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NOTE ON THE CONGRUENCE LATTICE OF A COMMUTATIVE SEPARATIVE SEMIGROUP

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Summary. Commutative separative semigroups whose lattice of congruences is a sublattice of the lattice of tolerances are described.

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AMS (MOS) subject classification: 20M10.

A tolerance on a semigroup S is a reflexive and symmetric subsemigroup of the direct product $S \times S$. The set $\mathcal{T}(S)$ of all tolerances on S forms a complete lattice with respect to set inclusion (see [1] and [2]). By $\mathcal{C}(S)$ we denote the complete lattice of all congruences on S. Clearly $\mathcal{C}(S)$ is a complete upper subsemilattice of $\mathcal{T}(S)$, but it need not be a lower subsemilattice of $\mathcal{T}(S)$. The aim of this paper consists in a characterization of a commutative separative semigroup S whose congruence lattice $\mathcal{C}(S)$ is a sublattice of $\mathcal{T}(S)$.

Let S be a commutative semigroup. For all $a, b, z \in S$ we shall use the following notation: $(a, b)\, z = (az, bz)$. Let $\emptyset \neq A \subseteqq S \times S$. By $T(A)$ $(C(A))$ we denote the least tolerance (congruence, respectively) on S containing A. The symbol S^1 stands for S if S has an identity, otherwise it stands for S with an identity adjoined. The following two lemmas are easy to verify.

Lemma 1. *Let S be a commutative semigroup. For $x, y \in S$, $x \neq y$, we have $(x, y) \in T(A)$ if and only if $x = x_1 x_2 \dots x_n z$ and $y = y_1 y_2 \dots y_n z$ where $z \in S^1$ and either $(x_i, y_i) \in A$ or $(y_i, x_i) \in A$ $(i = 1, 2, \dots, n)$.*

Lemma 2. *Let S be a commutative semigroup and $a, b \in S$, $a \neq b$. For $x, y \in S$, $x \neq y$, we have:*

1. *$(x, y) \in T(a, b)$ if and only if there exist $z \in S^1$ and a positive integer m such that either $(x, y) = (a^m, b^m)\, z$ or $(x, y) = (b^m, a^m)\, z$;*

2. *$(x, y) \in C(a, b)$ if and only if there exist $x_0, x_1, \dots, x_n \in S$ such that $x = x_0$, $y = x_n$ and for $i = 1, 2, \dots, n$ we have $x_{i-1} \neq x_i$ and either $(x_{i-1}, x_i) = (a, b)\, z_i$ or $(x_{i-1}, x_i) = (b, a)\, z_i$ for some $z_i \in S^1$.*

By $\vee$ and $\wedge$ we denote the join and meet, respectively, in the lattice $\mathscr{T}(S)$. Clearly we have $A \vee B = T(A \cup B)$ and $A \wedge B = A \cap B$ for all $A, B \in \mathscr{T}(S)$.

Recall that every commutative regular semigroup S is a semilattice of commutative groups. Terminology and notation not defined here may be found in [3]. The set of all idempotents of S is denoted by $E(S)$ and is partially ordered by: $e \leqq f$ if $ef = e$. We write $e < f$ for $e \leqq f$ and $e \neq f$. By $e \parallel f$ we denote the fact that idempotents e, f are incomparable. For any integer k, by x^k we denote the k-th power of an element x of S in the maximal subgroup G_e containing an idempotent $e = x^0$. It is known that for all $x, y \in S$ we have

$$(1) \qquad (xy)^0 = x^0 y^0 .$$

A commutative semigroup S is said to be separative if $a^2 = ab = b^2$ imply $a = b$ for $a, b \in S$.

Theorem 1. *Let S be a commutative separative semigroup. If the lattice $\mathscr{T}(S)$ is modular (distributive), then the lattice $\mathscr{C}(S)$ is modular (distributive).*

Proof. Suppose that the lattice $\mathscr{T}(S)$ of a commutative separative semigroup S is modular. It follows from Theorem 3 of [4] that S is regular. Theorem 1 of [4] (condition (M1)) implies that S has the following property: If e, f are two idempotents of S such that $e \parallel f$, then at least one of them is maximal with respect to the order in $E(S)$. Therefore $E(S)$ is a tree. Let $a, b \in S$ with $a^0 < b^0$. According to Theorem 1 of [4] (condition (M2)), we have $a^0 b = a^0$ and so $ab = aa^0 b = aa^0 = a$. Then Theorem 3 of [5] implies that the lattice $\mathscr{C}(S)$ is modular.

Assume that the lattice $\mathscr{T}(S)$ is distributive. It follows from Theorem 2 of [4] (condition (D1)) that every maximal subgroup G_e of S is locally cyclic. According to Ore's theorem (see [6]), the lattice $\mathscr{C}(G_e)$ is distributive for every $e \in E(S)$. Consequently, Theorem 3 of [5] implies that the lattice $\mathscr{C}(S)$ is distributive.

Note 1. According to Theorem 1, it seems that $\mathscr{C}(S)$ is a sublattice of $\mathscr{T}(S)$,

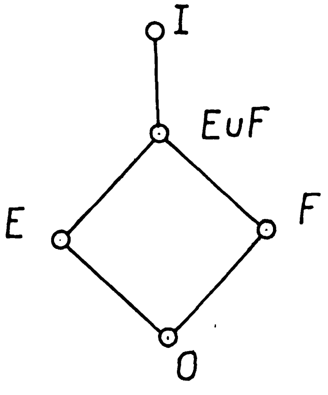

Fig. 1

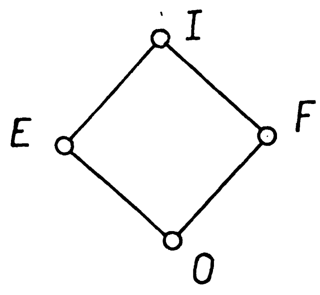

Fig. 2

whenever S is a commutative separative semigroup. But this need not be true. Consider for example the semigroup $P = \{e, f, 0\}$ given by the multiplicative table

$\cdot$	e	f	0
e	e	0	0
f	0	f	0
0	0	0	0

Put $O = \mathrm{id}_P$, $E = \{(e, 0), (0, e)\} \cup O$, $F = \{(f, 0), (0, f)\} \cup O$ and $I = P \times P$. It is easy to show that $\mathscr{T}(P)$ ($\mathscr{C}(P)$) is as in Fig. 1 (in Fig. 2, respectively).
Clearly $\mathscr{C}(P)$ is no sublattice of $\mathscr{T}(P)$.

Theorem 2. *Let S be a commutative separative semigroup. Then the following conditions are equivalent:*

1. *$\mathscr{C}(S)$ is a complete sublattice of $\mathscr{T}(S)$.*
2. *$\mathscr{C}(S)$ is a sublattice of $\mathscr{T}(S)$.*
3. *S is either a group or a group with zero.*

Proof. $1 \Rightarrow 2$. Clear.

$2 \Rightarrow 3$. Suppose that S is a commutative separative semigroup and $\mathscr{C}(S)$ is a sublattice of the lattice $\mathscr{T}(S)$.

I. First we shall show that S is regular. By way of contradiction, assume that there is an element a of S, which is not regular. Since S is separative, the element a is aperiodic. Let $A = C(a, a^3) \vee C(a^3, a^4)$ in $\mathscr{T}(S)$. By hypothesis, we have $A \in \mathscr{C}(S)$. It is clear that $(a, a^3), (a^3, a^4), (a^4, a^2) \in A$ and thus we obtain $(a, a^2) \in A = T(C(a, a^3) \cup C(a^3, a^4))$. Since $a \notin a^2S^1$ by hypothesis, we have by Lemma 1 and Lemma 2, $(a, a^2) \in C(a, a^3)$. It follows from Lemma 2 that there exist $x_0, x_1, \dots, x_n \in S$ such that $x_0 = a$, $x_n = a^2$ and for $i = 1, 2, \dots, n$ we have $x_{i-1} \neq x_i$ and either $(x_{i-1}, x_i) = (a, a^3) z_i$ or $(x_{i-1}, x_i) = (a^3, a) z_i$ for some $z_i \in S^1$.

We shall prove that

$$x_i\langle a^2\rangle \cap a\langle a^2\rangle \neq \emptyset \tag{2}$$

for $i = 0, 1, 2, \dots, n$, where $\langle a^2\rangle$ stands for the subsemigroup of S generated by a^2.

Evidently (2) is satisfied for $i = 0$, because $x_0 = a$.

Suppose that (2) is satisfied for $i \in \{0, 1, 2, \dots, n - 1\}$. It means that $x_i a^{2j} = a^{2k+1}$ for some positive integers j and k. We have $(x_i, x_{i+1}) = (a, a^3) z_{i+1}$ or $(x_i, x_{i+1}) = (a^3, a) z_{i+1}$, where $z_{i+1} \in S^1$. If $x_i = az_{i+1}$ and $x_{i+1} = a^3 z_{i+1}$, then $x_{i+1} = a^2 x_i$ and so $x_{i+1}a^{2j} = a^{2k+3}$. If $x_i = a^3 z_{i+1}$ and $x_{i+1} = az_{i+1}$, then $x_{i+1}a^{2j+2} = x_i a^{2j} = a^{2k+1}$. Consequently, (2) is satisfied for $i + 1$.

For $i = n$ we have $x_i = a^2$ and so (2) implies that $a^2\langle a^2\rangle \cap a\langle a^2\rangle \neq \emptyset$. This means that a is a periodic element of S, which is a contradiction. Therefore the semigroup S is regular.

II. We shall prove that the semilattice $E(S)$ is a chain. By way of contradiction, assume that there exist idempotents e, f of S such that $e \parallel f$. Let $A = C(e, ef) \vee$

$\vee\ C(ef, f)$ in $\mathscr{T}(S)$. By hypothesis, we have $A \in \mathscr{C}(S)$ and so $(e, f) \in A = T(C(e, ef) \cup \cup C(ef, f))$. According to Lemma 1 and Lemma 2, we have $(e, f) \in C(e, ef) \cup \cup C(ef, f)$. If $(e, f) \in C(e, ef)$, then by Lemma 2 we obtain $f \in eS^1$ and so $f \leqq e$, a contradiction. Then we have $(e, f) \in C(ef, f)$, which is analogously impossible. Therefore, $E(S)$ is a chain.

III. Now we shall prove that S is either simple or O-simple. By way of contradiction, assume that I is a proper ideal of S with card $I \geqq 2$. Choose $a \in S \setminus I$. If $e = a^0 \in I$, then $a = ae \in I$, which is a contradiction. Thus $e \in S \setminus I$. For any element x of I we have $x^0 = xx^{-1} \in I$ and so card $E(I) \geqq 1$.

We shall show that

$$f < e \tag{3}$$

for every $f \in E(I)$. If $e \leqq f$, then $e = ef \in I$, a contradiction. According to part II of the proof, we have (3).

Now we can distinguish two cases.

Case 1. card $E(I) \geqq 2$. Then we can choose two idempotents $f, g \in I$ such that $f > g$. Let $A = C(e, f) \vee C(f, g)$ in $\mathscr{T}(S)$. By hypothesis, we have $A \in \mathscr{C}(S)$ and so $(g, e) \in A = T(C(e, f) \cup C(f, g))$. According to Lemma 1, Lemma 2 and (3), we have $(g, e) \in C(e, f)$. Then it follows from Lemma 2 that there exist $x_0, x_1, \ldots, x_n \in S$ such that $x_0 = g$, $x_n = e$, and for $i = 1, 2, \ldots, n$ we have $x_{i-1} \neq x_i$ and either $(x_{i-1}, x_i) = (e, f)\, z_i$ or $(x_{i-1}, x_i) = (f, e)\, z_i$ for some $z_i \in S^1$.

We shall prove that

$$x_i^0 = g \tag{4}$$

for $i = 0, 1, 2, \ldots, n$. Clearly, (4) is satisfied for $i = 0$.

Suppose that (4) is satisfied for $i \in \{0, 1, 2, \ldots, n - 1\}$. We have $(x_i, x_{i+1}) = (e, f)\, z_{i+1}$ or $(x_i, x_{i+1}) = (f, e)\, z_{i+1}$. Assume that $(x_i, x_{i+1}) = (e, f)\, z_{i+1}$. Then, by (1), we have $g = x_i^0 = ez_{i+1}^0$. According to part II of the proof and (3), we have $g = z_{i+1}^0$ and so $x_{i+1}^0 = fz_{i+1}^0 = g$. If $(x_i, x_{i+1}) = (f, e)\, z_{i+1}$, then it can be proved in an analogous manner that $x_{i+1}^0 = g$.

Using (4) for $i = n$ we have $g = x_i^0 = e$, which is a contradiction.

Case 2. card $E(I) = 1$. Let $E(I) = \{f\}$. Since card $I \geqq 2$, we can choose an element b of I such that $b^0 = f$ and $b \neq f$. Let $A = C(e, f) \vee C(f, b)$ in $\mathscr{T}(S)$. By hypothesis, we have $A \in \mathscr{C}(S)$ and so $(e, b) \in A = T(C(e, f) \cup C(f, b))$. Using Lemma 2 we can easily show that $C(f, b) \subseteqq I \times I \cup \mathrm{id}_S$. Then, by Lemma 1 and (3), we have $(e, b) \in C(e, f)$. According to Lemma 2, there exist $x_0, x_1, \ldots, x_n \in S$ such that $x_0 = e$, $x_n = b$, and for $i = 1, 2, \ldots, n$ we have $x_{i-1} \neq x_i$ and

$$(x_{i-1}, x_i) = (e, f)\, z_i \quad \text{or} \quad (x_{i-1}, x_i) = (f, e)\, z_i \tag{5}$$

for some $z_i \in S^1$.

Now, we shall prove that $x_i = f$ for all odd i, where $1 \leqq i \leqq n$. First we shall show that $x_1 = f$. According to (5), we have $(x_0, x_1) = (e, f)\, z_1$ or $(x_0, x_1) = (f, e)\, z_1$. If $x_0 = fz_1$, then it follows from (1) that $e = x_0^0 = fz_1^0$ and so $e \leqq f$. This contradicts (3). Thus we obtain $(x_0, x_1) = (e, f)\, z_1$ and so $x_1 = fz_1 = fez_1 = fx_0 = fe = f$.

Suppose that $x_i = f$ for some odd $i \leqq n - 2$. If $(x_i, x_{i+1}) = (e, f)\, z_{i+1}$, then $x_i = fx_i = fez_{i+1} = fz_{i+1} = x_{i+1}$, a contradiction. According to (5), we have

$$(x_i, x_{i+1}) = (f, e)\, z_{i+1} \,. \tag{6}$$

If $(x_{i+1}, x_{i+2}) = (f, e)\, z_{i+2}$, then using (6) and (3) we have $x_{i+1} = fz_{i+2} = fx_{i+1} = fez_{i+1} = fz_{i+1} = x_i$, a contradiction. Therefore, by (5), we obtain

$$(x_{i+1}, x_{i+2}) = (e, f)\, z_{i+2} \,.$$

This, (6) and (3) imply that $x_{i+2} = fz_{i+2} = fez_{i+2} = fx_{i+1} = fez_{i+1} = fz_{i+1} = x_i = f$.

Since $x_n = b \neq f$, we see that n is even, $n \geqq 2$ and $x_{n-1} = f$. According to (5) we have either $(x_{n-1}, x_n) = (e, f)\, z_n$ or $(x_{n-1}, x_n) = (f, e)\, z_n$. If $(x_{n-1}, x_n) = (e, f)\, z_n$, then by (3) we have $f = fx_{n-1} = fez_n = fz_n = x_n = b$, a contradiction. If $(x_{n-1}, x_n) = (f, e)\, z_n$, then by (3) we have $b = b^0 b = fb = fx_n = fez_n = fz_n = x_{n-1} = f$, again a contradiction.

Consequently, S is either simple or 0-simple.

IV. It is well known that every commutative simple semigroup is a group. Clearly, it can be easily proved that every commutative regular 0-simple semigroup is a group with zero. See [3].

$3 \Rightarrow 1$. If S is a group, then it is known that $\mathscr{C}(S) = \mathscr{T}(S)$. Suppose that S is a group with zero. To show that $\mathscr{C}(S)$ is a complete sublattice of $\mathscr{T}(S)$ it suffices to prove that $\mathscr{C}(S)$ is a complete lower subsemilattice of $\mathscr{T}(S)$.

Let $A_i \in \mathscr{C}(S)$ $(i \in I)$. Put $A = \bigvee_{i \in I} A_i$ in $\mathscr{T}(S)$. We shall prove that $A \in \mathscr{C}(S)$. Let $(a, b), (b, c) \in A = T(\bigcup_{i \in I} A_i)$, where $a \neq b \neq c$. If $b \neq 0$, then $(a, c) = (a, b) \,.\, (b^{-1}, b^{-1})\, (b, c) \in A$. Assume that $b = 0$. It follows from Lemma 1 that $a = a_1 a_2 \dots a_m$ and $b = b_1 b_2 \dots b_m$, where $(a_k, b_k) \in A_{i_k}$ for $i_k \in I$ $(k = 1, 2, \dots, m)$. Since $b = 0$, there exists $j \in \{1, 2, \dots, m\}$ such that $b_j = 0$. We have $a_j \neq 0$ and $(a_j, 0) \in A_{i_j}$. Then $(a, 0)$, $(c, 0) \in A_{i_j}$ and so $(a, c) \in A_{i_j} \subseteq A$. Hence we have $A \in \mathscr{C}(S)$. Consequently, $\mathscr{C}(S)$ is a complete sublattice of $\mathscr{T}(S)$.

Corollary 1. *Let S be a commutative separative semigroup. If $\mathscr{C}(S)$ is a sublattice of $\mathscr{T}(S)$, then the lattices $\mathscr{C}(S)$ and $\mathscr{T}(S)$ are modular.*

Proof follows from Theorem 2 and Corollary 2 of [4].

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Souhrn

POZNÁMKA KE SVAZU KONGRUENCÍ NA KOMUTATIVNÍ SEPARATIVNÍ POLOGRUPĚ

BEDŘICH PONDĚLÍČEK

Tolerance na pologrupě je reflexivní, symetrická a kompatibilní relace. Svaz všech tolerancí [kongruencí] na pologrupě S označíme $\mathscr{T}(S)$ [$\mathscr{C}(S)$]. V práci je dokázána tato věta:

Tyto vlastnosti komutativní separativní pologrupy S jsou ekvivalentní:

1. *$\mathscr{C}(S)$ je úplný podsvaz svazu $\mathscr{T}(S)$.*
2. *$\mathscr{C}(S)$ je podsvaz svazu $\mathscr{T}(S)$.*
3. *S je grupa nebo grupa s nulou.*

Резюме

ЗАМЕЧАНИЕ О СТРУКТУРЕ КОНГРУЕНЦИЙ НА КОММУТАТИВНОЙ СЕПАРАТИВНОЙ ПОЛУГРУППЕ

BEDŘICH PONDĚLÍČEK

Для того, чтобы получить понятие толерантности на полугруппе, достаточно в определении конгруенции опустить условие транзитивности. Структуру всех толерантностей [конгруенций] на полугруппе S обозначим через $\mathscr{T}(S)$ [$\mathscr{C}(S)$]. В статье доказывается следующая теорема:

На коммутативной сепаративной полугруппе эквивалентны следующие свойства:

1. *$\mathscr{C}(S)$ — полная подструктура структуры $\mathscr{T}(S)$.*
2. *$\mathscr{C}(S)$ — подструктура структуры $\mathscr{T}(S)$.*
3. *S — группа или группа с нулем.*

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