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THE BIGRAPH DECOMPOSITION NUMBER OF A GRAPH

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Summary. The bigraph decomposition number $b(G)$ of a graph G is the minimum number of edge-disjoint complete bipartite graphs into which G can be decomposed. In the paper $b(G)$ is studied for weak products and direct products of graphs and is related to the domination number of G.

Keywords: Bipartite graph, decomposition of a graph, bigraph decomposition number.

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In [1], Problems 3.11 and 3.12, D. West introduced a new numerical invariant of a graph; he denoted it by $b(G)$. We shall call it the bigraph decomposition number.

We shall consider finite undirected graphs without loops and multiple edges. A bipartite graph will be shortly called a bigraph; this is a graph G whose vertex set is the union of two disjoint sets A, B (called the bipartition classes of G) which have the property that each edge of G joins a vertex of A with a vertex of B. If each vertex of A is joined by an edge with each vertex of B, such a graph is called a complete bigraph. The symbol $K_{m,n}$ denotes the complete bigraph in which $|A| = m$, $|B| = n$.

A bigraph decomposition of a graph G is a family of subgraphs of G which are complete bigraphs and have the property that each edge of G belongs to exactly one of them. The least cardinality of a bigraph decomposition of G is called the bigraph decomposition number of G and denoted by $b(G)$.

The problems of D. West from [1] are the following:

Determine $b(G)$ for special classes of graphs, or give a bound for it in terms of other parameters.

How does $b(G)$ behave under weak product and the other graph products?

We shall touch both the problems.

First we shall consider the weak product and the direct product of two graphs.

A weak product of two graphs G_1, G_2 is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ of the vertex sets $V(G_1)$, $V(G_2)$ of G_1 and G_2, and in which two vertices (u_1, u_2), (v_1, v_2) are adjacent if and only if u_1, v_1 are adjacent in G_1 and u_2, v_2 are adjacent in G_2.

Theorem 1. *Let G be the weak product of two graphs G_1, G_2, without isolated vertices.*

Then

$$b(G) \leqq 2\, b(G_1)\, b(G_2)\,,$$

and this bound cannot be improved.

Proof. Let $\mathscr{B}_1$ (or $\mathscr{B}_2$) be a bigraph decomposition of G_1 (or G_2, respectively) of the minimum cardinality. Let $H_1 \in \mathscr{B}_1$, $H_2 \in \mathscr{B}_2$, and let A_1, B_1 be the bipartition classes of H_1 and A_2, B_2 the bipartition classes of H_2. The subgraphs of G induced by $(A_1 \times A_2) \cup (B_1 \times B_2)$ and by $(A_1 \times B_2) \cup (A_2 \times B_1)$ are complete bigraphs; denote them by $L_1(H_1, H_2)$ and $L_2(H_1, H_2)$, respectively. Consider the family $\mathscr{B}$ of all graphs $L_1(H_1, H_2)$, $L_2(H_1, H_2)$ for $H_1 \in \mathscr{B}_1$, $H_2 \in \mathscr{B}_2$. Let e be an edge of G, let (u_1, u_2), (v_1, v_2) be its end vertices. Then there exists an edge u_1v_1 of G_1 and an edge u_2v_2 of G_2. There exists exactly one graph $H' \in \mathscr{B}_1$ containing u_1v_1 and exactly one graph $H'' \in \mathscr{B}_2$ containing u_2v_2 and evidently the edge e belongs to $L_1(H', H'')$ or to $L_2(H', H'')$. Conversely, if an edge f belongs to $L_1(H', H'')$ or to $L_2(H', H'')$, then f joins vertices (u', u''), (v', v'') such that the edge $u'v'$ belongs to H' and the edge $u''v''$ belongs to H''. Hence $\mathscr{B}$ is a bigraph decomposition of G. As $|\mathscr{B}| = 2|\mathscr{B}_1| \,.\, |\mathscr{B}_2| = 2b(G_1)\, b(G_2)$, the inequality from Theorem 1 holds.

If both G_1, G_2 are complete bigraphs, then $b(G_1) = b(G_2) = 1$. Let A_1, B_1 be the bipartition classes of G_1, let A_2, B_2 be the bipartition classes of G_2. Then the weak product of G_1 and G_2 is the union of two vertex-disjoint complete bigraphs; one of them has the bipartition classes $A_1 \times A_2$, $B_1 \times B_2$, the other $A_1 \times B_2$, $B_1 \times A_2$. Hence the bigraph decomposition number of this weak product is 2 and the equality occurs; the bound cannot be improved. □

Theorem 2. *There exist graphs G_1, G_2 such that for their weak product G the inequality $b(G) < 2\, b(G_1)\, b(G_2)$ holds.*

Proof. Let $G_1 \cong K_3$, $G_2 \cong K_2$ (the complete graphs with 3 and 2 vertices). We have $b(G_1) = 2$, $b(G_2) = 1$ and thus $2\, b(G_1)\, b(G_2) = 4$. But the weak product G of G_1 and G_2 is a circuit of length 6 and therefore $b(G) = 3$. □

Now we turn to the direct products of graphs. The direct product of the graphs G_1, G_2 is the graph G whose vertex set is $V(G_1) \times V(G_2)$ and in which two vertices (u_1, u_2), (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and u_2, v_2 are adjacent in G_2, or $u_2 = v_2$ and u_1, v_1 are adjacent in G_1.

Theorem 3. *Let G be the direct product of the graphs G_1, G_2, without isolated vertices. Then*

$$b(G) \leqq |V(G_1)|\, b(G_2) + |V(G_2)|\, b(G_1)\,,$$

and this bound cannot be improved.

Proof. For $u \in V(G_1)$ let $G_2)u)$ be the subgraph of G induced by the set of all vertices (u, x) for $x \in V(G_2)$. For $v \in V(G_2)$ let $G_1(v)$ be the subgraph of G induced by the set of all vertices (y, v) for $y \in V(G_1)$. These graphs will be called projections. Obviously all projections are edge-disjoint. We have $G_2(u) \cong G_2$, $G_1(v) \cong G_1$ for each u and v. The number of projections $G_2(u)$ (or $G_1(v)$) is $|V(G_1)|$ (or $|V(G_2)|$, respectively). If we take, in each $G_1(v)$, a bigraph decomposition of cardinality $b(G_1)$, and in each $G_2(u)$ a bigraph decomposition of cardinality $b(G_2)$, then the union of all these decompositions is a bigraph decomposition of G of cardinality $|V(G_1)|\, b(G_2) + |V(G_2)|\, b(G_1)$. This implies the inequality from Theorem 3.

Now let p, q be integers greater than 1. Let the vertex set of a graph G_1 be $V(G_1) = \bigcup_{i=1}^{2q} V_i$, where $V_1, \ldots, V_{2q}$ are pairwise disjoint sets of cardinality p. Two vertices of G_1 will be adjacent if and only if one of them belongs to V_i and the other to V_{i+1} for some $i \in \{1, \ldots, 2q - 1\}$, or one of them to V_1 and the other to V_{2q}. The graph G_2 will be a graph isomorphic to G_1. Let G be the direct product of G_1 and G_2. For each $j \in \{1, \ldots, q\}$ let H_j be the subgraph of G_1 induced by the set $V_{2j-2} \cup V_{2j-1} \cup V_{2j}$, where the subscripts are taken modulo $2q$. Each H_j is a complete bigraph with the bipartition classes V_{2j-1}, $V_{2j-2} \cup V_{2j}$, and no complete bigraph which is a subgraph of G_1 has so many edges as the graphs H_j; hence $b(G_1) = b(G_2) = q$. We consider the bigraph decomposition of G as described above; it has $4pq^2$ bigraphs, each of which is isomorphic to the graphs H_j. Any complete bigraph which is a subgraph of G and is not contained in any projection is a star or a circuit of length 4 and therefore it contains at most $4p$ edges, which is less than or equal to the number $2p^2$ of edges of any H_j. Hence $b(G) = 4pq^2 = |V(G_1)|\, b(G_2) + |V(G_2)|\, b(G_1)$. □

Theorem 4. *There exist graphs G_1, G_2 such that for their direct product G the inequality $b(G) < |V(G_1)|\, b(G_2) + |V(G_2)|\, b(G_1)$ holds.*

Proof. Let $G_1 \cong G_2 \cong K_2$. Then $b(G_1) = b(G_2) = 1$. The direct product G of G_1 and G_2 is $K_{2,2}$, and therefore $b(G) = 1$. □

At the end we relate $b(G)$ to the domination number of G. A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that for each $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. The minimum cardinality of a dominating set in G is called the domination number of G and denoted by $\delta(G)$.

Theorem 5. *Let G be a graph without isolated vertices. Then*

$$b(G) \geqq \tfrac{1}{2}\, \delta(G)\,.$$

Proof. Let $\mathscr{B}$ be a bigraph decomposition of G consisting of $b(G)$ graphs. In each graph $H \in \mathscr{B}$ we choose two vertices from distinct bipartition classes; then each vertex of H distinct from them is adjacent to one of them. The set of all chosen vertices for all $H \in \mathscr{B}$ is a dominating set in G and has at most $2\, b(G)$ vertices. Hence $2\, b(G) \geqq \delta(G)$, which implies $b(G) \geqq \frac{1}{2}\, \delta(G)$. □

Reference

[1] Problem Sessions. In: Graphs and Order. Proc. Conf. Banff 1984.

Souhrn

BIGRAFOVĚ ROZKLADOVÉ ČÍSLO GRAFU

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Bigrafově rozkladové číslo $b(G)$ grafu G je minimální počet hranově disjunktních úplných sudých grafů, na něž lze rozložit graf G. V článku se zkoumá $b(G)$ pro slabé součiny a direktní součiny grafů a porovnává se s dominačním číslem grafu G.

Резюме

ЧИСЛО БИГРАФОВОГО РАЗЛОЖЕНИЯ ГРАФА

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Числом биграфового разложения $b(G)$ графа G называется минимальное число реберно непересекающихся полных двудольных графов, на которые можно разложить G. В статье число $b(G)$ изучается для слабых произведений и прямых произведений графов и сравнивается с доминационным числом графа G.

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