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CONTINUITY OF LIFTINGS

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Summary. Conditions are given under which $L(M)\,\sigma_m(v_m)$ tend to $L(M)\,\sigma(v)$, where L is a lifting, M a manifold, σ_m and σ are sections defined in a neighbourhood of $x \in M$ such that $j_x^\infty(\sigma_m)$ tend to $j_x^\infty(\sigma)$, and v_m is a sequence of points over x tending to v.

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Let F and G be two natural bundles over n-dimensional manifolds. Let H be a natural bundle over dim (GR^n)-dimensional manifolds. ([4]). If U is an open subset of an n-manifold M, then a mapping $\sigma: U \to FM$ (or $\varrho: (\pi_M^G)^{-1}(U) \to HGM$) of class C^∞ such that $(\pi_M^F) \circ \sigma = \mathrm{id}_U$ $(\pi_{GM}^H) \circ \varrho = \mathrm{id}_{(\pi^G{}_M)^{-1}(U)})$ is called a section of $\pi_M^F: FM \to M$ $(\pi_{GM}^H: HGM \to GM)$. If M is an n-manifold, we denote by $\mathscr{F}M(\mathscr{H}\mathscr{G}M)$ the set of section of $FM \to M$ $(HGM \to GM)$. If φ is an embedding of an n-manifold M into an n-manifold N, we define $\varphi_*: \mathscr{F}M \to \mathscr{F}N$ and $(G\varphi)_*: \mathscr{H}\mathscr{G}N \to \mathscr{H}\mathscr{G}N$ by $\varphi_*\sigma = F\varphi \circ \sigma \circ \varphi^{-1}$ and $(G\varphi)_*\,\varrho = (HG\varphi) \circ \varrho \circ (G\varphi)^{-1}$. With each n-manifold M we associate a mapping $L(M): \mathscr{F}M \to \mathscr{H}\mathscr{G}M$, which is natural for embeddings. That is to say, for each embedding φ of an n-manifold M into an n-manifold N, we have $L(N) \circ \varphi_* = (G\varphi)_* \circ L(M)$.

A family $L = \{L(M)\}$ is called an (n, F, G, H)-lifting.

Examples. (1) Let F and H be two natural bundles over n-manifolds. Let G be the identity functor over n-manifolds. Let $D = \{D(M)\}$ be a natural differential operator ([6]) such that for each n-manifold M, $D(M): \mathscr{F}M \to \mathscr{H}M$. Then D is an (n, F, G, H)-lifting. In particular, if F is the functor of positive-defined symmetric $(0, 2)$-tensors and H is the functor of (p, q)-tensors, then D is called a natural tensor ([1]). Hence natural tensors are liftings.

(2) Let F be the functor of tangent bundles (or (0,0)-tensors) over n-manifolds. Let G be a natural bundle over n-manifolds. Let H be the functor of tangent bundles (or $(0, 0)$-tensors) over dim (GR^n)-manifolds. Let $L = \{L(M)\}$ be a lifting of vector fields to G (or a lifting of functions to G) (see [2], [3]). Then L is an (n, F, G, H)-lifting.

The main theorem of this paper reads as follows.

Theorem. *Let L be an (n, F, G, H)-lifting. Let M be an n-manifold and $\sigma \in \mathscr{F}M$ a section defined on a neighbourhood of $x \in M$ and satisfying the following condition:*

($*$) There exists a vector field X defined on a neighbourhood of x such that $X(x) \neq 0$ and $j_x^\infty(L_X\sigma) = j_x^\infty(0)$. Moreover, let $X(x) \neq 0$ and $j_x^\infty(L_X\sigma) = j_x^\infty(0)$.

Let $\sigma_m \in \mathscr{F}M$ $(m = 1, 2, 3, \ldots)$ be a sequence of sections such that $j_x^\infty(\sigma_m)$ tend to $j_x^\infty(\sigma)$ if m tends to infinity. Let $v_m \in (\pi_M^G)^{-1}(x)$ $(m = 1, 2, 3, \ldots)$ be a sequence of points tending to v. Then $L(M)\,\sigma_m(v_m)$ tend to $L(M)\,\sigma(v)$.

Remark. $L_X\sigma$ is the Lie derivative of σ with respect to X. If $y \in \operatorname{dom}(X) \cap \cap \operatorname{dom}(\sigma)$, then $L_X\,\sigma(y)$ is the vector from $T_{\sigma(y)}FM$ given by the curve $t \to (\varphi_{-t})_* \,.\, . \,\sigma(y)$, where $\{\varphi_t\}$ is a local 1-parameter group of X.

If φ is an embedding of an n-manifold M into an n-manifold N, then $\varphi_*(L_X\sigma) = = L_{\varphi_*X}\varphi_*\sigma$ (see [6]). We denote by 0 the mapping given by $M \ni y \to 0 \in T_{\sigma(y)}FM$.

Remark. The counterexample of D. B. A. Epstein [1, p. 638–641] shows why we insist that σ should satisfy ($*$).

From now on, we denote by π the given map from GR^n to R^n. We write F_0 instead of $(\pi_{R^n}^F)^{-1}(0)$ and G_0 instead of $\pi^{-1}(0)$. Let $s = \dim(F_0)$. If $x \in R^n$, we denote by τ_x the translation by $x (\tau_x: R^n \to R^n, \tau_x(y) = x + y)$. We have the C^∞-diffeomorphism $T: R^n \times F_0 \to FR^n$ given by $(x, f) \to F\,\tau_x(f)$. We write L instead of $L(R^n)$. We denote by P the projection $R^n \times F_0 \to F_0$, and by $p: R^n \to R$ the projection $(x_1, \ldots, x_n) \to x_1$.

We prove two lemmas.

Lemma 1. *Let $\sigma_1, \sigma_2 \in \mathscr{F}R^n$ be two sections such that $0 \in \operatorname{dom}(\sigma_t)$ $(t = 1, 2)$ and $j_0^\infty(\sigma_1) = j_0^\infty(\sigma_2)$. Then $L\sigma_1$ is equal to $L\sigma_2$ on G_0.*

Proof. Choose a chart (U, ψ) on F_0 such that $P \circ T^{-1} \circ \sigma_0(0) \in U$. Putting $f_t = \psi \circ P \circ T^{-1} \circ \sigma_t$ $(t = 1, 2)$ we find that $j_0^\infty(f_1) = j_0^\infty(f_2)$. By Whitney's extension theorem [5] there exist a C^∞-mapping $f: R^n \to R^s$ and an open neighbourhood W of 0 such that $f = f_t$ on $V_t = \{(x_1, \ldots, x_n) \in \overline{W}: (-1)^t x_1 \geqq n|x_i| \text{ for } 2 \leqq i \leqq n\}$ for $t = 1, 2$. Let $\bar{\sigma} \in \mathscr{F}R^n$ be given by $\bar{\sigma}(x) = T(x, \psi^{-1} \circ f(x))$. Then $\bar{\sigma} = \sigma_t$ on V_t for $t = 1, 2$. Hence $L\bar{\sigma} = L\sigma_t$ on $\pi^{-1}(\operatorname{int} V_t)$ for $t = 1, 2$. Since $G_0 \subset \operatorname{cl}(\pi^{-1}(\operatorname{int} V_t))$ we obtain that $L\sigma_1 = L\sigma_2$ on G_0.

Lemma 1 is proved.

Lemma 2. *Let $\sigma \in \mathscr{F}R^n$ be a section such that $0 \in \operatorname{dom}(\sigma)$ and $j_0^\infty(L_{\partial/\partial x_1}\sigma) = j_0^\infty(0)$. Then there exist a section $\tilde{\sigma} \in \mathscr{F}R^n$ and a chart (U, φ) on F_0 such that $\tilde{\sigma}(0) \in U$, $j_0^\infty(\tilde{\sigma}) = j_0^\infty(\sigma)$ and $\partial/\partial x_1 \tilde{f} \equiv 0$, where $\tilde{f} = \varphi \circ P \circ T^{-1} \circ \tilde{\sigma}$.*

Proof. Choose a chart (U, φ) on F_0 such that $\sigma(0) \in U$. Let $\psi = = (T \circ (\mathrm{id}_{R^n} \times \varphi^{-1}))^{-1}$. Putting $f = \varphi \circ P \circ T^{-1} \circ \sigma$, we find $\varepsilon > 0$ such that

$\psi \circ (\tau_{(-t,0,\ldots,0)})_* \sigma(x) = (x, f \circ \tau_{(t,0,\ldots,0)}(x))$ for $\|x\| < \varepsilon$, $|t| < \varepsilon$. It follows (since $j_0^\infty(L_{\partial/\partial x_1}\sigma) = j_0^\infty(0)$) that $j_0^\infty(\partial/\partial x_1 f) = 0$. On some open neighbourhood W of $0 \in \in \boldsymbol{R}^n$, define $\tilde{f}$: $W \to \boldsymbol{R}^s$ by $\tilde{f}(x_1, \ldots, x_n) = f(0, x_2, \ldots, x_n)$. Then $j_0^\infty(\tilde{f}) = j_0^\infty(f)$. Let $\tilde{\sigma} \in \mathscr{F}\boldsymbol{R}^n$ be given by $\tilde{\sigma}(x) = T(x, \varphi^{-1} \circ \tilde{f}(x))$. It is easy to verify that $j_0^\infty(\tilde{\sigma}) = j_0^\infty(\sigma)$ and $\tilde{f} = \varphi \circ P \circ T^{-1} \circ \tilde{\sigma}$.

Lemma 2 is proved.

Proof of the theorem. Since $X(x) \neq 0$, we may of course assume that $M = \boldsymbol{R}^n$, $x = 0$ and $X = \partial/\partial x_1$. By Lemmas 1 and 2 we may assume that there exists a chart (U, φ) on F_0 such that $\sigma(0) \in U$ and $\partial/\partial x_1 f \equiv 0$, where $f = \varphi \circ P \circ T^{-1} \circ \sigma$. We show that any subsequence of $L\sigma_m(v_m)$ contains another subsequence tending to $L\sigma(v)$. This is sufficient to establish the result.

Let $f_m = \varphi \circ P \circ T^{-1} \circ \sigma_m$ $(m = 1, 2, 3, \ldots)$. By passing to subsequences, we may assume that $\|D(f_m - f)(0)\| < \exp(-m)$ for each differential operator obtained by partially differentiating at most m-times (so D is a monomial in the $\partial/\partial x_i$). Let $x_m = (1/m, 0, \ldots, 0) \in \boldsymbol{R}^n$. By Whitney's extension theorem [5] there is a C^∞-mapping $h: \boldsymbol{R}^n \to \boldsymbol{R}^s$ such that $j_{x_m}^\infty(h)$ is equal to 0 if m is odd and to $j_{x_m}^\infty((f_m - f) \circ \tau_{-x_m})$ if m is even, for m sufficiently large. Let $\tilde{h} = h + f$.

Since $\partial/\partial x_1 f \equiv 0$, we obtain that $j_{x_m}^\infty(\tilde{h})$ is equal to $j_{x_m}^\infty(f)$ if m is odd and to $j_{x_m}^\infty(f_m \circ \tau_{-x_m})$ if m is even, for m sufficiently large. Define $\tilde{\sigma} \in \mathscr{F}\boldsymbol{R}^n$ by $\tilde{\sigma}(x) = T(x, \varphi^{-1} \circ \tilde{h}(x))$. Then $j_0^\infty((\tau_{-x_m})_* \tilde{\sigma})$ is equal to $j_0^\infty((\tau_{-x_m})_* \sigma)$ if m is odd and to $j_0^\infty(\sigma_m)$ if m is even, for m sufficiently large. By Lemma 1, we obtain that $HG\tau_{-x_m} \circ L\tilde{\sigma} \circ G\,\tau_{x_m}(v_m)$ is equal to $HG\tau_{-x_m} \circ L\sigma \circ G\,\tau_{x_m}(v_m)$ if m is odd and to $L\sigma_m(v_m)$ if m is even, for m sufficiently large. Therefore $L\sigma_{2m}(v_{2m})$ tends to $L\sigma(v)$ as required.

The theorem is proved.

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Souhrn

SPOJITOST LIFTŮ

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Jsou udány podmínky, za kterých $L(M)\,\sigma_m(v_m)$ konverguje k $L(M)\,\sigma(v)$, kde L je lift, M varieta, σ_m, σ jsou řezy definované na okolí bodu $x \in M$ a splňující $j_x^\infty(\sigma_m) \to j_x^\infty(\sigma)$, a v_m je posloupnost bodů nad x konvergující k v.

Резюме

НЕПРЕРЫВНОСТЬ ЛИФТИНГОВ

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В работе даны условия, при которых $L(M)\,\sigma_m(v_m)$ стремится к $L(M)\,\sigma(v)$, где L — лифтинг, M — многообразие, σ_m, σ — сечения, определеные на окрестности точки $x \in M$ и такие, что $j_x^\infty(\sigma_m) \to j_x^\infty(\sigma)$, и v_m — сходящаяся последовательность лежащая над x с пределом v.

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