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ON L_p — SOLUTIONS OF n-TH ORDER NONLINEAR DIFFERENTIAL EQUATION

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Summary. In this paper, $L_p(R_+)$ — integrability of solutions of the n-th order nonlinear differential equation with righthand side satisfying certain conditions is studied.

Keywords: L_p — solutions, Gronwall's lemma, Bihari's lemma.

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Let an n-th order nonlinear differential equation

$$y^{(n)} + \sum_{k=0}^{n-1} f_k(t)\, y^{(k)} = g(t, y, y', \ldots, y^{(n-1)})\,, \tag{1}$$

be given, where $f_k\colon R_+ \to R$, $k = 0, 1, \ldots, n-1$ and $g\colon D = R_+ X R^n \to R$ are continuous functions and n is a natural number.

Throughout the paper we assume that any solution of the differential equation (1) exists on R_+. We give some sufficient conditions which imply that all solutions $y(t)$ of the differential equation (1) are of the class $L_p(R_+)$, $1 < p < \infty$. In this paper some results of [1] will be generalised.

Let $x_i(t)$, $i = 1, 2, \ldots, n$ be linear by independent solutions of the differential equation

$$x^{(n)} + \sum_{k=0}^{n-1} f_k(t)\, x^{(k)} = 0\,. \tag{2}$$

It is well-known (e.g. [3]) that an arbitrary solution $y(t)$ of the differential equation (1) satisfies

$$y^{(k)}(t) = \sum_{i=1}^{n} C_i\, x_i^{(k)}(t) + \int_0^t \frac{W_k(t, s)}{W(s)}\, g(s, y(s), y'(s), \ldots, y^{(n-1)}(s))\, \mathrm{d}s\,, \tag{3}$$

$t \in R_+$, $k = 0, 1, \ldots, n-1$, where C_i are arbitrary real numbers, $i = 1, 2, \ldots, n$, $W(t)$ is the Wronskian of the solutions $x_1, x_2, \ldots, x_n$ and $W_k(t, s)$ denotes the determinant

$$
(4) \qquad W_k(t, s) = \begin{vmatrix} x_1(s) & x_2(s) & \dots & x_n(s) \\ x_1'(s) & x_2'(s) & \dots & x_n'(s) \\ \dots & \dots & \dots & \dots \\ x_1^{(n-2)}(s) & x_2^{(n-2)}(s) & \dots & x_n^{(n-2)}(s) \\ x_1^{(k)}(t) & x_2^{(k)}(t) & \dots & x_n^{(k)}(t) \end{vmatrix},
$$

$k = 0, 1, \dots, n - 1$. It is clear that

$$
W_k(t, s) = \frac{\partial^k W_0(t, s)}{\partial t^k} \quad \text{for all} \quad t, s \in R_+, \quad s \leqq t, \quad k = 1, 2, \dots, n - 1.
$$

For all $s \in R_+$ let us define a function

$$
(5) \qquad G(s) = \max \{|W_{ki}(s)|, \ i = 1, 2, \dots, n; \ k = 0, 1, \dots, n - 1\}
$$

where W_{ki}, $i = 1, 2, \dots, n$ are the determinants obtained from (4) by deleting the i-th column and the n-th row.

From (3) using (5) we get

$$
6) \qquad |y^{(k)}(t)| \leqq \left\{C + \int_0^t \left|\frac{G(s)}{W(s)} g(s, y(s), y'(s), \dots, y^{(n-1)}(s))\right| \mathrm{d}s\right\} \sum_{i=1}^n |x_i^{(k)}(t)|,
$$

$t \in R_+$, $k = 0, 1, \dots, n - 1$, where $C = \max \{|C_i|, \ i = 1, 2, \dots, n\}$.

Theorem 1. *Let m be a fixed integer, $0 \leqq m \leqq n - 2$, and suppose that the following conditions are satisfied:*

1) *for all solutions $x(t)$ of the differential equation (2) we have*

$$
x^{(j)}(t) \in L_p(R_+), \quad 1 < p < \infty, \quad j = 0, 1, \dots, m;
$$

2) *for all solutions $y(t)$ of the differential equation (1) we have*

$$
y^{(j)}(t) \in L_p(R_+), \quad 1 < p < \infty, \quad j = m + 1, m + 2, \dots, n - 1;
$$

$$
3) \qquad \left|\frac{G(t)}{W(t)} g(t, u_1, u_2, \dots, u_n)\right| \leqq h(t) + \sum_{i=1}^n S(t) |u_i| \quad \text{in} \quad D,
$$

where $h: R_+ \to R^+$ are continuous functions;

$$
4) \qquad h(t) \in L_1(R_+), \quad S(t) \in L_r(R_+) \quad \textit{and} \quad \frac{1}{p} + \frac{1}{r} = 1.
$$

Then for all solutions $y(t)$ of the differential equation (1) we have

$$
y^{(j)}(t) \in L_p(R_+), \quad j = 0, 1, \dots, m; \quad 1 < p < \infty.
$$

Proof. From (6) using the conditions 2), 3), 4) of the above theorm, Hölder's inequality and the inequality

$$
(*) \qquad \left(\sum_{i=1}^n a_i\right)^p \leqq 2^{(n-1)(p-1)} \sum_{i=1}^n a_i^p, \quad a_i > 0, \quad i = 1, 2, \dots, n
$$

we get

$$(7)\qquad |y^{(k)}(t)| \leqq \{C + \textstyle\int_0^t [h(s) + \sum_{j=0}^{n-1} S(s)\,|y^{(j)}(s)|]\,\mathrm{d}s\} \sum_{i=1}^{n} |x_i^{(k)}(t)| \leqq$$

$$\leqq \{C + \textstyle\int_0^t h(s)\,\mathrm{d}s + \sum_{j=0}^{n-1} (\int_0^t S^r(s)\,\mathrm{d}s)^{1/r} (\int_0^t |y^{(j)}(s)|^p\,\mathrm{d}s)^{1/p}\} \sum_{i=1}^{n} |x_i^{(k)}(t)| \leqq$$

$$\leqq \{m_1 + m_2 \textstyle\sum_{j=0}^{m} (\int_0^t |y^{(j)}(s)|^p\,\mathrm{d}s)^{1/p}\} \sum_{i=1}^{n} |x_i^{(k)}(t)|\,,$$

$t \in R_+$, $k = 0, 1, \ldots, n - 1$ and $m_1, m_2 > 0$ are constants.

Now by taking the p-th power of (7) and applying the inequality $(*)$ we get

$$(8)\qquad |y^{(k)}(t)|^p \leqq \{m_3 + m_4 \textstyle\sum_{j=0}^{m} \int_0^t |y^{(j)}(s)|^p\,\mathrm{d}s\} \sum_{i=1}^{n} |x_i^{(k)}(t)|^p\,,$$

$t \in R_+$, $k = 0, 1, \ldots, n - 1$ and $m_3, m_4 > 0$ are constants.

Putting $k = 0, 1, \ldots, m$ in (8) we obtain a system of $m + 1$ inequalities and summing them we obtain

$$(9)\qquad \sum_{k=0}^{m} |y^{(k)}(t)|^p \leqq \{m_3 + m_4 \textstyle\sum_{j=0}^{m} \int_0^t |y^{(j)}(s)|^p\,\mathrm{d}s\} \sum_{k=0}^{m} \sum_{i=1}^{n} |x_i^{(k)}(t)|^p \leqq$$

$$\leqq m_3 \sum_{k=0}^{m} \sum_{i=1}^{n} |x_i^{(k)}(t)|^p + \sum_{k=0}^{m} \sum_{i=1}^{n} |x_i^{(k)}(t)|^p\, m_4 \int_0^t \sum_{j=0}^{m} |y^{(j)}(s)|^p\,\mathrm{d}s\,,$$

$$t \in R_+\,.$$

Now we may integrate (9) from 0 to t, and by using the condition 1) of Theorem 1 we get

$$(10)\qquad \int_0^t \sum_{k=0}^{m} |y^{(k)}(s)|^p\,\mathrm{d}s \leqq m_5 + m_4 \int_0^t \sum_{k=0}^{m} \sum_{i=1}^{n} |x_i^{(k)}(s)|^p \int_0^s \sum_{j=0}^{m} |y^{(j)}(u)|^p\,\mathrm{d}u\,\mathrm{d}s\,,$$

$t \in R_+$ and $m_5 > 0$ is a constant.

From (10) applying Gronwall's lemma and using the condition 1) of Theorem 1 we obtain the inequality

$$(11)\qquad \int_0^t \sum_{k=0}^{m} |y^{(k)}(s)|^p\,\mathrm{d}s \leqq m_5 \exp m_4 \int_0^t \sum_{k=0}^{m} \sum_{i=1}^{n} |x_i^{(k)}(s)|^p\,\mathrm{d}s \leqq K\,,$$

$t \in R_+$, where $K > 0$ is a constant.

Now, it follows from (11) that $y^{(k)}(t) \in L_p(R_+)$, $k = 0, 1, \ldots, m$, $1 < p < \infty$, and the proof is complete.

Remark 1. Theorem 1 in [1] is a special case of the above theorem for $n = 2$, $f_1(t) \equiv 0$.

Analogously we can prove

Theorem 2. *Let m be a fixed integer, $0 \leqq m \leqq n - 2$, and suppose that the following conditions are satisfied:*

1) *for all solutions* $x(t)$ *of the differential equation* (2) *we have* $x^{(j)}(t) \in L_p(R_+)$, $j = m + 1, m + 2, \ldots, n - 1$ *and* $1 < p < \infty$;

2) *for all solutions* $y(t)$ *of the differential equation* (1) *we have* $y^{(j)}(t) \in L_p(R_+)$, $j = 0, 1, \ldots, m$, $1 < p < \infty$;

3) $$\left|\frac{G(t)}{W(t)}\, g(t, u_1, u_2, \ldots, u_n)\right| \leqq h(t) + \sum_{i=1}^{n} S(t)\,|u_i| \quad \text{in} \quad D\,,$$

where $h\colon R_+ \to R^+$ *and* $S\colon R_+ \to R^+$ *are continuous functions;*

4) $$h(t) \in L_1(R_+)\,, \quad S(t) \in L_r(R_+) \quad \text{and} \quad \frac{1}{p} + \frac{1}{r} = 1\,.$$

Then for all solutions $y(t)$ *of the differential equation* (1) *we have* $y^{(k)}(t) \in L_p(R_+)$, $k = m + 1, m + 2, \ldots, n - 1$, $1 < p < \infty$.

Theorem 3. *Let* j *be a fixed integer,* $0 \leqq j \leqq n - 1$, *and suppose that*

1) *for all solutions* $x(t)$ *of the differential equation* (2) *we have* $x^{(j)}(t) \in L_p(R_+)$, $1 < p < \infty$;

2) $$\left|\frac{G(t)}{W(t)}\, g(t, u_0, u_1, \ldots, u_{n-1})\right| \leqq h(t) + S(t)\,|u_j|^a\,, \quad 0 < a \leqq 1$$

on D, *where* $h\colon R_+ \to R^+$ *and* $S\colon R_+ \to R^+$ *are continuous functions;*

3) $$h(t) \in L_1(R_+)\,, \quad S(t) \in L_r(R_+) \quad \textit{and} \quad \frac{a}{p} + \frac{1}{r} = 1\,.$$

Then for all solutions $y(t)$ *of the differential equation* (1) *we have* $y^{(j)}(t) \in L_p(R_+)$, $1 < p < \infty$.

Proof. It is readily seen from (6) by using the conditions 2) and 3) of Theorem 3 and Hölder's inequality that the following inequality holds:

$$(12)\qquad |y^{(j)}(t)| \leqq \{C + \textstyle\int_0^t [h(s) + S(s)\,|y^{(j)}(s)|^a]\,ds\} \displaystyle\sum_{i=1}^{n} |x_i^{(j)}(t)| \leqq$$

$$\leqq \{C + \textstyle\int_0^t h(s)\,ds + (\int_0^t S^r(s)\,ds)^{1/r} (\int_0^t [|y^{(j)}(s)|^a]^{p/a}\,ds)^{a/p}\} \displaystyle\sum_{i=1}^{n} |x_i^{(j)}(t)| \leqq$$

$$\leqq \{m_1 + m_2(\textstyle\int_0^t |y^{(j)}(s)|^p\,ds)^{a/p}\} \displaystyle\sum_{i=1}^{n} |x_i^{(j)}(t)|\,,$$

$t \in R_+$ and $m_1, m_2 > 0$ are constants.

By taking the p-th power of (12) and applying the inequality $(*)$ we get

$$(13)\qquad |y^{(j)}(t)|^p \leqq \{m_3 + m_4(\textstyle\int_0^t |y^{(j)}(s)|^p\,ds)^a\} \displaystyle\sum_{i=1}^{n} |x_i^{(j)}(t)|^p \leqq$$

$$\leqq m_3 \sum_{i=1}^{n} |x_i^{(j)}(t)|^p + m_4 \sum_{i=1}^{n} |x_i^{(j)}(t)|^p \left(\int_0^t |y^{(j)}(s)|^p \,ds\right)^a ,$$

$t \in R_+$ and $m_3, m_4 > 0$ are constants.

Now, integrating (13) from 0 to t, $t \in R_+$, and using the condition 1) of Theorem 3 we obtain

$$(14) \qquad \int_0^t |y^{(j)}(s)|^p \,ds \leqq m_5 + m_4 \int_0^t \sum_{i=1}^{n} |x_i^{(j)}(s)|^p \left(\int_0^s |y^{(j)}(u)|^p \,du\right)^a ds ,$$

$t \in R_+$ and $m_5 > 0$ is a constant.

For $a = 1$ we may use Gronwall's lemma and from (14) by using the condition 1) of Theorem 3 we obtain

$$(15) \qquad \int_0^t |y^{(j)}(s)|^p \,ds \leqq m_5 \exp m_4 \int_0^t \sum_{i=1}^{n} |x_i^{(j)}(s)|^p \,ds \leqq m_6 ,$$

where $m_6 > 0$ is a constant. Now, it follows from (15) that $y^{(j)}(t) \in L_p(R_+)$, $1 < p < \infty$.

For $0 < a < 1$ we may use Bihari's lemma and from (14) using the condition 1) of Theorem 3 we get

$$(16) \qquad \int_0^t |y^{(j)}(s)|^p \,ds \leqq \left[m_5^{(1-a)} + (1 - a)\, m_4 \sum_{i=1}^{n} |x_i^{(j)}(s)|^p \,ds\right]^{1/(1-a)} \leqq$$

$\leqq m_7$, where $m_7 > 0$ is a constant. It follows from (16) that $y^{(j)}(t) \in L_p(R_+)$, $1 < p < \infty$. The theorem is proved.

Remark 2. Corollary 1 in [1] is a special case of the above theorem for $n = 2$, $f_1(t) \equiv 0$, $a = 1$, $j = 0$.

Remark 3. Corollary 2 in [1] is a special case of the above theorem for $n = 2$, $f_1(t) \equiv 0$. $0 < a < 1$, $j = 0$.

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Souhrn

O L_p-RIEŠENIACH NELINEÁRNEJ DIFERENCIÁLNEJ ROVNICE n-TÉHO RÁDU

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V práci sa študuje L_p-integrovateľnosť riešení nelineárnej diferenciálnej rovnice n-tého rádu (a ich derivácií) ak pravá strana rovnice spĺňa isté podmienky.

Резюме

О L_p-РЕШЕНИЯХ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ПОРЯДКА n

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В работе изучается принадлежность классу $L_p(R_+)$ решений и их производных нелинейного дифференциального уравнения n-того порядка, правая часть которого удовлетворяет определенным условиям.

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