Shigeyoshi Owa

On certain classes of $p$-valent functions

Persistent URL: http://dml.cz/dmlcz/118352

# ON CERTAIN CLASSES OF $p$-VALENT FUNCTIONS

Shigeyoshi Owa, Osaka

*Summary.* We introduce the classes $T^*(n + p - 1, \alpha)$ of analytic $p$-valent functions with negative coefficients by using the symbol $D^{n+p-1} f(z)$ defined by $\{z^p/(1 - z)^{n+p}\} * f(z)$. The object of the present paper is to show distortion theorems and some closure theorems for functions $f(z)$ belonging to the class $T^*(n + p - 1, \alpha)$. Further, we consider the modified Hadamard product of functions $f(z)$ in $T^*(n + p - 1, \alpha)$.

*Keywords:* analytic function, $p$-valent function, distortion theorem, closure theorem, Hadamard product.

*AMS (MOS) subject classification* (1980): 30C45.

## 1. INTRODUCTION

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$(1.1) \qquad f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, 3, \ldots\})$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Further, let $f(z) \in A(p)$ and $g(z) \in A(p)$. Then we denote by $f * g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by $(1.1)$ and $g(z)$ is given by

$$(1.2) \qquad g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (p \in N),$$

then

$$(1.3) \qquad f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

With the aid of the Hadamard product, we introduce the symbol

$$(1.4) \qquad D^{n+p-1} f(z) = \left\{ \frac{z^p}{(1 - z)^{n+p}} \right\} * f(z)$$

for $n$, where $n$ is any integer greater than $-p$. Then we can observe that

$$(1.5) \qquad D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n + p - 1)!}.$$

In particular, the symbol $D^n f(z)$ was introduced by Ruscheweyh [7] and was named the $n$th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1].

Now, we introduce the following classes by using the symbol $D^{n+p-1} f(z)$.

**Definition.** We say that $f(z)$ is in the class $T^*(n + p - 1, \alpha)$, if $f(z)$ defined by

$$(1.6) \qquad f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geqq 0)$$

satisfies the condition

$$(1.7) \qquad Re \left\{ \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} \right\} > \alpha \quad (z \in U)$$

for $p \in N$, $n > -p$ and $0 \leqq \alpha \leqq \frac{1}{2}$.

Particularly, $T^*(n + p - 1, \alpha)$ is a subclass of $T(n, \alpha)$ which was studied by Goel and Sohi [3]. Further, by using the symbol $D^{n+p-1} f(z)$, other classes were studied by Goel and Sohi [4], [5], Sohi [8], Fukui and Sakaguchi [2] and Owa [6].

## 2. DISTORTION THEOREMS

**Theorem 1.** Let the function $f(z)$ be defined by $(1.6)$. Then $f(z)$ is in the class $T^*(n + p - 1, \alpha)$ if and only if

$$(2.1) \quad \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \{(n + p)(1 - \alpha) + k\}}{k!} a_{p+k} \leqq (n + p)! (1 - \alpha).$$

*Equality holds for the function $f(z)$ given by*

$$(2.2) \qquad f(z) = z^p - \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} z^{p+1}.$$

Proof. Suppose that the inequality $(2.1)$ holds and let $|z| = 1$. Then we obtain

$$(2.3) \qquad \left| \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - 1 \right| =$$

$$= \left| \frac{\sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots k a_{p+k} z^k}{(n + p)! - (n + p)\sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots (k + 1) a_{p+k} z^k} \right| \leqq$$

$$\leqq \frac{\sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots k a_{p+k} |z|^k}{(n + p)! - (n + p)\sum_{k=1}^{\infty} (n + p + k - 1)(n + p + k - 2) \ldots (k + 1) a_{p+k} |z|^k} \leqq$$

$$\leqq \frac{\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\ldots ka_{p+k}}{(n+p)!-(n+p)\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\ldots(k+1)a_{p+k}} \leqq$$

$$\leqq 1-\alpha.$$

Hence we can see that the values of $D^{n+p}f(z)/D^{n+p-1}f(z)$ lie in a circle centered at $w=1$ whose radius is $(1-\alpha)$. Consequently, we can observe that the function $f(z)$ satisfies the condition $(1.7)$, hence $f(z)$ belongs to the class $T^*(n+p-1,\alpha)$.

For the converse, suppose that the function $f(z)$ belongs to the class $T^*(n+p-1,\alpha)$. Then we get

(2.4)
$$Re\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\}=$$

$$= Re\left\{\frac{(n+p)!-\sum_{k=1}^{\infty}(n+p+k)(n+p+k-1)\ldots(k+1)a_{p+k}z^k}{(n+p)!-(n+p)\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\ldots(k+1)a_{p+k}z^k}\right\}$$

$$> \alpha$$

for $z \in U$. Choose values of $z$ on the real axis so that $D^{n+p}f(z)/D^{n+p-1}f(z)$ is real. Clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we have

(2.5)
$$(n+p)!-\sum_{k=1}^{\infty}(n+p+k)(n+p+k-1)\ldots(k+1)a_{p+k} \geqq$$

$$\geqq \alpha\{(n+p)!-(n+p)\sum_{k=1}^{\infty}(n+p+k-1)(n+p+k-2)\ldots(k+1)a_{p+k}\}$$

which implies (2.1).

Finally, we can see that the function $f(z)$ given by (2.2) is an extreme one for the theorem. This completes the proof of the theorem.

**Corollary 1.** *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n+p-1,\alpha)$. Then*

(2.6)
$$a_{p+k} \leqq \frac{k!(n+p)!(1-\alpha)}{(n+p+k-1)!\{(n+p)(1-\alpha)+k\}}$$

*for $k \geqq 1$. The equality holds for the function $f(z)$ of the form*

(2.7)
$$f(z) = z^p - \frac{k!(n+p)!(1-\alpha)}{(n+p+k-1)!\{(n+p)(1-\alpha)+k\}}z^{p+k}.$$

**Theorem 2.** *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n+p-1,\alpha)$.*

344

*Then*

(2.8)
$$|f(z)| \geqq |z|^p - \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} |z|^{p+1}$$

*and*

$$|f(z)| \leq |z|^p + \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} |z|^{p+1}$$

*for $z \in U$. The results are sharp.*

Proof. Since $f(z)$ belongs to the class $T^*(n + p - 1, \alpha)$, in view of Theorem 1 we obtain

(2.10)
$$(n + p)! \{(n + p)(1 - \alpha) + 1\} \sum_{k=1}^{\infty} a_{p+k} \leqq$$

$$\leqq \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \{(n + p)(1 - \alpha) + k\}}{k!} a_{p+k} \leqq$$

$$\leqq (n + p)!(1 - \alpha)$$

which gives

(2.11)
$$\sum_{k=1}^{\infty} a_{p+k} \leqq \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} .$$

Consequently, we see that

(2.12)
$$|f(z)| \geqq |z|^p - |z|^{p+1} \sum_{k=1}^{\infty} a_{p+k} \geqq$$

$$\geqq |z|^p - \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} |z|^{p+1}$$

and

(2.13)
$$|f(z)| \leqq |z|^p + |z|^{p+1} \sum_{k=1}^{\infty} a_{p+k} \leqq$$

$$\leqq |z|^p + \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} |z|^{p+1}$$

for $z \in U$.

Further, by taking the function $f(z)$ given by

(2.14)
$$f(z) = z^p - \frac{1 - \alpha}{(n + p)(1 - \alpha) + 1} z^{p+1}$$

we can observe that the results of the theorem are sharp.

Corollary 2. *Let the function $f(z)$ defined by $(1.6)$ be in the class $T^*(n + p - 1, \alpha)$.*

*Then f(z) is included in a disk with its center at the origin and radius r given by*

$$(2.15) \qquad r = \frac{(n + p + 1)(1 - \alpha) + 1}{(n + p)(1 - \alpha) + 1}.$$

**Theorem 3.** *Let the function $f(z)$ defined by $(1.6)$ be in the class $T^*(n + p - 1, \alpha)$. Then*

$$(2.16) \qquad |f'(z)| \geqq p|z|^{p-1} - \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p$$

*and*

$$(2.17) \qquad |f'(z)| \leqq p|z|^{p-1} + \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p$$

*for $z \in U$. The results are sharp.*

Proof. Since $f(z)$ is in the class $T^*(n + p - 1, \alpha)$, by virtue of Theorem 1 we have

$$(2.18) \qquad \frac{(n + p)! \{(n + p)(1 - \alpha) + 1\}}{p + 1} \sum_{k=1}^{\infty} (p + k) a_{p+k} \leqq$$

$$\leqq \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \{(n + p)(1 - \alpha) + k\}}{k!} a_{p+k} \leqq$$

$$\leqq (n + p)! (1 - \alpha)$$

which implies

$$(2.19) \qquad \sum_{k=1}^{\infty} (p + k) a_{p+k} \leqq \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1}.$$

Hence, by using $(2.19)$, we obtain

$$(2.20) \qquad |f'(z)| \geqq p|z|^{p-1} - |z|^p \sum_{k=1}^{\infty} (p + k) a_{p+k} \geqq$$

$$\geqq p|z|^{p-1} - \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p$$

and

$$(2.21) \qquad |f'(z)| \leqq p|z|^{p-1} + |z|^p \sum_{k=1}^{\infty} (p + k) a_{p+k} \leqq$$

$$\leqq p|z|^{p-1} + \frac{(p + 1)(1 - \alpha)}{(n + p)(1 - \alpha) + 1} |z|^p$$

for $z \in U$.

Further, we can see that the results of the theorem are sharp for the function $f(z)$ given by $(2.14)$.

**Corollary 3.** *Let the function $f(z)$ defined by (1.6) be in the class $T^*(n + p - 1, \alpha)$. Then $f'(z)$ is included in a disk with its center at the origin and radius $R$ given by*

$$(2.22) \qquad R = \frac{(np + p^2 + p + 1)(1 - \alpha) + p}{(n + p)(1 - \alpha) + 1}.$$

## 3. CLOSURE THEOREMS

**Theorem 4.** *Let the functions*

$$(3.1) \qquad f_i(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} \qquad (a_{p+k,i} \geqq 0)$$

*be in the class $T^*(n + p - 1, \alpha)$ for every $i = 1, 2, 3, \ldots, m$. Then the function $h(z)$ defined by*

$$(3.2) \qquad h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geqq 0)$$

*is also in the same class $T^*(n + p - 1, \alpha)$, where*

$$(3.3) \qquad \sum_{i=1}^{m} c_i = 1.$$

**Proof.** By means of the definition of $h(z)$, we can write

$$(3.4) \qquad h(z) = z^p - \sum_{k=1}^{\infty} \left( \sum_{i=1}^{m} c_i a_{p+k,i} \right) z^{p+k}.$$

Now, since $f_i(z) \in T^*(n + p - 1, \alpha)$ for every $i = 1, 2, 3, \ldots, m$, we obtain

$$(3.5) \qquad \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \, (n + p)(1 - \alpha) + k\}}{k!} a_{p+k,i} \leqq (n + p)! \, (1 - \alpha)$$

for every $i = 1, 2, 3, \ldots, m$, by virtue of Theorem 1. Consequently, with the aid of (3.5) we can see that

$$(3.6) \qquad \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \, \{(n + p)(1 - \alpha) + k\}}{k!} \left( \sum_{i=1}^{m} c_i a_{p+k,i} \right) =$$

$$= \sum_{i=1}^{m} c_i \left\{ \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \, \{(n + p)(1 - \alpha) + k\}}{k!} a_{p+k,i} \right\} \leqq$$

$$\leqq \left( \sum_{i=1}^{m} c_i \right) (n + p)! \, (1 - \alpha) = (n + p)! \, (1 - \alpha).$$

This proves that the function $h(z)$ belongs to the class $T^*(n + p - 1, \alpha)$.

347

**Theorem 5.** *Let*

$$(3.7) \qquad\qquad f_p(z) = z^p$$

*and*

$$(3.8) \qquad f_{p+k}(z) = z^p - \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\, z^{p+k}$$

*for* $p \in N$, $n > -p$, $k \in N$ *and* $0 \leqq \alpha \leqq 1/2$. *Then* $f(z)$ *is in the class* $T^*(n + p - 1, \alpha)$ *if and only if it can be expressed in the form*

$$(3.9) \qquad\qquad f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z),$$

*where* $\lambda_{p+k} \geqq 0$ *and*

$$(3.10) \qquad\qquad \sum_{k=0}^{\infty} \lambda_{p+k} = 1.$$

Proof. Assume that

$$(3.11) \qquad f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z) =$$

$$= z^p - \sum_{k=1}^{\infty} \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\, \lambda_{p+k} z^{p+k}.$$

Then we get

$$(3.12) \qquad \sum_{k=1}^{\infty} \left\{ \frac{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}{k!} \right.$$

$$X \left. \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}\, \lambda_{p+k} \right\} \leqq (n+p)!\,(1-\alpha).$$

By virtue of Theorem 1 this shows that $f(z)$ is in the class $T^*(n + p - 1, \alpha)$.

Conversely, assume that $f(z)$ belongs to the class $T^*(n + p - 1, \alpha)$. Again, by virtue of Theorem 1, we have

$$(3.13) \qquad a_{p+k} \leqq \frac{k!\,(n+p)!\,(1-\alpha)}{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}$$

for $p \in N$, $n > -p$, $k \in N$ and $0 \leqq \alpha \leqq 1/2$. Next, setting

$$(3.14) \qquad \lambda_{p+k} = \frac{(n+p+k-1)!\,\{(n+p)(1-\alpha)+k\}}{k!\,(n+p)!\,(1-\alpha)}\, a_{p+k}$$

for $p \in N$, $n > -p$, $k \in N$ and $0 \leqq \alpha \leqq 1/2$, and

$$(3.15) \qquad\qquad \lambda_p = 1 - \sum_{k=1}^{\infty} \lambda_{p+k},$$

we have the representation (3.9). This completes the proof of the theorem.

348

## 4. MODIFIED HADAMARD PRODUCT

Let $f(z)$ be defined by $(1.6)$ and let $g(z)$ be defined by

$$(4.1) \qquad g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \qquad (b_{p+k} \geqq 0).$$

Further, let $f * g(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is,

$$(4.2) \qquad f * g(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

**Theorem 6.** *Let the functions $f_i(z)$ defined by $(3.1)$ be in the classes $T^*(n_i + p - 1, \alpha)$ for each $i = 1, 2, 3, \ldots, m$, respectively. Then the modified Hadamard product $f_1 * f_2 * \ldots * f_m(z)$ is in the class $T^*(n + p - 1)$, where $n = \operatorname*{Min}_{1 \leqq i \leqq m} \{n_i\}$.*

Proof. Since $f_i(z) \in T^*(n_i + p - 1, \alpha)$ for each $i = 1, 2, 3, \ldots, m$, by using Theorem 1 we get

$$(4.3) \qquad a_{p+k,i} \leqq \frac{1 - \alpha}{(n_i + p)(1 - \alpha) + 1}$$

for each $i = 1, 2, 3, \ldots, m$. Hence we can show that

$$(4.4) \qquad \sum_{k=1}^{\infty} \frac{(n + p + k - 1)! \{(n + p)(1 - \alpha) + k\}}{k!} \left( \prod_{i=1}^{m} a_{p+k,i} \right) \leqq$$

$$\leqq (n + p)! (1 - \alpha) \frac{(n + p)(1 - \alpha) + 1}{1 - \alpha} \prod_{i=1}^{m} \left( \frac{1 - \alpha}{(n_i + p)(1 - \alpha) + 1} \right) \leqq$$

$$\leqq (n + p)! (1 - \alpha)$$

with the aid of $(4.3)$ and Theorem 1. This shows that the modified Hadamard product $f_1 * f_2 * \ldots * f_m(z)$ is in the class $T^*(n + p - 1, \alpha)$.

**Corollary 4.** *Let the functions $f_i(z)$ defined by $(3.1)$ be in the same class $T^*(n + p - 1, \alpha)$ for every $i = 1, 2, 3, \ldots, m$. Then the modified Hadamard product $f_1 * f_2 * \ldots * f_m(z)$ is also in the class $T^*(n + p - 1, \alpha)$.*

### References

[1] *H. S. Al-Amiri:* On Ruscheweyh derivative. Ann. Polon. Math. *38* (1980), 87—94.

[2] *S. Fukui* and *K. Sakaguchi:* An extension of a theorem of S. Ruscheweyh. Bull. Fac. Edu. Wakayama Univ. Nat. Sci. *29* (1980), 1—3.

[3] *R. M. Goel* and *N. S. Sohi:* A new criterion for univalence and its applications. Glas. Mat. *16* (1981), 39—49.

[4] *R. M. Goel* and *N. S. Sohi:* A new criterion for $p$-valent functions. Proc. Amer. Math. Soc. *78* (1980), 353—357.

[5] *R. M. Goel* and *N. S. Sohi:* New criteria for *p*-valence. Indian J. Pure Appl. Math. *11* (1980), 1356—1360.

[6] *S. Owa:* On new criteria for *p*-valent functions. Indian J. Pure Appl. Math. *13* (1982), 920—930.

[7] *St. Ruscheweyh:* New criteria for univalent functions. Proc. Amer. Math. Soc. *49* (1975), 109—115.

[8] *N. S. Sohi:* On a subclass of *p*-valent functions. Indian J. Pure Appl. Math. *11* (1980), 1504—1508.

Souhrn

O JISTÝCH TŘÍDÁCH *p*-ZNAČNÝCH FUNKCÍ

SHIGEYOSHI OWA

Autor zavádí třídy $T^*(n + p - 1, \alpha)$ analytických *p*-značných funkcí s negativními koeficienty s použitím symbolu $D^{n+p-1} f(z)$ definovaného pomocí $\{z^p/(1 - z)^{n+p}\} * f(z)$. Cílem práce je dokázat jisté věty o distorsi a o uzávěru pro funkce $f(z)$ patřící do třídy $T^*(n + p - 1, \alpha)$. Je studován rovněž modifikovaný Hadamardův součin funkcí $f(z)$ v $T^*(n + p - 1, \alpha)$.

Резюме

О НЕКОТОРЫХ КЛАССАХ *p*-ЗНАЧНЫХ ФУНКЦИЙ

SHIGEYOSHI OWA

Автор, пользуясь символом $D^{n+p-1} f(z)$, определённым при помощи $\{z^n/(1 - z)^{n+p}\} * f(z),^p$ вводит классы $T^*(n + p - 1, \alpha)$ *p*-значных аналитических функций с отрицательными коэффициентами. Целью статьи является доказательство некоторых теорем о дисторсии и замыкании для функций $f(z)$ из класса $T^*(n + p - 1, \alpha)$. Кроме того изучается модифицированное произведение Адамарда функций из $T^*(n + p - 1, \alpha)$.

*Author's address:* Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577, Japan.