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ON OSCILLATORY SOLUTIONS OF THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

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Summary. The subject of this paper are the third order differential equations which have the solution space with bases consisting of 0, 1, 2 or 3 oscillatory solutions. To study such equations we use [3] and seek the possibility of perturbing the self-adjoint differential equation in such a way that both equations be asymptotically equivalent.

Keywords: third order linear differential equation, asymptotic equivalence, oscillatory solutions.

AMS classification: 34C10, 34E10.

1. INTRODUCTION

It will be assumed that the coefficients of the differential equations considered are real continuous functions on $[t\ , \infty)$. We shall call a function $f(t)$ oscillatory when the set of its zero-points is infinite and unbounded from above. Otherwise, we shall call it non-oscillatory.

The third-order linear differential euqation $L_3 y = 0$ can be

I. non-oscillatory, when all its solutions are non-oscillatory;

II. strictly oscillatory, when all its solutions are oscillatory;

III. oscillatory: IIIa. there is only one non-oscillatory solution (up to a constant multiplication factor);

IIIb. there is a two-parameter set of oscillatory solutions;

IIIc. there is only one oscillatory solution.

The equations of types I, II, IIIa and IIIc were studied by several authors (e.g. [2], [7], [8]). The subject of our paper will be the equations of type IIIb.

Let us consider the differential equation

$$y''' + 2\,q(t)\,y' + (q'(t) + r(t))\,y = 0 \tag{1}$$

and its adjoint

$$y''' + 2\,q(t)\,y' + (q'(t) - r(t))\,y = 0\,. \tag{2}$$

If $r(t) = 0$ we have the self-adjoint equation

$$x''' + 2\,q(t)\,x' + q'(t)\,x = 0 \tag{3}$$

and all its solutions are given by

$$x = c_1 z_1^2 + c_2 z_1 z_2 + c_3 z_2^2$$

where z_1, z_2 are linearly independent solutions of

$$z'' + \tfrac{1}{2}\, q(t)\, z = 0\,, \quad q \in C^1[t_0, \infty)\,. \tag{4}$$

In [3, 4] Jones described the types of bases possible for the solution space of (1) with respect to the number of oscillatory solutions possible in a given basis. It is well-known [e.g. 2, Theorem 2.52] that if (4) is oscillatory then (3) has bases consisting of i oscillatory solutions, $i = 0, 1, 2, 3$.

We seek the possibility of perturbing (3) to (1) in such a way that this property be preserved, i.e. the solution space of (1) has bases consisting of exactly i oscillatory solutions, $i = 0, 1, 2, 3$ (Theorem 2). First we shall find sufficient conditions under which (1), (3) are asymptotically equivalent (Theorem 1) and in particular, (1) has a solution such that $\liminf y(t) > 0$ (Corollary 1). Our results include the case when $r(t)$ is oscillatory.

We shall consider equations which are of Class I or Class II as defined by Hanan in [1]. We say that (1) is of Class I or Class II if every solution of (1) satisfying $y(\alpha) = y'(\alpha) = 0$, $y''(\alpha) > 0$, $\alpha > 0$ satisfies also $y(t) > 0$ for $t \in (t_0, \alpha)$ or $t > \alpha$, respectively. In [8] M. Švec studied the effects of these properties on the existence of a solution without zeros.

2. ASYMPTOTIC EQUIVALENCE

Denote by $X = X(t_0)$ and $Y = Y(t_0)$ the sets of all solutions of (3) and (1) on $[t_0, \infty)$, respectively. The continuity of coefficients of equations (1), (3) ensures $X \neq \emptyset$, $Y \neq \emptyset$ and thus X, Y are linear spaces of the dimenion 3.

Theorem 1. *Let every solution of* (4) *be bounded on* $[t_0, \infty)$ *and let*

$$\int^{\infty} |r(t)|\, \mathrm{d}t < \infty\,. \tag{5}$$

Then (1) *and* (3) *are asymptotically equivalent, i.e. there exists a one-to-one mapping* $T: X \to Y$ *such that*

$$\lim_{t \to \infty} |x(t) - Tx(t)| = 0 \quad \textit{for every} \quad x(t) \in X\,.$$

Proof. Our assumptions imply that every solution $y \in Y$ is bounded (see e.g. [2, Theorem 3.16]).

From (1), (3) we get

$$(y - x)''' + 2q(y - x)' + q'(y - x) = -ry$$

and putting $u = y - x$,

$$u''' + 2qu' + q'u = -ry\,. \tag{6}$$

Using the variation-of-constants formula we can write each solution $u(t)$ of (6) in the form

$$u(t) = c_1 z_1^2(t) + c_2 z_1(t) z_2(t) + c_3 z_2^2(t) - \int_{t_0}^{t} K(t, s) r(s) y(s) \, ds ,$$

where the kernel

$$K(t, s) = \frac{1}{2} \begin{vmatrix} z_1(t) & z_2(t) \\ z_1(s) & z_2(s) \end{vmatrix}^2$$

and z_1, z_2 are arbitrary solutions of (4) subject to the Wronskian condition $z_1(t) z_2'(t) - z_1'(t) z_2(t) \equiv 1$.

Thus

$$\begin{aligned} u(t) = {} & z_1^2(t) \left[c_1 - \tfrac{1}{2} \int_{t_0}^{\infty} z_2^2(s) r(s) y(s) \, ds\right] + \\ & + z_1(t) z_2(t) \left[c_2 - \int_{t_0}^{\infty} z_1(s) z_2(s) r(s) v(s) \, ds\right] + \\ & + z_2^2(t) \left[c_3 - \tfrac{1}{2} \int_{t_0}^{\infty} z_1^2(s) r(s) y(s) \, ds\right] + \\ & + \int_t^{\infty} K(t, s) r(s) y(s) \, ds . \end{aligned}$$

Let $y \in Y$ and let

$$c_1 = \tfrac{1}{2} \int_{t_0}^{\infty} z_2^2 r y \, ds , \quad c_2 = \int_{t_0}^{\infty} z_1 z_2 r y \, ds , \quad c_3 = \tfrac{1}{2} \int_{t_0}^{\infty} z_1^2 r y \, ds .$$

Then

(7) $$u(t) = \int_t^{\infty} K(t, s) r(s) y(s) \, ds$$

with the property $\lim_{t \to \infty} u(t) = 0$.

We define a mapping $V: Y \to X$ by the relation

(8) $$(Vy)(t) = y(t) - u(t) = y(t) - \int_t^{\infty} K(t, s) r(s) y(s) \, ds$$

and prove that V is an injection. Note that by virtue of the linearity of the mapping V the function $(Vy)(t)$ is really a solution of (3), i.e. if $y \in Y$ then $Vy \in X$. Suppose on the contrary that there exist $y_1, y_2 \in Y$, $y_1 \neq y_2$ on $[t_0, \infty)$ such that for $x_1 = Vy_1$, $x_2 = Vy_2$ we have $x_1 \doteq x_2$ on $[t_0, \infty)$. Then according to (8)

$$y_1(t) - \int_t^{\infty} K(t, s) r(s) y_1(s) \, ds = y_2(t) - \int_t^{\infty} K(t, s) r(s) y_2(s) \, ds ,$$

thus

(9) $$y_1(t) - y_2(t) = \int_t^{\infty} K(t, s) r(s) (y_1(s) - y_2(s)) \, ds , \quad t \in [t_0, \infty) .$$

Next we prove that the integral equation

$$f(t) = \int_t^{\infty} K(t, s) r(s) f(s) \, ds , \quad t \in [t_0, \infty)$$

has only the trivial solution on $[t_0, \infty)$.

As $|K(t, s)| \leqq A$ for some real A and for all $t \geqq t_0$, $s \geqq t_0$, we have

$$|f(t)| \leqq A \int_t^{\infty} |f(s)| \, |r(s)| \, ds .$$

Put $x = R(t) = \int_t^\infty |r(s)|\, ds$. Then we have

$$|f(R^{-1}(x))| \leqq A \int_0^x |f(R^{-1}(s))|\, ds$$

and the Gronwall inequality yields $|f(R^{-1}(x))| = 0$, i.e. $|f(t)| = 0$.

We conclude that V is a linear injection of the vector spaces X, Y of the same dimension 3, i.e. V is a one-to-one mapping. Thus the mapping $T: X \to Y$ defined by the relation $T = V^{-1}$ has the property required in Theorem 1. Indeed,

$$\lim_{t\to\infty} |x(t) - Tx(t)| = \lim_{t\to\infty} |Vy(t) - V^{-1}(Vy(t))| = \lim_{t\to\infty} |Vy(t) - y(t)| =$$
$$= \lim_{t\to\infty} u(t) = 0 . \quad \square$$

Corollary 1. *Let $q(t) > 0$ be such that q, q^{-1} are bounded and there exists a $\gamma \neq 0$ such that q^γ is either convex or concave. Let (5) hold.*

Then (1) has a nonoscillatory solution $y(t)$ such that $\liminf_{t\to\infty} y(t) > 0$. Furthermore, every solution of (1) is bounded.

Proof. Under the assumptions on q (4) is oscillatory and we can use the asymptotic formulas for the solutions $z_1(t)$, $z_2(t)$ of (4) derived in [6]

$$\begin{aligned} z_1(t) &\sim \quad q^{-1/4}(t) \sin\left(\textstyle\int_{t_0}^t q^{1/2} + o\right), \\ z_1'(t) &\sim \quad q^{+1/4}(t) \cos\left(\textstyle\int_{t_0}^t q^{1/2} + o\right), \\ z_2(t) &\sim \quad q^{-1/4}(t) \cos\left(\textstyle\int_{t_0}^t q^{1/2} + o\right), \\ z_2'(t) &\sim -q^{+1/4}(t) \sin\left(\textstyle\int_{t_0}^t q^{1/2} + o\right). \end{aligned}$$

This implies that every solution of (4) and its derivative are bounded, and a solution $x(t)$ of (3) satisfies

$$\liminf_{t\to\infty} x(t) = \liminf\,(z_1^2(t) + z_2^2(t)) = \liminf q^{-1/2}(t) = [\limsup q(t)]^{-1/2} > 0 .$$

Theorem 1 yields the existence of a solution $y(t)$ of (1) such that $\liminf y(t) \geqq \geqq \liminf x(t) + \lim u(t) > 0$.

Corollary 2. *Let $\lim_{t\to\infty} q(t) = c > 0$ and let there exist a $\gamma \neq 0$ such that q^γ is either convex or concave. Let (5) hold. Then (1) has a nonoscillatory solution $y(t)$ such that*

$$\lim y(t) = 1/\sqrt{c} .$$

Proof. It is similar to that of Corollary 1.

3. OSCILLATORY SOLUTIONS

Theorem 2. *Let (1) be of Class I or Class II, oscillatory and let the assumptions of Corollary 1 be fulfilled.*

Then the solution space of (1) *has bases consisting of exactly i oscillatory solutions,* $i = 0, 1, 2, 3$.

For the proof of Theorem 2 we need the following

Proposition 1 [3, Theorem 1]. *If* (1) *is of Class I and if some of its solutions oscillates then the solution space of* (1) *has a basis with three oscillatory solutions and a basis with exactly two oscillatory solutions.*

Proposition 2 [3, Theorem 2]. *If* (1) *is of Class II and if some of its solutions oscillates then the solution space of* (1) *has a basis consisting of exactly i oscillatory solutions, for* $i = 0, 1, 2$.

Proposition 3 [3, Theorem 4]. *If* (1) *is of Class I, if some of its solutions oscillates and if it has a basis with two or three nonoscillatory elements then* (2) *has a basis with three oscillatory elements.*

Proof of Theorem 2. We will need the fact that if y_1, y_2, y_3 are linearly independent solutions of (1) then so are $y_1, y_1 + y_2, y_2 + y_3$ or $y_1 + y_2, y_1 + y_3, y_1 + y_2 + y_3$. This easily follows from the fact that they have the same wronskian.

Suppose (1) is of Class I. Since (1) has an oscillatory solution according to Proposition 1 the equation (1) has bases with i oscillatory solutions, $i = 3, 2$. Thus it remains to prove the existence of bases with i oscillatory solutions, $i = 0, 1$.

By Corollary 1 the equation (1) has a nontrivial nonoscillatory solution $w(t)$ such that $\liminf w(t) = c > 0$. On the other hand, by Proposition 1 we have two linearly independent oscillatory solutions $u(t), v(t)$ which together with $w(t)$ form a basis for the solutions of (1). According to Corollary 1 the solutions $u(t), v(t)$ are bounded by $N > 0$. If we take the nonoscillatory solution $w^*(t) := 2N/c\, w(t)$ then the solution $u + w^*$, $v + w^*$ both are nonoscillatory and together with $u(t)$ form a basis for (1). Indeed,

$$\liminf (u + w^*) \geqq \liminf u + \liminf w^* =$$
$$= \liminf u + \frac{2N}{c} c \geqq -N + 2N = N > 0 .$$

Analogously, if we put $w^{**} := 3N/c\, w(t)$ then the solutions $u + w^*$, $v + w^*$, $u + v + w^{**}$ are nonoscillatory and form a basis for (1).

Now let (1) be of Class II. By Proposition 2 we get that (1) has a basis consisting of exactly i oscillatory solutions, $i = 0, 1, 2$. Let us prove the existence of a basis with three oscillatory solutions. It was shown in [1] that (1) is of Class II if and only if (2) is of Class I. Considering the equation (2) which also has an oscillatory solution (see [1]) we have from the first part of the proof that (2) has a basis with two and three nonoscillatory elements. Now, Proposition 3 gives the existence of a basis with three oscillatory elements which was to prove.

Corollary 3. *Let $q(t)$ satisfy the assumptions of Corollary 1 and let $r(t)$ be such that either $r(t) \geqq 0$ or $r(t) \leqq 0$ on $[t_0, \infty)$, $r(t) \equiv 0$ does not hold on any subinterval, and $\int^\infty r(t)\,\mathrm{d}t$ converges.*

Then the conclusion of Theorem 2 holds.

Proof. If $r(t) \geqq 0$ then (1) is of Class I and (1), (2) are oscillatory. Indeed, this follows e.g. from [2, Theorem 2.61] because q, q^{-1} are bounded.

If $r(t) \leqq 0$ then (2) is of Class I and (1) of Class II. Thus all assumptions of Theorem 2 are fulfilled.

Concluding remark. The problem of structure of the solution space of (1) with respect to the number of oscillatory solutions remains open in the following cases:

1) $r(t)$ satisfies (5) and

i) $q(\infty) = \infty$, or

ii) $q(\infty) = 0$;

2) $\int^\infty r(t)\,\mathrm{d}t$ diverges.

Suppose that 1i) holds. Let $\gamma \in (0, 1/2)$ exist such that $q^{-\gamma}$ is either convex or concave. Then by the same asymptotic formulas as in the proof of Corollary 1 we get that every solution of (4) tends to zero and thus every solution $x(t)$ of (3) tends to zero. In this case the problem of existence of two and three nonoscillatory solutions is open.

Suppose that 2ii) holds. Let $\int^\infty q^{-5/2} q'^2 < \infty$ and let there exist a $\gamma > 0$ such that q^γ is convex. Then every solution $x(t)$ of (3) satisfies $\limsup_{t \to \infty} x(t) = \infty$. The validity of Theorem 1 for this case would entail the validity of Theorem 2.

As concerns the case 2, we mention that the following theorem (see [2, Theorem 3.6] or [5]) holds. If $q(t) \geqq 0$, $q'(t) + r(t) \geqq d > 0$, $r(t) - q'(t) \geqq 0$ then every solution of (1) is oscillatory on (t_0, ∞) except one solution $y(t)$ with the property $y \to 0$, $y' \to 0$ as $t \to \infty$, i.e. (1) is of type IIIa.

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Souhrn

O OSCILATORICKÝCH ŘEŠENÍCH LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 3. ŘÁDU

ZUZANA DOŠLÁ

Předmětem článku jsou lineární diferenciální rovnice 3. řádu, jejichž prostor řešení má báze obsahující právě i oscilatorických řešení, pro všechna $i = 0, 1, 2, 3$. Nejprve hledáme asymptoticky ekvivalentní perturbaci samoadjungované rovnice, odkud dostaneme existenci jistého neoscilatorického řešení, a pak použijeme výsledků [3].

Резюме

О КОЛЕБЛЮЩИХСЯ РЕШЕНИЯХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ 3-ГО ПОРЯДКА

ZUZANA DOŠLÁ

Предмет статьи — линейные дифференциальные уравнения 3-го порядка, пространство решений которых обладает базисом, содержащим ровно i колеблющихся решений, для всех $i = 0, 1, 2, 3$. Сначала ищется асимптотически эквивалентное возмущение самосопряженного уравнения, откуда следует существование некоторого неосцилляторического решения, и затем применяется работа [3].

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