $\bar{\xi}_J \in \{\bar{\xi}_{<M}\}$. Then $\beta^i_j = \gamma^i_j \equiv 0$ follows from the identity $d^2\xi_j = 0$ by a simple induction on j. The second group of differentials $d\bar{\xi}_k$ is seemingly more complicated, but the vanishing of $\bar{\alpha}_k = \bar{\beta}^i_k = \bar{\gamma}^i_k \equiv 0$ can be verified even more easily than before by induction on the lower index k.

At last, one can see that $\tau_j \wedge \tau_{j+1} = 0\,(j = 0, \ldots, N-1)$, hence $\tau_0 = \ldots = \tau_N = \tau \in \operatorname{Adj} \Xi$ which corresponds to the relevant results of Section 21. However, only the relations of the type $\bar{\tau}_k \wedge \bar{\tau}_{k+1} = \xi_N \wedge \beta\,(k = 0, \ldots, M-1)$ with some form β can be read from the identity $d^2\bar{\xi}_k = 0$ so that we conclude $\bar{\tau}_k = \tau + g_k\xi_N$ $(k = 0, \ldots, M-1)$ with appropriate functions $g_k \in \Phi_0$. The final result

$$d\xi_j = \tau \wedge \xi_{j+1} + \sum_{i=0}^{j} \alpha^i_j \wedge \xi_i \quad (i = 0, \ldots, N-1)\,, \tag{65}$$

$$d\bar{\xi}_k = (\tau + g_k\xi_k) \wedge (\bar{\xi}_{k+1} + f_k\xi_N) + \sum_{i=0}^{N-1} \bar{\alpha}^i_k \wedge \xi_i + \sum_{i=0}^{k} \bar{\beta}^i_k \wedge \bar{\xi}_i \tag{66}$$
$$(k = 0, \ldots, M-1)$$

is a little more complicated than (33).

39. The canonical formulae. Since (65) is identical with (33), we may suppose the validity of formulae (34) for the generators $\xi_0, \ldots, \xi_{N-1}$. And, keeping these generators, we shall successively modify $\bar{\xi}_0, \ldots, \bar{\xi}_{M-1}$ in order to obtain some simple expressions for them. In doing so, $\bar{\xi}_k$ may be replaced by any form of the type

$$f\bar{\xi}_k + h_{k-1}\bar{\xi}_{k-1} + \ldots + h_0\bar{\xi}_0 + h'_{N-1}\xi_{N-1} + \ldots + h'_0\xi_0 \quad (f \neq 0)$$

without disturbing the structural formulae (66), but some data (as $f_k, \ldots, \bar{\beta}^i_k$) need a slight adaptation (which need not be explicitly explained here).

We begin with $\bar{\xi}_0$. Let us look at the submodule $\Xi' = \{\xi_{<N}, \bar{\xi}_0\} \subset \Xi$. Clearly $\operatorname{Adj} \Xi' = \{\tau, \xi_{\leq N}, \bar{\xi}_{\leq 1}\} = \{dx, dy_{\leq N}, \bar{\xi}_{\leq 1}\}$. As follows from Section 9, there are generators of Ξ' expressible by the adjoint variables to Ξ', that is, by the variables $x, y_0, \ldots, y_N$ and two additional functions, say z_0, z_1. In particular,

$$f\bar{\xi}_0 = h'\,dz_0 + h''\,dz_1 + h_N\,dy_N + \ldots + h_0\,dy_0 + h\,dx \tag{67}$$

where the coefficients $h', \ldots, h$ are functions of the above mentioned adjoint variables. Then, $\bar{\xi}_0$ being replaced by the new generator $f\bar{\xi}_0 - h_{N-1}\xi_{N-1} - \ldots - h_0\xi_0$, the formula (67) reduces to $\bar{\xi}_0 = h'\,dz_0 + h''\,dz_1 + h_N\,dy_N + h\,dx$. At last, if we take an appropriate multiple of the last form $\bar{\xi}_0$, the expression $h'\,dz_0 + h''\,dz_1$ turns into a complete differential which may be identified with dz_0. After all these changes, we have

$$\bar{\xi}_0 = dz_0 - p_0\,dx - q_0\,dy_N \quad (-p_0 = h,\ -q_0 = h_N)\,,$$

where p_0, q_0 are functions of the variables $x, y_0, \ldots, y_N, z_0, z_1$.

Keeping this form $\bar{\xi}_0$, we may continue with $\bar{\xi}_1$ and the submodule $\Xi' = \{\xi_{<N}, \bar{\xi}_0, \bar{\xi}_1\}$ with $\operatorname{Adj} \Xi' = \{\tau, \xi_{\leq N}, \bar{\xi}_{\leq 1}\} = \{dx, dy_{\leq N}, \bar{\xi}_{\leq 2}\} = \{dx, dy_{\leq N}, dz_0, dz_1, dz_2\}$, where z_2 is a new function. Using similar arguments as above, one can find a new generator of the type $\bar{\xi}_1 = dz_1 - p_1\, dx - q_1\, dy_N$, where p_1, q_1 are functions of $x, y_0, \ldots, y_N, z_0, z_1, z_2$.

Keeping this form $\bar{\xi}_1$, we continue with $\bar{\xi}_2$, and so on, to the final result

$$(68)_{1,2} \qquad \xi_j = dy_j - y_{j+1}\, dx\,, \quad \bar{\xi}_k = dz_k - p_k\, dx - q_k\, dy_N$$
$$(j = 0, \ldots, N-1,\ k = 0, \ldots, M-1)$$

where x, y_j, z_k, p_k, q_k are appropriate functions. In particular,

$$(69)_k \quad p_k = p_k(x, y_0, \ldots, y_N, z_0, \ldots, z_{k+1})\,, \quad q_k = q_k(x, y_0, \ldots, y_N, z_0, \ldots, z_{k+1})$$

are composed functions of the above mentioned type. One can verify (looking at $\operatorname{Adj}\{\xi_{<N}, \bar{\xi}_{\leq k}\}$) that z_{k+1} is really present in formulae $(69)_k$. If, for instance, z_{k+1} is really present in the functions p_k, one can ensure $p_k \equiv z_{k+1}$ by a simple change of variables and $(68)_2$ reduces to $\bar{\xi}_k = dz_k - z_{k+1}\, dx - q_k\, dy_N$ resembling the contact forms of the first group $(68)_1$.

The formulae (68), (69) are sufficient in all explicit calculations. Nevertheless, the next (essentially weaker) consequence is also of certain interest.

40. Theorem. *Every factordiffiety of the diffiety Θ of Section* 30 *is isomorphic either to the same diffiety or to the diffiety Θ from Section* 18.

Proof. Consider the submodule $\Xi' \subset \iota^*\Omega$ generated by (68) together with the forms $dy_j - y_{l+1}\, dx$ $(j = N, \ldots, N + M;\ y_{s+1} \equiv L_X y_s;\ X \in \mathscr{H}(\Theta))$.

41. A note on the equivalence problem. Given a diffiety Ω, we are interested whether $\iota^*\Omega^0 = \Xi$ for some module Ξ with the generators (68). Instead of employing the general equivalence theory of exterior systems, one may proceed more elementarily as follows. In view of the existence of generators (68), there are chains

$$(70) \qquad \{\xi_0\} \subset \ldots \subset \{\xi_0, \ldots, \xi_{N-1}\} \subset$$
$$\subset \{\xi_0, \ldots, \xi_{N-1}, \bar{\xi}_0\} \subset \ldots \subset \{\xi_0, \ldots, \xi_{N-1}, \bar{\xi}_0, \ldots, \bar{\xi}_{M-1}\} = \Xi$$

of submodules in Ξ of a very special type. Namely, recalling (68), (69), one can check the properties

$$\ell(\operatorname{Adj}\{\xi_0, \ldots, \xi_j\}) = j + 2, \quad \ell(\operatorname{Adj}\{\xi_0, \ldots, \xi_{N-1}, \bar{\xi}_0, \ldots, \bar{\xi}_k\}) = N + k + 3\,,$$
$$\{\xi_0, \ldots, \xi_{j+1}\} \subset \operatorname{Adj}\{\xi_0, \ldots, \xi_j\} \quad (j = 0, \ldots, N-2)\,,$$
$$\{\xi_0, \ldots, \xi_{N-1}, \bar{\xi}_0, \ldots, \bar{\xi}_{k+1}\} \subset \operatorname{Adj}\{\xi_0, \ldots, \xi_{N-1}, \bar{\xi}_0, \ldots, \bar{\xi}_k\} \quad (k = 0, \ldots, M-2)\,.$$

Since ι^* identifies Ω^0 with Ξ, a quite analogous chain must be present in the given module Ω^0; this is the chain $\{\iota^{*-1}\xi_0\} \subset \{\iota^{*-1}\xi_0, \iota^{*-1}\xi_1\} \subset \dots$, of course. This method resembles the approach of Sections 26 and 32 based on the sequence of derived submodules $\Xi \subset \mathrm{Der}\,\Xi \subset \dots$ with the property (35). However, in our problem the chains (70) with the above properties need not be uniquely determined so that the calculations are more interesting.

42. Example. Quite in the spirit of Sections 28 and 29, let us have a brief look at the explicit solvability of the equation $\mathrm{d}z/\mathrm{d}t = h(\mathrm{d}x/\mathrm{d}t,\ \mathrm{d}y/\mathrm{d}t)$ involving three unknown functions. According to the general theory, the diffiety Ω with the Cartan filtration (1) where the first term Ω^0 is generated by the forms

$$(71) \qquad \xi = \mathrm{d}x - x'\,\mathrm{d}t\,,\quad \eta = \mathrm{d}y - y'\,\mathrm{d}t\,,\quad \zeta = \mathrm{d}z - z'\,\mathrm{d}t \quad (z' = h(x', y'))$$

is coming into play, and we are going to determine other generators that are of the kind (68) for the module Ω^0. In principle, two subcases may occur. Instead of the original variables t, x, y, z, x, y, either there are some new coordinates $u, v_0, v_1, w_0, w_1, w_2$ such that we have the generators

$$(72) \qquad \mathrm{d}v_0 - v_1\,\mathrm{d}u\,,\quad \mathrm{d}w_0 - p_0\,\mathrm{d}u - q_0\,\mathrm{d}v_1\,,\quad \mathrm{d}w_1 - p_1\,\mathrm{d}u - q_1\,\mathrm{d}v_1$$

(with p_0, q_0 functions of u, v_0, v_1, w_0, w_1), or there are coordinates $u, v_0, v_1, v_2, w_0, w_1$ such that Ω^0 possesses generators

$$(73) \qquad \mathrm{d}v_0 - v_1\,\mathrm{d}u\,,\quad \mathrm{d}v_1 - v_2\,\mathrm{d}u\,,\quad \mathrm{d}w_0 - p_0\,\mathrm{d}u - q_0\,\mathrm{d}v_2$$

with appropriate functions p_0, q_0.

Let us begin with the first subcase. Let us look for the submodule $\Omega' \subset \Omega^0$ generated by the first two forms of (72). As follows from (72), we have

$$(74) \qquad \ell(\mathrm{Adj}\,\Omega') = 5\,,\quad \Omega^0 \subset \mathrm{Adj}\,\Omega'\,.$$

On the other hand, Ω' may be generated by two linear combinations of the forms (71). Assuming $\Omega' = \{\xi + f\zeta, \eta + g\zeta\}$ for definiteness, we are led to the question whether there exist functions f, g such that (74) is satisfied. The calculations are easy provided the auxiliary basis of forms

$$\mathrm{d}t,\ \zeta,\ \mathrm{d}x',\ \mathrm{d}y',\ \xi + f\zeta,\ \xi + g\eta$$

and the dual vector fields are employed. Abbreviating $p = \partial h/\partial x'$, $q = \partial h/\partial y'$, we have

$$\begin{aligned} \mathrm{d}(\xi + f\zeta) &= \mathrm{d}t \wedge ((1 + fp)\,\mathrm{d}x + fq\,\mathrm{d}y') + \mathrm{d}f \wedge \zeta\,, \\ \mathrm{d}(\xi + g\eta) &= \mathrm{d}t \wedge (gp\,\mathrm{d}x' + (1 + gq)\,\mathrm{d}y) + \mathrm{d}g \wedge \zeta\,, \end{aligned}$$

and specifying $\mathrm{d}f = a\,\mathrm{d}t + b\,\mathrm{d}x' + c\,\mathrm{d}y' + \dots$, $\mathrm{d}g = A\,\mathrm{d}t + B\,\mathrm{d}x' + C\,\mathrm{d}y' + \dots$, one can explicitly find the generators of $\mathrm{Adj}\,\Omega'$ and see that (74) is satisfied if and only if the matrix

$$\begin{pmatrix} 0 & 0 & a & A & 1+fp & gp & fq & 1+gq & 0 \\ a & A & 0 & 0 & -b & -B & -c & -C & 1 \\ 1+fp & gp & b & B & 0 & 0 & 0 & 0 & 0 \\ fq & 1+gq & c & C & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is of rank three. Then, a simple reasoning shows that this is the case if and only if the submatrix

$$\begin{pmatrix} 1+fp & gp & b & B \\ fq & 1+gq & c & C \end{pmatrix}$$

is of rank one. The last condition can be expressed by

$$(75)_{1,2,3} \qquad 1 + fp + gq = 0\,, \quad gc + fb = 0\,, \quad bC - Bc = 0\,.$$

Since $b = \partial f/\partial x'$, $c = \partial f/\partial y'$, $B = \partial g/\partial x'$, $C = \partial g/\partial y'$, the identity $(75)_3$ implies

$$g(t, x, y, z, z', y') = G(t, x, y, z, f(t, x, y, z, x', y', z'))$$

(briefly $g = G(f)$), where G is an arbitrary function. In virtue of this result, the identity $(75)_2$ reduces to the equation $G(f)\,\partial f/\partial y' + f\,\partial f/\partial x' = 0$ for the function f. One can easily check the general solution f. According to the well-known arguments, it is implicitly determined by a relation of the type

$$(76) \qquad G(f)\,x' - fy' + H(f) = 0\,,$$

where $H = H(t, x, y, z, \cdot)$ is an arbitrary function. Finally, the identity $(75)_1$ can be rewritten as

$$(77) \qquad 1 + f\,\partial h/\partial x' + G(f)\,\partial h/\partial y' = 0$$

which is a condition on the function h in order that (75) may be satisfied.

It is not much difficult to prove that (77), (78) are also sufficient for the existence of generators of the type (72) to the module Ω^0. Moreover, investigation of the second subcase with generators (73) immediately leads to the problem of determining a submodule $\Omega' = \{\xi + f\zeta, \eta + g\zeta\} \subset \Omega^0$ with the property $\ell(\mathrm{Adj}\,\Omega') = 4$. This proves to be an even more restrictive requirement. In particular, the conditions (76), (77) must be satisfied, too. So we conclude that *there are nontrivial requirements on the function h in any case.*

43. Aside. The last result is in contradiction with the so called *Darboux method* as presented in [2]. This method was invented for solving the equation of the type $f(y'_1, \ldots, y'_n) = 0$ $(' = \mathrm{d}/\mathrm{d}x)$ and proceeds as follows.

(i) Functions $y'_i(x)$ $(i = 1, \ldots, n-1)$ are chosen quite arbitrarily and $y'_n(x)$ is calculated from the given equation (we suppose $\partial f/\partial y'_n \neq 0$).

(ii) New auxiliary functions $Y_i = y_i - xy'_i$ $(i = 1, \ldots, n)$ are introduced.

(iii) A function $u(x)$ is arbitrarily chosen.

(iv) We denote

$$\Delta = \begin{vmatrix} Y_1 & \dots & Y_n \\ y'_1 & \dots & y'_n \\ \dots & & \\ y_1^{(n)} & \dots & y_n^{(n)} \end{vmatrix}.$$

(v) By means of the equations $u(x) = \Delta(x)$, $u'(x) = \Delta'(x), \dots, u^{(n-1)}(x) = \Delta^{(n-1)}(x)$, the functions $Y_1(x), \dots, Y_n(n)$ are expressed in terms of the functions $y''_1(x), \dots, y''_n(x), \dots, y_1^{(2n-1)}(x), \dots, y_n^{(2n-1)}(x), u(x), \dots, u^{(n-1)}(x)$. This is possible since $Y'_i = -xy''_i$, cf. (ii).

(vi) Using (v) and (ii), one arrives at the formulae $y_i(x) = Y_i(x) + x\, y'_i(x)$ giving an explicit expression for $y_i(x)$ in terms of quite arbitrary functions $y'_1(x), \dots, y'_{n-1}(x)$, $u(x)$ and their derivatives.

But in fact, the method fails even for the simplest case $n = 2, f(y'_1, y'_2) = (y_1)^2 + (y_2)^2$ resolved already by Euler. The point is that the above chain (i) ... (vi) cannot be run in the reverse order: (v) implies only $Y_i = zy''_i$ with a function $z \neq -x$.

EXPLICIT SOLVABILITY WITH QUADRATURE

44. Preliminaries. We are interested in factordiffieties of the diffiety $\Theta = \{dv - h(t, u_0, \dots, u_a)\, dt,\ du_i - u_{i+1}\, dt;\ i = 0, 1, \dots\}$ on the underlying space I with the coordinates $t, v, u_0, u_1, \dots$. Clearly $\mu(\Theta) = 1$ and $\mathscr{H}(\Theta)$ is generated by the vector field

$$X = \partial + h\, \partial/\partial v = \partial/\partial t + \Sigma u_{i+1}\, \partial/\partial u_i + h\, \partial/\partial v\,.$$

If we use the abbreviations $\pi = dv - h\, dt$, $\vartheta_i = du_i - u_{i+1}\, dt$, $\delta = \Sigma \partial/\partial u_i\, \vartheta_i$, $h_i = \partial h/\partial u_i, f_v = \partial f/\partial v$, the formula playing the role of the previous (27), (54), (57) reads as follows:

$$\text{(78)} \qquad d\xi = dt \wedge \Sigma(f^{i-1} + Xf^i + gh_i)\, \vartheta_i + \Sigma \delta f^i \wedge \vartheta_i + + (Xg\, dt + \delta g - \Sigma f_v^i \vartheta_i) \wedge \pi\,,$$

where $\xi = \Sigma f^i \vartheta_i + g\pi$. As usual, we are interested in submodules $\Xi \subset \Theta$ with the property $\ell(\operatorname{Adj} \Xi) \leqq \ell(\Xi) + 2$. Since the story is already long enough, only the cases $a \leqq 3$ will be analyzed. In order to make the exposition more interesting, the previous scheme of reasoning will not be followed with much servility.

45. Some results on diffieties. We interrupt the main subject for a moment to mention some properties of filtrations and morphisms of general diffieties Ω and Θ under the supposition $\mu(\Omega) = \mu(\Theta) = 1$. The results presented in this section are only slight (but very essential) modifications of the ideas in [3] and call for a thorough analysis and generalizations in a separate paper. However, we pursue only a modest goal, to clarify the arguments in the subsequent exposition.

(i) Let (1), (9) be semi-involutive filtrations, let K be the lowest index satisfying $\Omega^0 \subset \bar{\Omega}^K$. Assume $K \geqq 1$. There exists $\omega \in \Omega^0$ with $\omega \notin \bar{\Omega}^{K-1}$. If $X \in \mathscr{H}$, $X \neq 0$, then clearly $\mathscr{L}_X\omega \in \Omega^1$, $\mathscr{L}_X\omega \notin \bar{\Omega}^K$, hence $\mathscr{L}_X\omega \notin \Omega^0$ and the semi-involutiveness of (9) implies $\mathscr{L}_X^l\omega \notin \bar{\Omega}^K$ for any $l \geqq 1$. But $\mu(\Omega) = 1$, thus the classes of the forms $\mathscr{L}_X\omega, \mathscr{L}_X^2\omega, \ldots$ are free generators of the module Ω/Ω^0, cf. the semi-involutiveness of (1). Consequently $\bar{\Omega}^K/\Omega^0 = \{0\}$, $\bar{\Omega}^K = \Omega^0$.

(ii) *Given two semi-involutive filtrations of the same diffiety, one of these filtrations is a c-normal prolongation of the other for an appropriate c* (immediately from (i)).

(iii) Since a normal prolongation of a Cartan filtration is again a Cartan filtration, *there exists a unique Cartan filtration of a given diffiety which is the finest one in the sense that any other Cartan filtration of the diffiety is its c-normal prolongation* for an appropriate $c \geqq 0$.

(iv) Let (1) be a Cartan filtration. Then $\ell(\mathrm{Adj}\,\Omega^0) = \ell(\Omega^0) + 2$ (see (36)), hence $\ell(\Omega^0) \geqq \ell(\mathrm{Der}\,\Omega^0) + 1$ (use point (iv), Section 25). The completely integrable case $\Omega^0 = \mathrm{Der}\,\Omega^0$ is not interesting (it implies $\Omega = \Omega^0$), so that we will assume $\ell(\Omega^0) = = \ell(\mathrm{Der}\,\Omega^0) + 1$ in future. Now, let us look at the module $\mathrm{Der}^2(\Omega^0)$. Obviously $\ell(\mathrm{Der}\,\Omega^0) \geqq \ell(\mathrm{Der}^2\,\Omega^0) + 2$ (a consequence of (iv) Section 25). The equality $\ell(\mathrm{Der}\,\Omega^0) = \ell(\mathrm{Der}^2\,\Omega^0)$ means $\mathrm{Der}\,\Omega^0 = \mathrm{Der}^2\,\Omega^0$, that is, (35) with $e = 1$ is valid for the module Ω^0 instead of Ξ. Denoting $c = \ell(\Omega^0)$, we have $\Omega^0 = \{\mathrm{d}v^1, \ldots, \mathrm{d}v^{c-1}, \mathrm{d}y_0 - y_1\,\mathrm{d}x\}$ (cf. (41) for the module Ω^0) and according to Theorem 11, the diffiety Ω is isomorphic to the diffiety Θ from Section 30 with the constant $b = \ell(\Omega^0) - 1$.

(v) Continuing the preceding point, we shall consider the case $\ell(\mathrm{Der}\,\Omega^0) = = \ell(\mathrm{Der}^2\,\Omega^0) + 1$. Then we have the equalities

$$(79) \qquad \ell(\Omega^0) = \ell(\mathrm{Der}\,\Omega^0) + 1 = \ell(\mathrm{Der}^2\,\Omega^0) + 2$$

identical with the initial part of the chain (35). One can easily find that the reasoning in Section 25 gives also the initial part of the relations (38) and (39), that is,

$$(80)_{1,2} \qquad \ell(\mathrm{Adj}\,\mathrm{Der}\,\Omega^0) = \ell(\mathrm{Der}\,\Omega^0) + 2\,, \quad \mathrm{Adj}(\mathrm{Der}\,\Omega^0) \supset \Omega^0\,.$$

Moreover, a combination of arguments of Sections 10 and 25 leads to several important conclusions: first, according to $(80)_1$, there are some free generators of the module $\mathrm{Der}\,\Omega^0$ of the kind

$$(81)^0 \qquad \mathrm{d}y_0^k - h^k\,\mathrm{d}y_0^c - \bar{h}^k\,\mathrm{d}x \quad (k = 0, .., c - 1;\ c - 1 = \ell(\mathrm{Der}\,\Omega^0))\,,$$

where $y_0^1, \ldots, y_0^c$, x are adjoint variables to the module $\mathrm{Der}\,\Omega^0$ and we suppose $\mathrm{d}x \notin \Omega^0$; note that $h^k, \bar{h}^k$ are certain functions expressible in terms of these adjoint variables (compare this with the particular case $m - c = \mu(\Omega) = 1$ of $(18)^0$). According to $(80)_2$, the forms $(81)^0$ together with an appropriate form of the type $\mathrm{d}y_0^c - y_1^c\,\mathrm{d}x$ freely generate the module Ω^0. Clearly y_1^c belongs to the adjoint variables of the module Ω^0 but not of $\mathrm{Der}\,\Omega^0$. Altogether we have free generators

$$(81)^1 \qquad \mathrm{d}y_0^k - g^k\,\mathrm{d}x \quad (k = 1, \ldots, c - 1;\ g^k = h^k y_1^c + \bar{h}^k)\,, \quad \mathrm{d}y_0^c - y_1^c\,\mathrm{d}x$$

to the module Ω^0 exactly corresponding to the family $(18)^1$. Continuing in this way, we shall look at the next module Ω^1 using (17). It follows that there is a form of the type $\mathrm{d}y_1^c - y_2^c\,\mathrm{d}x$ generating together with $(81)^1$ the module Ω^1. The new function y_2^c appearing here belongs to the adjoint variables of the module Ω^1, but not of Ω^0. Quite analogously, owing to the semi-involutiveness, one can see that the forms

$$(81)^{l+1} \qquad \mathrm{d}y_0^k - g^k\,\mathrm{d}x\,, \quad \mathrm{d}y_s^c - y_{s+1}^c\,\mathrm{d}x \quad (k = 1, \ldots, c-1;\ s = 1, \ldots, l)$$

(where $y_{s+1}^c = Xy_s^c$, $X \in \mathscr{H}(\Omega)$ is normalized by $Xx = 1$) are free generators of the module Ω^l ($l \geqq 1$). Besides, note that the variables $x, y_0^1, \ldots, y_0^c, y_1^c, y_2^c, \ldots$ may be chosen for coordinates on the underlying space J.

(vi) Summarizing the results of the preceding point we draw the following conclusions: *If* (1) *is a Cartan filtration satisfying* (79), *then the new filtration* (9) *defined by*

$$\bar{\Omega}^0 = \operatorname{Der}\Omega^0\,, \quad \bar{\Omega}^l = \Omega^{l-1} \quad (l \geqq 1)$$

is again a Cartan filtration. This follows directly from the explicit expressions (81) for the generators. Consequently, (79) *cannot be true if* (1) *is the finest Cartan filtration so that necessarily* $\ell(\operatorname{Der}\Omega^0) = \ell(\operatorname{Der}^2\Omega^0) + 2$ *in this case.*

(vii) We shall derive some preparatory results needed in the most interesting concluding point (viii) to follow. Two diffieties will be considered at the same time. Let

$$(82) \qquad \Theta^*\colon \Theta^0 \subset \Theta^1 \subset \ldots \subset \Theta^l \subset \ldots \subset \Theta = \bigcup \Theta^l$$

be a Cartan filtration of Θ, let $\Xi \subset \Theta$ be a finitely generated submodule, and K the lowest index satisfying $\Xi \subset \Theta^K$. Assume $K \geqq 1$. One can see that there exist two vector fields $X, Y \in \Theta^{K\perp}$ linearly independent modulo the space $\operatorname{Adj}\Xi^\perp$. (Indeed, in the other case $\Xi \subset \operatorname{Der}\Theta^K = \Theta^{K-1}$ as follows from point (iv) Section 25 applied to the definition of the module $\operatorname{Der}\Theta^K$.) Assume moreover $\ell(\operatorname{Adj}\Xi) = \ell(\Xi) + 2$. Then clearly $\Theta^{K\perp} \subset \Xi^\perp = \{X, Y, \operatorname{Adj}\Xi^\perp\}$ and the equalities $\ell(\Theta^K) = \ell(\operatorname{Der}\Theta^K) + 1 = \ell(\operatorname{Der}^2\Theta^K) + 2$ (valid for every Cartan filtration provided $K \geqq 1$) imply the inequalities

$$\ell(\Xi) \geqq \ell(\operatorname{Der}\Xi) + 1 \geqq \ell(\operatorname{Der}^2\Xi) + 2\,.$$

(Look at the vectors generating the modules $\operatorname{Der}\Xi$, $\operatorname{Der}^2\Xi$ using (iv) Section 25.)

(viii) *Let Ω be a factordiffiety of Θ, let* (1) *be the finest Cartan filtration,* (82) *a Cartan filtration, $\iota\colon I \to J$ the relevant morphism. Then either $\iota^*\Omega^0 \subset \Theta^0$, or the diffiety Ω is isomorphic to the diffiety Θ from Section* 30 *with the constant* $b = \ell(\Omega^0) - 1$. Proof. Let the inclusion be not true and choose $\Xi = \iota^*\Omega^0$. Then (vii) may be applied. Since ι^* is injective and (79) cannot hold, we obtain either $\Omega^0 = \operatorname{Der}\Omega^0$, or $\operatorname{Der}\Omega^0 = \operatorname{Der}^2\Omega^0$. Then point (iv) concludes the proof.

46. The nearest order cases $a \leqq 2$ will be mentioned here, but let us recall the problem in a manner employing the achievements of Section 45: We look for the factordiffieties Ω of the diffiety Θ from Section 44. Let (82) be the Cartan filtration with terms

$$\Theta^l = \{\mathrm{d}v - h(t, u_0, \ldots, u_a)\,\mathrm{d}t,\ \mathrm{d}u_j - u_{j+1}\,\mathrm{d}t;\ j = 0, \ldots, a + l - 1\}\,,$$

let (1) be the finest Cartan filtration of Ω. Omitting the simple case of Ω isomorphic to the diffiety Θ from Section 30 (which is already easy for us), we have $\iota^*\Omega^0 \subset \Theta^0$ with an injective ι^* (cf. (viii) Section 45). So it is sufficient to deal only with the submodules $\Xi = \iota^*\Omega^0 \subset \Theta^0$, which essentially simplifies the problem. We need $\ell(\mathrm{Adj}\,\Xi) \leqq \ell(\Xi) + 2$, of course, exactly as before.

If $a < 0$, that is, $h = h(t)$, then $\mathrm{d}v - h\,\mathrm{d}t$ is a multiple of a differential $\mathrm{d}w$ and $\Theta = \{\mathrm{d}w, \mathrm{d}u_j - u_{j+1}\,\mathrm{d}t;\ j = 0, 1, \ldots\}$ is only the diffiety from Section 30 with $b = 1$.

If $a = 0$, that is, $h = h(t, u_0)$ with $\partial h/\partial u_0 \neq 0$, then Θ is even isomorphic to the diffiety Θ from Section 18.

If $a = 1$, that is, $h = h(t, u_0, u_1)$ with $\partial h/\partial u_1 \neq 0$, then $\Theta^0 = \{\mathrm{d}v - h\,\mathrm{d}t, \mathrm{d}u_0 - u_1\,\mathrm{d}t\}$, $\ell(\Theta^0) = \ell(\mathrm{Der}\,\Theta^0) + 1 = \ell(\mathrm{Der}^2\,\Theta^0) + 2$ and $\mathrm{Der}^2\,\Theta^0 = \{0\}$ so that we have the chain (35) for the module Θ^0 (instead of Ξ) with the constant $e = 2$. Hence $\Theta^0 = \{\mathrm{d}y_0 - y_1\,\mathrm{d}x, \mathrm{d}y_1 - y_2\,\mathrm{d}x\}$ for appropriate functions x, y_0, y_1, y_2 (cf. (42)) and Θ is again isomorphic to the diffiety from Section 18.

Look at the case $a = 2$, that is, $h = h(t, u_0, u_1, u_2)$ with $\partial h/\partial u_2 \neq 0$. Then the inclusion $\Xi \subset \Theta^0$ may be proper or not. If $\Xi \neq \Theta^0$, then $\ell(\Xi) < \ell(\Theta^0) = 3$, hence $\ell(\Xi) \leqq 2$ and $\ell(\mathrm{Adj}\,\Xi) \leqq \ell(\Xi) + 2 \leqq 4$. Since $\ell(\Xi) = \ell(\mathrm{Der}\,\Xi) + 1$ (cf. the beginning of (iv) Section 45 for the module Ξ instead of Ω^0), (35) is valid with $e \leqq 2$ for the module Ξ and Ω proves to be isomorphic to the diffiety from Section 30. In the other case $\iota^*\Omega^0 = \Theta^0$, the diffieties Ω and Θ are isomorphic and we have a trivial case of a factordiffiety of Θ.

47. The lowest nontrivial case occurs if $a = 3$, that is, $h = h(t, u_0, \ldots, u_3)$ with $\partial h/\partial u_3 \neq 0$. In this case $\Theta^0 = \{\pi, \vartheta_0, \vartheta_1, \vartheta_2\}$, cf. the notation introduced in Section 44, and our aim is to find submodules $\Xi \subset \Theta^0$ with $\ell(\mathrm{Adj}\,\Xi) \leqq \ell(\Xi) + 2$. Clearly $\ell(\Xi) \leqq \ell(\Theta^0) = 4$ and we may promptly omit the extremal cases $\ell(\Xi) = 4$ (Ω is isomorphic to Θ) and $\ell(\Xi) = 1$ (which is very easy). If $\ell(\Xi) = 2$, then $\ell(\mathrm{Adj}\,\Xi) \leqq 4$ and one can easily prove the existence of a chain (35) with $e \leqq 2$ so that there are generators (41) and Ω is isomorphic to the diffiety from Section 30 with an appropriate constant b. It follows that only the case $\ell(\Xi) = 3$ may be of certain interest.

Assuming $\ell(\Xi) = 3$, we have three linearly independent generators of the module Ξ, linear combinations of the forms $\pi, \vartheta_0, \vartheta_1, \vartheta_2$. Let us recall the fundamental principle: if $\xi \in \Xi$, then $\xi' = \partial/\partial t \urcorner \mathrm{d}\xi$ belongs to $\mathrm{Adj}\,\Xi$, and if even $\xi' \in \Xi$, then $\xi'' = \partial/\partial t \urcorner$ $\urcorner\, \mathrm{d}\xi' \in \mathrm{Adj}\,\Xi$, and so on. Following this principle and looking at (78), one can find only three types (say: **A, B, C**) of these submodules with the relevant generators as follows:

A: $\xi_0 = \vartheta_1 + f\vartheta_0$, $\xi_1 = \vartheta_2 + f\vartheta_1 + Xf \,.\, \vartheta_0, \pi$,

B: ϑ_0, ϑ_1, $\xi = g\vartheta_2 + \pi \quad (g \neq 0)$,

C: ξ_0 and ξ_1 are as above, $\xi = g\vartheta_0 + \pi \quad (g \neq 0)$.

Here f and g are appropriate functions and it is supposed that $\pi \notin \Xi$ in the cases **B**, **C**. The corresponding modules Adj Ξ are generated by the forms from Ξ together with the forms

A: $\mathrm{d}t + (\ldots)$, $\xi_2 = \vartheta_3 + f\vartheta_2 + 2Xf \,.\, \vartheta_1 + X^2 f \,.\, \vartheta_0$,

B: $\mathrm{d}t + (\ldots)$, ϑ_3(or, alternatively, $\xi' = (g + h_3)\,\vartheta_3 + (Xg + h_2)\,\vartheta_2$),

C: $\mathrm{d}t + (\ldots)$, ξ_2 as above (or $\xi' = g\vartheta_1 + Xg \,.\, \vartheta_0 + \Sigma h_i \vartheta_i$).

We shall now consider these cases separately to derive canonical forms for the generators. We should like to employ the method of cross-sections (cf. Section 24) since it proves to be the most effective one. To this aim, some preliminary information concerning the functions f, g is needed.

48. The subcase A. The calculations will be made using the basis π, $\mathrm{d}t$, ϑ_0, ξ_0, ξ_1, ξ_2, ϑ_4, ϑ_5, ... of the space Φ_1 and the dual basis denoted by $\partial/\partial\pi$, $\partial/\partial t$, $\partial/\partial\vartheta_0$, $\partial/\partial\xi_0$, ... for the vector fields. Since $\pi \in \Xi$ and $\mathrm{d}\pi = \mathrm{d}t \wedge \Sigma h_i \vartheta_i$, we have

$$\partial/\partial\xi_2 \urcorner \mathrm{d}\pi = \mathrm{d}t \in \mathrm{Adj}\,\Xi\,, \quad \partial/\partial t \urcorner \mathrm{d}\pi = \Sigma h_i \vartheta_i \in \mathrm{Adj}\,\Xi\,.$$

On the other hand one can directly find a decomposition of the type

$$h_3\vartheta_3 + h_2\vartheta_2 + h_1\vartheta_1 + h_0\vartheta_0 = P\xi_2 + Q\xi_1 + R\xi_0 + S\vartheta_0\,,$$

where P, Q, R are unimportant coefficients (which need not be specified), but the last one necessarily vanishes:

$$\begin{aligned} 0 = S = {} & -h_3 X^2 f + 2h_3 (Xf)^2 + \\ & + (h_2 - h_1 + (h_2 - h_3(1 + f))\,f)\, Xf + h_0 \end{aligned} \tag{83}$$

(since, as we know, $\Sigma h_i \vartheta_i \neq \mathrm{Adj}\,\Xi$).

Assume $f = f(t, v, u_0, \ldots, u_m)$ with $\partial f/\partial u_m \neq 0$. There is a summand $-h_3 u_{m+2}\, \partial f/\partial u_m$ in (83) and the functions $h_1, \ldots, h_3$ can be expressed only in terms of the variables $t, u_0, \ldots, u_3$. So we conclude $m + 2 \leqq 3$, hence $m \leqq 1$ and $f = f(t, v, u_0, u_1)$.

Let us now look at the formula

$$\mathrm{d}\xi_1 = \mathrm{d}t \wedge \xi_2 + \delta f \wedge \vartheta_1 + \delta(Xf) \wedge \vartheta_0 + \pi \wedge (f_v \vartheta_1 + (Xf)_v\, \vartheta_0)\,,$$

a particular case of (78). As one can see,

$$\partial/\partial\xi_2 \urcorner \mathrm{d}\xi_1 = \partial/\partial\xi_2 \urcorner \delta(Xf) \wedge \vartheta_0 + (\ldots) = f_v h_3 \vartheta_0 + (\ldots)$$

with the summand $(\ldots) \in \mathrm{Adj}\,\Xi$. This implies $f_v = 0$, hence $f = f(t, u_0, u_1)$.

At this stage, the cross-section $u_0 = \text{const.}$ can be applied. Denoting by tildes the results of constraining (at least partly, whenever necessary for better understanding), we obtain the generators

$$\tilde{\pi} = \mathrm{d}v - \tilde{h}\,\mathrm{d}t\,,\quad \tilde{\xi}_0 = \mathrm{d}u_1 - (u_2 + u_1 f)\,\mathrm{d}t\,,$$
$$\tilde{\xi}_1 = \mathrm{d}u_2 + f\,\mathrm{d}u_1 - (u_3 + u_2 f + u_1 Xf)\,\mathrm{d}t =$$
$$= \mathrm{d}u_2 - (u_3 + u_1(Xf - f^2))\,\mathrm{d}t + f\tilde{\xi}_0\,.$$

In terms of the variables $t = t$, $v = v$, $v_0 = u_1$, $v_1 = u_2 + u_1 f$, $v_2 = u_3 + u_1(Xf - f^2)$, the generators are expressed as

$$\mathrm{d}v - \tilde{h}\,\mathrm{d}t\,,\quad \mathrm{d}v_0 - v_1\,\mathrm{d}t\,,\quad \mathrm{d}v_1 - v_2\,\mathrm{d}t\,,$$

where $\tilde{h}$ is a function of t, u_1, u_2, u_3, that is, of the variables t, v_0, v_1, v_2. It follows that the factordiffiety Ω *is isomorphic to the diffiety from Section* 44 *but with another function h under the sign of quadrature with the lowered order* $a = 2$.

49. The subcase B leads to the same results, but simpler arguments are sufficient. The calculations can be made in the basis $\mathrm{d}t, \vartheta_0, \vartheta_1, \xi, \vartheta_2, \vartheta_3, \ldots$ for the differential forms and the dual basis $\partial/\partial t, \partial/\partial\vartheta_0, \ldots$ for the vector fields. Clearly $\mathrm{d}t = \partial/\partial t \urcorner \urcorner\, \mathrm{d}\vartheta_2 \in \mathrm{Adj}\,\Xi$. Moreover,

$$\xi' = (g + h_3)\,\vartheta_3 + (Xg + h_2)\,\vartheta_2 \in \mathrm{Adj}\,\Xi = \{\vartheta_0, \vartheta_1, \mathrm{d}t, \vartheta_2\}\,,$$

hence $g + h_3 = 0$. Then the cross-section $u_3 = \text{const.} = c$ immediately applies and gives the generators

$$\tilde{\vartheta}_0 = \vartheta_0\,,\quad \tilde{\vartheta}_1 = \vartheta_1\,,\quad \tilde{\xi} = -h_3\vartheta_3 + \pi = \mathrm{d}v - \tilde{h}_3\,\mathrm{d}u_2 - (\tilde{h} - c\tilde{h}_3)\,\mathrm{d}t\,.$$

Replacing v by the new variable $w = v - \int \tilde{h}_3\,\mathrm{d}u_2$ (a primitive function of $\tilde{h}_3$ with respect to u_2), one can see that the last generator of the module Ξ may be replaced by a form of the type $\mathrm{d}w - \bar{h}\,\mathrm{d}t$, where $\bar{h}$ is a certain function of t, u_0, u_1, u_2. It follows that *the conclusions are exactly the same as in the preceding case A.*

50. The remaining subcase C is the most interesting one and will conclude this paper. For the reader's convenience, let us recall the main part of the problem: In the space of variables $t, v, u_0, u_1, \ldots$, we consider the module Ξ generated by the forms

$$(84)\qquad \xi_0 = \vartheta_1 + f\vartheta_0\,,\quad \xi_1 = \vartheta_2 + f\vartheta_1 + Xf\,.\,\vartheta_0\,,\quad \xi = g\vartheta_0 + \pi\,,$$

and we are interested in necessary and sufficient conditions on the functions f, g ($g \neq 0$) which ensure $\ell(\mathrm{Adj}\,\Xi) \leqq 5$. We shall use the basis

$$\mathrm{d}t, \vartheta_0, \xi, \xi_0, \xi_1, \xi_2, \vartheta_4, \vartheta_5, \ldots$$

and the relevant dual basis of vector fields, where

$$(85)\qquad \xi_2 = \partial/\partial t \urcorner \mathrm{d}\xi_1 = \vartheta_3 + f\vartheta_2 + 2Xf\,.\,\vartheta_1 + X^2 f\,.\,\vartheta_0 \in \mathrm{Adj}\,\Xi\,.$$

In terms of this basis, the formula (78) easily gives

$$\mathrm{d}\xi = \mathrm{d}t \wedge \xi' + \delta g \wedge \vartheta_0 + \xi \wedge g_v\vartheta_0\,, \quad \mathrm{d}\xi_0 = \mathrm{d}t \wedge \xi_1 + \delta f \wedge \vartheta_0 + \xi \wedge f_v\vartheta_0\,,$$

$$\mathrm{d}\xi_1 = \mathrm{d}t \wedge \xi_2 + \delta f \wedge (\xi_0 - f\vartheta_0) + \delta Xf \wedge \vartheta_0 +$$
$$+ (\xi - g\vartheta_0) \wedge (f_v\xi_0 + ((Xf)_v - ff_v)\,\vartheta_0)\,.$$

Since $\xi' = \partial/\partial t \urcorner \mathrm{d}\xi = g\vartheta_1 + Xg\vartheta_0 + \Sigma h_i\vartheta_i \in \mathrm{Adj}\,\Xi$, the form ξ' is a linear combination of ξ_0, ξ_1, ξ_2, say,

$$(86) \qquad g\vartheta_1 + Xg\vartheta_0 + h_0\vartheta_0 + \ldots + h_3\vartheta_3 = h_3(\xi_3 + P\xi_1 + Q\xi_0)$$

with unknown coefficients P, Q. Then, inserting (84), (85) into (86), one can eliminate P, Q and the existence of a relation of the type (86) proves to be equivalent to the identity

$$(87) \qquad hX^2f + (-3h_3f + h_2)\,Xf + h_3f^3 - h_2f^2 + h_1f - h_0 - Xg + fg = 0$$

which is of fundamental importance. This identity provides (a seemingly very complicated) partial differential equation for the function f and will be analyzed in a moment.

In the meantime, let us return to the primary requirement $\ell(\mathrm{Adj}\,\Xi) \leqq 5$. As follows from the general theory of adjoint variables, beside the forms (84), (85) also

$$(88) \qquad \delta g = \partial/\partial\vartheta_0 \urcorner \mathrm{d}\xi\,, \quad \delta f = \partial/\partial\vartheta_0 \urcorner \mathrm{d}\xi_0\,, \quad \delta Xf = \partial/\partial\vartheta_0 \urcorner \mathrm{d}\xi_1\,,$$

$$(89) \qquad -h_3\,\mathrm{d}t + \frac{\partial g}{\partial u_3}\,\vartheta_0 = \partial/\partial\xi_2 \urcorner \mathrm{d}\xi\,,$$

$$-\mathrm{d}t + \left(\frac{\partial f}{\partial u_3} f + \frac{\partial Xf}{\partial u_3}\right)\vartheta_0 = \partial/\partial\xi_2 \urcorner \mathrm{d}\xi_1$$

(and $\partial/\partial\xi_2 \urcorner \mathrm{d}\xi_0 = \partial f/\partial u_3\vartheta_0$, but it does not matter) belong to the module $\mathrm{Adj}\,\Xi$. (We omit some summands from the module $\mathrm{Adj}\,\Xi$ to abbreviate the formulae.) This implies

$$\partial g/\partial u_i \equiv 0 \quad (i \geqq 4)\,, \qquad \partial f/\partial u_i \equiv 0 \quad (i \geqq 3)\,,$$

that is, we have

$$(90) \qquad g = g(t, v, u_0, u_1, u_2, u_3)\,, \quad f = f(t, v, u_0, u_1, u_2)$$

since the summands with ϑ_i $(i \geqq 4)$ cannot be present in (88). Moreover, the forms (89) must be proportional:

$$(91) \qquad \frac{\partial g}{\partial u_3} = h_3\left(\frac{\partial f}{\partial u_2} + (Xf)_3\right) = h_3\left(2\,\frac{\partial f}{\partial u_2} + h_3f_v\right).$$

Conversely, the conditions (87), (90), (91) together imply

$$\mathrm{Adj}\,\Xi = \{\xi, \xi_0, \xi_1, \xi_2, h_3\,\mathrm{d}t - \partial g/\partial u_3\,.\,\vartheta_0\}$$

so that the requirement $\ell(\mathrm{Adj}\ \Xi) \leqq 5$ is satisfied (even with equality). We are going to look at these conditions in more detail.

Inserting $Xf = f' + hf_v, X^2f = X(Xf)$ into (87), one obtains a rather long expression which is a function of the variables $t, v, u_0, \ldots, u_4$; cf. (90). However, the variable u_4 appears only in the group of terms $u_4(\partial f/\partial u_2 + h_3 f_v - \partial g/\partial u_3)$ which must vanish, of course. According to (91), we obtain $\partial f/\partial u_2 = 0$, and (91) reduces to $\partial g/\partial u_3 = (h_3)^2 f_v$. It follows that (90), (91) may be replaced by simpler relations

$$(92)_{1,2} \qquad g = \int (h_3)^2 f_v \, \mathrm{d}u_3 + G(t, v, u_0, u_1, u_2)\,, \quad f = f(t, v, u_0, u_1)$$

with a certain fixed primitive function but arbitrary G.

Inserting (92) into (87), the arising expression is a function of the variables $t, v, u_0, \ldots, u_3$ of a very special kind: It is a linear combination of certain functions depending on the variable u_3 and expressible by h (as are, for instance, $h_i, h_i h$, $\int (h_3)^2 \, \mathrm{d}u_3, u_3 h_i$, and so on) with coefficients expressible by the functions f and G and hence independent of u_3. It follows that under some linear independence hypothesis on $h_i, h_i h, \ldots$ each coefficient must vanish, which permits to specify the functions f and G. If there are some linear relations between h_i $h_i h, \ldots$, they may be taken into account with an analogous final effect. However, such relations are possible only for some particular (and very interesting) cases of the function h. To perform this program to the full extent is rather long. For this reason, we conclude the paper with two very particular examples $h = \mathrm{e}^{u_3}$ and $h = u_3^{2/3}$.

Let us denote $f_i = \partial f/\partial u_i$ and analogously for $f_{ti}, f_{vi}, f_{ii}, f_{tt}$. Let us moreover put

$$A = \partial f_t + \Sigma u_{i+1} u_{j+1} f_{ij} + u_2 f_0 - 3f \, \partial f_t + f^3\,,$$
$$B = 2\partial f_v - 3ff_v\,, \quad C = -\partial f_v + ff_v\,, \quad D = -\partial G + fG\,.$$

Assuming the function $h = h(u_3)$ independent of the variables t, v, u_0, u_1, u_2 and abbreviating $H = \int (h_3)^2 \, \mathrm{d}u_3$ for a fixed primitive function, we obtain as a result of inserting (92) into (87) the equality

$$(93) \qquad Ah_3 + Bhh_3 + CH + D + f_1 u_3 h_3 - f_{vv} hH + f_{vv} h_3 h^2 - G_v h - G_2 u_3 = 0\,.$$

For a „quite general" function h, each coefficient must vanish which immediately gives $f = f(t)$, $G = \text{const.} \exp\left(\int f \, \mathrm{d}t\right)$ with the function f satisfying the ordinary differential equation

$$(94) \qquad \mathrm{d}^2 f/\mathrm{d}t^2 - 3f \, \mathrm{d}f/\mathrm{d}t + f^3 = 0\,.$$

Then the cross-section method leads to the same result as above: *The diffiety Ω is isomorphic to Θ from Section* 44 *but with the reduced order $a = 2$.*

Now, let us specify $h = \mathrm{e}^{u_3}$. Then (93) turns into

$$(95) \qquad \tfrac{1}{2} f_{vv} \, \mathrm{e}^{3u_3} + \left(B + \tfrac{1}{2}C\right) \mathrm{e}^{2u_3} + \left(A - G_v\right) \mathrm{e}^{u_3} + f_1 u_3 \mathrm{e}^{u_3} + G_2 u_3 + D = 0\,.$$

The vanishing of the first, fourth, and the last but one coefficients easily gives $f = P(t, u_0)\, v + Q(t, u_0)$, $G = G(t, v, u_0)$. Moreover, a simple analysis of the con-

condition $A = G_v$ implies $f_0 = 0$, $P = 0$, hence $f = f(t)$ and then the condition turns into the equation $\mathrm{d}^2 f/\mathrm{d}t^2 - 3f\,\mathrm{d}f/\mathrm{d}t + f^3 = G_v$, that is, we have

$$(96) \qquad g = G(t, v, u_0) = (f'' - 3ff' + f^3)\, v + R(t, u_0)\,,$$

according to $(92)_1$. Only the last coefficient in (95) remains, and it reduces to $G_t + u_1 G_0 - fG = 0$. Using (96), one can derive $G_0 = R_0 = 0$ (hence $R = R(t)$) and $G_t = fG$. Inserting (96) for the function G into the last condition, the final result is

$$(f'' - 3ff' + f^3)' = f(f'' - 3ff' + f^3)\,, \quad R = \text{const.}\, \mathrm{e}^{\int f(t)\,\mathrm{d}t}\,.$$

In the particular case when (94) is valid, the resulting diffiety Ω is exactly the same as in the case of a „general" function h and is isomorphic to the diffiety Θ from Section 44 with a reduced order. However, in other cases, the cross-section method can be applied with quite another conclusion (cf. the last example below).

Finally, let us assume $h = u_3^{2/3}$. Then (93) turns into

$$(97) \qquad (2f_{vv} - G_2)\, u_3 + (\tfrac{2}{3}f_1 - G_v)\, u_3^{2/3} + (\tfrac{2}{3}B + \tfrac{4}{3}C)\, u^{1/3} + D + \tfrac{2}{3}Au_3^{-1/3} = 0\,.$$

The vanishing of D implies $G_2 = G_1 = 0$, hence $G = G(t, v, u_0)$. Then, the vanishing of the first coefficient in (97) means $f_{vv} = 0$, that is, $f = P(t, u_0, u_1)\, v + Q(t, u_0, u_1)$. But looking at the summand f^3 in the condition $A = 0$ we immediately conclude $P = Q$, hence $f = Q = f(t, u_0, u_1)$ and the third coefficient in (97) turns to zero. Due to the information already available, the second coefficient vanishes if

$$(98) \qquad f = M(t, u_0)\, u_1 + N(t, u_0)\,, \quad G = \tfrac{2}{3}M(t, u_0)\, v + R(t, u_0)\,,$$

and only two coefficients in (97) remain. Now, (98) inserted into $D = 0$ easily gives $M = (T(t) - u_0)^{-1}$, $N = -T'(t)\,(T(t) - u_0)^{-1}$, $R = -C(u_0)\,(T(t) - u_0)^{-1}$ with C, T quite arbitrary functions (or $M = 0$ identically, but this assumption leads to the same formulae as in the above case of a „quite arbitrary" function h and is of little interests). So we have

$$f = (u_1 - T'(t))/(T(t) - u_0)\,, \quad g = (\tfrac{2}{3}v - C(u_0))/(T(t) - u_0)\,,$$

as follows from (98), $(92)_1$. Lastly, the requirement $A = 0$ leads (after some unpleasant calculations) to the condition $T'' = 0$, hence $T(t) = at + b$. (Note that the same result can be derived for the function $h = u_3^{1/2}$, but not for any other power of u_3.) This result completely solves the problem.

Let us mention the cross-sections. After applying $u_0 = \text{const.}$, one obtains the generators

$$\mathrm{d}u_1 - (u_2 + u_1(u_1 - a)/(at + d))\,\mathrm{d}t\,, \quad \mathrm{d}u_2 - (u_3 + u_1 u_2/(at + d))\,\mathrm{d}t\,,$$
$$\mathrm{d}v - (u_3^{2/3} + u_1(\tfrac{2}{3}v - c)/(at + d))\,\mathrm{d}t$$

($d = b - \text{const.}$, $c = C\,(\text{const.})$). They may be expressed more simply as

$$\mathrm{d}v_1 - v_2\,\mathrm{d}t\,, \quad \mathrm{d}v_2 - v_3\,\mathrm{d}t\,, \quad \mathrm{d}v - \bar{h}\,\mathrm{d}t$$

if one uses the variables t, $v_0 = u_1$, $v_1 = u_2 + u_1(u_1 - a)/(at + d)$, $v_2 = u_3 + u_2(3u_1 - a)/(at + d) + 2u_1((u_1 - a)/(at + d))^2$. But the function $\bar{h}$ (not specified here) depends also on the variable v and one cannot easily find out whether the relevant factordiffiety Ω (with $\iota^*\Omega^0 = \Xi$) is isomorphic to the diffiety Θ from Section 44 with a lowered order a. This is a typical equivalence problem for the system of three Pfaff's equations in a five dimensional space, the problem completely discussed by Cartan in one of his most important (but not very popular) paper [13] on the theory of pseudogroups. It seems that the development of his ideas still belongs to the most urgent tasks.

51. Added in proof. First, we use the opportunity to correct some misprints in [1]. The inclusion $(4)_1$ should be opposite, the line 357_9 should read $\ldots 0 \in \mathscr{G}^{l+1}$, 358_4 should read $\ldots \subset \mathscr{G}^{l+1}_{r,s+1}$, the word „set" should be inserted between 358_{8-7}, the formula (21) should be corrected to $X_k: (M/(X)_{k-1} M)^c \to (M/(X)_{k-1} M)^{c+1}$, the line 364_9 should begin by $(N \cap \ldots$, the bracket should be inserted into 365_{18} to give $\ldots$ Ass $M/M(Q))$, the last term in 368_7 should be $\ldots \mathscr{G}^{l,k+1}$, the inequality at the end of 377_5 should be $l \geqq c$, the location of several indices to the letter M on the pages 379–381 should be easily repaired, and the reference 384_{11} should be to Section 8. Secondly, the Section 25 erroneously appears twice in [1]. The first appearance should be numbered 24 and the original Section 24 of [1] (devoted to Cauchy characteristics) must be cancelled (since the setting is a misleading one). Thirdly, the author found that the expression $f'' - 3ff' + f^3$ occuring in Section 50 of the present paper turns into $-v'''/v$ after the substitution $f = -v'/v$ and moreover (which is the most surprising point!) it is identical with one of the basic operators of the Painlevé theory (cf. [V. V. Golubjev: Lectures on analytical theory of differential equations, Moskva–Leningrad 1950 (Russian), page 179, formula (I)]). Lastly, we should like to note with pleasure that the fundaments of the theory will be essentially clarified in the next part III (devoted to pseudogroups): the clumsy limits of spaces and mappings will be eliminated, a general concept of regularity will be introduced, and (owing to the corrected Cauchy characteristics) the axiom $\mathscr{D}im$ will be omitted.

References

[1] *J. Chrastina:* On formal theory of differential equations I. Časopis pěst. mat. *111* (1986), 353–383.

[2] *M. P. Zervos:* Le probléme de Monge. Mémorial des sciences mathématiques, Paris 1932.

[3] *E. Cartan:* Sur l'equivalence absolu de certain systémes d'equations différentielles et sur certaines familles de courbes. Oeuvres complétes II 2, Paris 1953, 1133–1167.

[4] *A. Kumpera, C. Ruiz:* Sur l'equivalence locale des systémes de Pfaff en drapeau. Monge-Ampére equations and related topics, Inst. Naz. Alta Francesco Severi, Rome 1982, 201–247.

[5] *A. M. Vinogradov, I. S. Krasilščik, V. V. Lyčagin:* Introduction into the geometry of nonlinear differential equations, Moskva 1986 (Russian).
[6] *J. F. Pommaret:* Systems of partial differential equations and Lie pseudogroups. Mathematics and Its Applications 14 (1978); Russian translation 1983.
[7] *S. P. Finikov:* Method of Cartan's exterior forms in differential geometry, Moskva 1953 (Russian).
[8] *H. Goldschmidt:* Integrability criteria for systems of nonlinear partial differential equations. J. Diff. Geom. *1* (1967), 269—307.
[9] *Ch. Riquier:* Les systémes d'equations aux dérivées partielles. Cahiers Scientifiques, Paris 1910.
[10] *M. Janet:* Lecons sur les systémes d'equations aux dérivées partielles. Gauthier-Villars, Paris 1921.
[11] *E. Cartan:* Les systémes différentiels extérieurs et leurs applications geometriques. Gauthier-Villars, Paris 1945.
[12] *E. Cartan:* Sur l'integration de certain systémes indeterminés d'equations différentielles, Oeuvres complétes II 2, Paris 1953, 1169—1174.
[13] *E. Cartan:* Les systémes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre. Oeuvres complétes II 2, Paris 1953, 927—1010.

Souhrn

O FORMÁLNÍ TEORII DIFERENCIÁLNÍCH ROVNIC II

JAN CHRASTINA

Jako aplikace nekonečných prodloužení diferenciálních rovnic, v článku je navržena metoda pro studium Mongeova problému. Tento dávný problém (~ 1780) se týká toho, zda je možno řešit daný nedourčený systém obyčejných diferenciálních rovnic (více neznámých funkcí, než rovnic) pomocí explicitních vzorců algebraického typu obsahujících vyšší derivace některých libovolných (parametrických) funkcí. Navrhovaná metoda redukuje Mongeův problém na klasický problém ekvivalence Pfaffových systémů (a připouští mnohá zobecnění).

Резюме

О ФОРМАЛЬНОЙ ТЕОРИИ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ II

JAN CHRASTINA

В качестве приложения бесконечных продолжений дифференциальных уравнений (диффиэт), в стате приводится метод исследования проблемы Монжа. Эта давняя проблема (~ 1780) касается вопроса, можно ли решить данную неопределённую систему обыкновенных дифференциальных уравнений (больше неизвестных функций, чем уравнений) явными формулами алгебраического вида, содержащими высшие производные некоторых произвольных (параметрических) функций. Предлагаемый метод сводит проблему Монжа к классической проблеме эквивалентности систем Пфаффа (и допускает многие обобщения).

Autor's address: Katedra aplikované matematiky UJEP, Janáčkovo nám. 2a, 662 95 Brno.