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INTERCHANGEABILITY OF UNBOUNDED OPERATORS: SPECIAL CRITERIA OF TRANSMUTATIVITY

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Summary. In this paper, we continue the study of transmutativity of unbounded resolventive and scalar operators started in [1]. Thus the acquaintance with this paper is indispensable even if, for the sake of completeness, we repeat below the definitions of some basic notions.

Our main aim is to give some criteria of transmutativity of special classes of the operators mentioned above, expressing them as far as possible most directly in terms of the given operators themselves.

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1. PRELIMINARIES

1.1. We frequently use the notation, notions and results from [1] in the extent of Sections 1, 2, 4 and 5.

1.2. For $A \in \mathsf{L}^+(E)$ and $M \subseteq E$, we denote $A(M) = \{Ax: x \in \mathsf{D}(A) \cap M\}$.

1.3. Let $A \in \mathsf{L}^+(E)$. The operator A is called a *projector* if $A^2 = A$.

1.4. Let $A, B \in \mathsf{L}^+(E)$. The operators A, B are called *transmutative* if $ST = TS$ for every $S, T \in \mathsf{L}(E)$ such that

(I) $SA \subseteq AS$,

(II) $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,

(III) $TB \subseteq BT$,

(IV) $TH = HT$ for every $H \in \mathsf{L}(E)$ with $HB \subseteq BH$.

1.5. Let $A \in \mathsf{L}^+(E)$ and $z \in \mathrm{C}$. The number z is called a *resolvent point of the operator* A if the operator $zI + A$ is one-to-one and $(zI + A)^{-1} \in \mathsf{L}(E)$.

1.6. Let $A \in \mathsf{L}^+(E)$. The operator A is called *resolventive* if there exists a number $z \in C$ which is a resolvent point of the operator A.

1.7. By $\mathsf{B}(\mathrm{C})$ we denote the set of all Borel subsets of C. Further, the set of all mappings of $\mathsf{B}(\mathrm{C})$ into $\mathsf{L}(E)$ will be denoted by $\mathsf{B}(\mathrm{C}) \to \mathsf{L}(E)$.

1.8. Let $A \in \mathsf{L}^+(E)$ and $\mathscr{E} \in \mathsf{B}(\mathrm{C}) \to \mathsf{L}(E)$. The function $\mathscr{E}$ is called a (*spectral*) *resolution of the operator* A if

(I) the function $\mathscr{E}(\cdot)\, x$ is a σ-additive vector-valued measure on $\mathsf{B}(\mathrm{C})$ for every $x \in E$,

(II) $\mathscr{E}(X \cap Y) = \mathscr{E}(X)\, \mathscr{E}(Y)$ for every $X, Y \in \mathsf{B}(\mathrm{C})$,

(III) $\mathscr{E}(\mathrm{C}) = I$,

(IV) $x \in \mathsf{D}(A)$ if and only if the integral $\int_{\mathrm{C}} \sigma\, \mathscr{E}(\mathrm{d}\sigma)\, x$ exists,

(V) $Ax = \int_{\mathrm{C}} \sigma\, \mathscr{E}(\mathrm{d}\sigma)\, x$ for every $x \in \mathsf{D}(A)$.

1.9. Let $A \in \mathsf{L}^+(E)$. The operator is called *scalar* if there exists a function $\mathscr{E} \in \mathsf{B}(\mathrm{C}) \to \mathsf{L}(E)$ which is a resolution of the operator A.

2. RESOLVENTIVE OPERATORS AND THEIR TRANSMUTATIVITY

2.1. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If the operators A, B are densely defined resolventive, then the following statements* (A) *and* (B) *are equivalent*:

(A) *there exists a dense subset D of* $\mathrm{cl}\, \mathsf{D}(A) \cap \mathrm{cl}\, \mathsf{D}(B)$ *such that for every $d \in D$ we can find an operator $U \in \mathsf{L}(E)$ for which*

(I) $d \in \mathsf{R}(U^2)$,

(II) $\mathsf{R}(U) \subseteq \mathsf{D}(A) \cap \mathsf{D}(B)$,

(III) $UA \subseteq AU$, $UB \subseteq BU$,

(IV) $AUBU \doteq BUAU$;

(B) *the operators A, B are transmutative.*

Proof. Let us first fix, according to the assumption (α), two numbers $\alpha, \beta \in \mathrm{C}$ such that

(1) α is a resolvent point of the operator A,

(2) β is a resolvent point of the operator B.

(A) $\Rightarrow$ (B): We choose a fixed dense subset D of $\mathrm{cl}\, \mathsf{D}(A) \cap \mathrm{cl}\, \mathsf{D}(B)$ such that the condition (A) holds.

Since the operators A, B are densely defined by the assumption (β), we have

(3) the set D is a dense subset of E.

Let us now consider an arbitrary but fixed element $d \in D$. For this $d \in D$, let us fix a $U \in \mathsf{L}(E)$ for which (A) (I)–(IV) hold.

By (A) (III), we have

(4) $ABU^2 \supseteq AUBU$,

(5) $BAU^2 \supseteq BUAU$.

By (4) and (5), we can write

(6) $(\alpha I + A)(\beta I + B)U^2 \supseteq \alpha\beta U^2 + \alpha BU^2 + \beta AU^2 + ABU^2 \supseteq \alpha\beta U^2 + \alpha BU^2 + \beta AU^2 + AUBU$,

(7) $(\beta I + B)(\alpha I + A)U^2 \supseteq \alpha\beta U^2 + \alpha BU^2 + \beta AU^2 + BAU^2 \supseteq \alpha\beta U^2 + \alpha BU^2 + \beta AU^2 + BUAU$.

Using now (A) (II), we see that the right hand sides of (6) and (7) are everywhere defined and hence

(8) $(\alpha I + A)(\beta I + B)U^2 = \alpha\beta U^2 + \alpha BU^2 + \beta AU^2 + AUBU$,

(9) $(\beta I + B)(\alpha I + A)U^2 = \alpha\beta U^2 + \alpha BU^2 + \beta AU^2 + BUAU$.

By (A) (IV), (8) and (9) immediately imply

(10) $(\alpha I + A)(\beta I + B)U^2 = (\beta I + B)(\alpha I + A)U^2$.

On the other hand, from (A) (II) we easily obtain that

(11) $(\alpha I + A)U^2 = \alpha U^2 + AU^2 \supseteq \alpha U^2 + U^2 A = U^2(\alpha I + A)$,

(12) $(\beta I + B)U^2 = \beta U^2 + BU^2 \supseteq \beta U^2 + U^2 B = U^2(\beta I + B)$.

It follows from (11) and (12) that

(13) $(\alpha I + A)(\beta I + B)U^2 \supseteq U^2(\alpha I + A)(\beta I + B)$.

Having the relations (10) and (13) at hand, we can continue as follows.

First, it is clear from (1) and (2) that

(14) $\mathsf{R}((\alpha I + A)(\beta I + B)) = E$.

We see from (A) (I) and (14) that

(15) $d \in \mathsf{R}(U^2(\alpha I + A)(\beta I + B))$.

By (10), (13) and (15), we obtain

(16) $d \in \mathsf{R}((\alpha I + A)(\beta I + B)U^2)$,

(17) $d \in \mathsf{R}((\beta I + B)(\alpha I + A)U^2)$.

Let us now denote

(18) $N = \{x : U^2 x = 0\}$.

It is easy to see from (18) that there exists a unique operator Θ on E/N into E such that

(19) $\Theta \boldsymbol{x} = U^2 x$ for every $\boldsymbol{x} \in E/N$ and $x \in E$ such that $x \in \boldsymbol{x}$.

From (18) and (19) we immediately obtain that

(20) the operator Θ is a one-to-one operator on E/N into E.

It follows from (10), (16), (17) and (19) that

(21) $(\alpha I + A)(\beta I + B)\Theta = (\beta I + B)(\alpha I + A)\Theta$,

(22) $d \in \mathsf{R}(\alpha I + A)(\beta I + B)\Theta)$,

(23) $d \in \mathsf{R}(\beta I + B)(\alpha I + A)\Theta)$.

By (1), (2) and (20)–(23), we obtain

(24) $\Theta^{-1}(\alpha I + A)^{-1}(\beta I + B)^{-1} = \Theta^{-1}(\beta I + B)^{-1}(\alpha I + A)^{-1}$,

(25) $d \in \mathsf{D}(\Theta^{-1}(\alpha I + A)^{-1}(\beta I + B)^{-1})$,

(26) $d \in \mathsf{D}(\Theta^{-1}(\beta I + B)^{-1}(\alpha I + A)^{-1})$.

Since $d \in D$ was supposed aribtrary, we conclude from (24)–(26) that

(27) $(\alpha I + A)^{-1}(\beta I + B)^{-1} d = (\beta I + B)^{-1}(\alpha I + A)^{-1} d$ for every $d \in D$.

It follows from (1), (2), (3) and (27) that

(28) $(\alpha I + A)^{-1}(\beta I + B)^{-1} = (\beta I + B)^{-1}(\alpha I + A)^{-1}$.

By virtue of [1] 4.4, the desired implication (A) $\Rightarrow$ (B) is an immediate consequence of (28).

(B) $\Rightarrow$ (A): By virtue of [1] 4.4, we obtain from (B) and (1), (2) that

(29) $(\alpha I + A)^{-1}(\beta I + B)^{-1} = (\beta I + B)^{-1}(\alpha I + A)^{-1}$.

Let us denote

(30) $U = (\alpha I + A)^{-1}(\beta I + B)^{-1}$.

We begin with proving that

(31) the set $\mathsf{R}(U)$ is a dense subset of E.

Indeed, by (30), we have $\mathsf{R}(U) = \mathsf{R}((\alpha I + A)^{-1}(\beta I + B)^{-1} = (\alpha I + A)^{-1}(\mathsf{D}(B))$. Since the set $\mathsf{D}(B)$ is dense in E by the assumption (β), we obtain that $\mathsf{R}(U)$ is a dense subset of $\mathsf{D}(A)$. But by (β), the set $\mathsf{D}(A)$ itself is also dense in E and this gives (31).

As an immediate consequence of (1), (2), (30) and (31) we have

(32) the set $\mathsf{R}(U^2)$ is a dense subset of E.

Let us now put

(33) $D = \mathsf{R}(U^2)$.

Let us write down the following evident consequences of (β), (32) and (33):

(34) the set D is a dense subset of $\mathrm{cl}\,\mathsf{D}(A) \cap \mathrm{cl}\,\mathsf{D}(B)$,

(35) $d \in \mathsf{R}(U^2)$ for every $d \in D$.

We obtain easily from (29) and (30) that

(36) $\mathsf{R}(U) \subseteq \mathsf{D}(A) \cap \mathsf{D}(B)$.

Let us now note two easy consequences of (1) and (2), namely

(37) $(\alpha I + A)^{-1} A \subseteq A(\alpha I + A)^{-1}$, $A(\alpha I + A)^{-1} = I - \alpha(\alpha I + A)^{-1}$,

(38) $(\beta I + B)^{-1} B \subseteq B(\beta I + B)^{-1}$, $B(\beta I + B)^{-1} = I - \beta(\beta I + B)^{-1}$.

Now it follows from (29), (30), (37) and (38) that

$$(39)\quad UA = (\beta I + B)^{-1} (\alpha I + A)^{-1} A \subseteq (\beta I + B)^{-1} A(\alpha I + A)^{-1} =$$
$$= (\beta I + B)^{-1} (I - \alpha(\alpha I + A)^{-1}) = (I - \alpha(\alpha I + A)^{-1} (\beta I + B)^{-1} =$$
$$= A(\alpha I + A)^{-1} (\beta I + B)^{-1} = AU,$$

and by the similar pattern,

$$(40)\quad UB \subseteq BU.$$

Finally, by (29), (30), (37) and (39), we obtain that

$$(41)\quad AUBU = A(\alpha I + A)^{-1} (\beta I + B)^{-1} B(\beta I + B)^{-1} (\alpha I + A)^{-1} =$$
$$= (I - \alpha(\alpha I + A)^{-1}) (\beta I + B)^{-1} (I - \beta(\beta I + B)^{-1}) (\alpha I + A)^{-1} =$$
$$= (I - \beta(\beta I + B)^{-1}) (\alpha I + A)^{-1} (I - \alpha(\alpha I + A)^{-1}) (\beta I + B)^{-1} =$$
$$= B(\beta I + B)^{-1} (\alpha I + A)^{-1} A(\alpha I + A)^{-1} (\beta I + B)^{-1} = BUAU.$$

The implication (B) $\Rightarrow$ (A) now follows from (1), (2), (30), (34), (35), (36), (39), (40) and (41).

2.2. Proposition. *Let $A \in \mathsf{L}^+(E)$. If the operator A is resolventive, then it is closed.*

Proof. Immediately from [1] 4.1 and 4.2.

2.3. Lemma. *Let $A \in \mathsf{L}^+(E)$. Then*

(a) *the set of all resolvent points of the operator A is open,*

(b) *the function $(zI + A)^{-1}$ is analytic on the set of resolvent points of the operator A.*

Proof. Well-known – cf. [2], Part I.

2.4. Let $A \in \mathsf{L}^+(E)$. The operator A is called *extensively resolventive* if there exists an unbounded, connected subset of resolvent points of the operator A.

2.5. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If the operators A, B are densely defined extensively resolventive, then the following statement* (A) *implies the statement* (B):

(A) *there exists a dense subset Λ of* $\operatorname{cl} \mathsf{D}(A) \cap \operatorname{cl} \mathsf{D}(B)$ *such that*

(I) $\Lambda \subseteq \operatorname{cl} \mathsf{D}(AB) \cap \operatorname{cl} \mathsf{D}(BA)$,

(II) $ABx = BAx$ *for every* $x \in \Lambda$,

(III) *for every $x \in \Lambda$, we can find a closed linear subspace Γ of E satisfying $x \in \Gamma$, $\Gamma \subseteq \Lambda$, $A(\Gamma) \subseteq \Gamma$ and $B(\Gamma) \subseteq \Gamma$;*

(B) *the operators A, B are transmutative.*

Proof. By assumption, we have

(1) $\operatorname{cl} \mathsf{D}(A) = E$, $\operatorname{cl} \mathsf{D}(B) = E$.

Further, in view of 2.4, we can fix subsets $X_A \subseteq \mathrm{C}$, $X_B \subseteq \mathrm{C}$ such that

(2) the sets X_A, X_B are unbounded connected,

(3) all $\alpha \in X_A$, $\beta \in X_B$ are resolvent points of the operators A, B, respectively.

By [1] 4.2 and 2.2, we get from (2) and (3) that

(4) the operators A, B are closed.

Now let us fix a dense subset Λ of $\mathrm{cl}\,\mathsf{D}(A) \cap \mathrm{cl}\,\mathsf{D}(B)$ such that the statement (A) holds.

By (1), we have

(5) the set Λ is a dense subset of E.

Let us now consider an arbitrary but fixed $x \in \Lambda$.

By (A) (III) we can fix a closed linear subspace Γ of E such that (A) (III) holds. Consequently, we have

(6) $x \in \Gamma$,

(7) $\Gamma \subseteq \Lambda$,

(8) $A(\Gamma) \subseteq \Gamma$, $B(\Gamma) \subseteq \Gamma$.

It follows from (A) (I), (II) and (7) that

(9) $\Gamma \subseteq \mathsf{D}(A)B \cap \mathsf{D}(BA)$,

(10) $ABy = BAy$ for every $y \in \Gamma$.

Let us now denote by A_Γ, B_Γ the restrictions of the operators A, B to the subspace Γ.

We see from (8) and (9) that

(11) $\mathsf{D}(A_\Gamma) = \mathsf{D}(B_\Gamma) = \Gamma$, $\mathsf{R}(A_\Gamma) \subseteq \Gamma$, $\mathsf{R}(B_\Gamma) \subseteq \Gamma$.

By the closed graph theorem, we obtain from (1) and (4) that

(12) $A_\Gamma, B_\Gamma \in \mathsf{L}(\Gamma)$.

Moreover, by (10) and (11) we have

(13) $A_\Gamma B_\Gamma = B_\Gamma A_\Gamma$.

It follows from (12) that

(14) the operators $\alpha I + A_\Gamma$, $\beta I + B_\Gamma$ are one-to-one and $(\alpha I + A_\Gamma)^{-1}$, $(\beta I + B_\Gamma)^{-1} \in \mathsf{L}(\Gamma)$ for sufficiently large $\alpha, \beta \in C$.

Moreover, we obtain from (13) that

(15) $(\alpha I + A_\Gamma)^{-1} (\beta I + B_\Gamma)^{-1} = (\beta I + B_\Gamma)^{-1} (\alpha I + A_\Gamma)^{-1}$ for sufficiently large $\alpha, \beta \in C$.

On the other hand, it is easy to see that (2), (3), (11) and (14) imply

(16) $(\alpha I + A)^{-1} y = (\alpha I + A_\Gamma)^{-1} y$, $(\beta I + B)^{-1} y = (\beta I + B_\Gamma)^{-1} y$ for every $y \in \Gamma$ and for sufficiently large $\alpha \in X_A$ and $\beta \in X_B$.

Now we get from (15) and (16) that

(17) $(\alpha I + A)^{-1} (\beta I + B)^{-1} y = (\beta I + B)^{-1} (\alpha I + A)^{-1} y$ for every $y \in \Gamma$ and for sufficiently large $\alpha \in X_A$ and $\beta \in X_B$.

Since $x \in \Lambda$ was chosen fixed but arbitrary, we conclude from (6) and (17) that

(18) $(\alpha I + A)^{-1} (\beta I + B)^{-1} x = (\beta I + B)^{-1} (\alpha I + A)^{-1} x$ for every $x \in \Lambda$ and for sufficiently large $\alpha \in X_A$ and $\beta \in X_B$, possibly in dependence on x.

By virtue of 2.3 it follows from (3) that

(19) the functions $(\alpha I + A)^{-1}$, $(\beta I + B)^{-1}$ are analytic for $\alpha \in X_A$ and $\beta \in X_B$.

By analytic continuation, we easily prove that (2), (18) and (19) give

(20) $(\alpha I + A)^{-1} (\beta I + B)^{-1} x = (\beta I + B)^{-1} (\alpha I + A)^{-1} x$ for every $x \in \Lambda$ and every $\alpha \in X_A$ and $\beta \in X_B$.

By (3), (5) and (20), we get

(21) $(\alpha I + A)^{-1} (\beta I + B)^{-1} = (\beta I + B)^{-1} (\alpha I + A)^{-1}$ for every $\alpha \in X_A$ and $\beta \in X_B$.

Using now [1] 4.4 with arbitrary $\alpha \in X_A$ and $\beta \in X_B$, we get the statement (B) immediately from (21).

2.6. Examples. It is easy to see and well-known that every everywhere defined continuous operator is resolventive and even extensively resolventive. The same is true for generators of semigroups or cosine functions and for scalar operators with real spectra, especially for selfadjoint operators in Hilbert spaces.

2.7. Remark. The condition (A) (III) in 2.5 cannot be replaced by a simpler one, namely

(A) (III′) $A(\Lambda) \subseteq \Lambda$, $B(\Lambda) \subseteq \Lambda$.

This condition (A) (III′) was shown to be insufficient by an example constructed by Nelson. Compare [3], VIII.5 — in particular p. 273.

2.8. Remark. The statements (A) and (B) of 1.5 become equivalent for scalar extensively resolventive operators (especially for selfadjoint operators in Hilbert spaces) as shown in 3.8.

3. SCALAR OPERATORS AND THEIR TRANSMUTATIVITY

3.1. Proposition. *Let $A \in \mathsf{L}^+(E)$. If the operator A is scalar, then it is closed and densely defined.*

Proof. See [2], XVIII.

3.2. Lemma. *Let $A, P \in \mathsf{L}^+(E)$ and $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$. If*
(α) *$PA \subseteq AP$,*
(β) *the operator P is a projector from $\mathsf{L}(E)$,*
(γ) *the function E is a resolution of the operator A,*
then the function EP is a resolution of the operator AP.

Proof. Easy, by virtue of [1] 5.4(b).

3.3. Lemma. *Let $A \in \mathsf{L}^+(E)$ and $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$. If the function $\mathscr{E}$ is a resolution of the operator A, then $\mathsf{R}(\mathscr{E}(X)) \subseteq \mathsf{D}(A)$ for every bounded set $X \in \mathrm{B}(\mathrm{C})$.*

Proof. Immediately from [1] 5.2.

3.4. Lemma. *Let $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to E$. If there exists an operator $A \in \mathsf{L}^+(E)$ such that the function $\mathscr{E}$ is a resolution of the operator A, then*
(a) *$\mathscr{E}(X)$ is a projector from $\mathsf{L}(E)$ for every $X \in \mathrm{B}(\mathrm{C})$,*
(b) *$\mathscr{E}(X_1)\,\mathscr{E}(X_2) = \mathscr{E}(X_2)\,\mathscr{E}(X_1)$ for every $X_1, X_2 \in \mathrm{B}(\mathrm{C})$.*

Proof. Immediately from [1] 5.2.

3.5. Lemma. *Let $\mathscr{E}, \mathscr{F} \in \mathrm{B}(\mathrm{C}) \to E$. If there exist operators $A, B \in \mathsf{L}^+(E)$ such that the functions $\mathscr{E}, \mathscr{F}$ are resolutions of the operators A, B, then the set $\{\mathscr{E}(X)\,\mathscr{F}(Y)\,x$: $x \in E, X, Y \in \mathrm{B}(\mathrm{C})$, the sets X, Y bounded$\}$ is dense in E.*

Proof. Immediately from [1] 5.2.

3.6. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If the operators A, B are scalar, then the following statements* (A) *and* (B) *are equivalent:*

(A) *there exists a dense subset D of $\mathrm{cl}\,\mathsf{D}(A) \cap \mathrm{cl}\,\mathsf{D}(B)$ such that for every $d \in D$ we can find a projector $P \in \mathsf{L}(E)$ for which*
  (I) *$d \in \mathsf{R}(P)$,*
  (II) *$\mathsf{R}(P) \subseteq \mathsf{D}(A) \cap \mathsf{D}(B)$,*
  (III) *$PA \subseteq AP$, $PB \subseteq BP$,*
  (IV) *$APBP = BPAP$;*

(B) *the operators A, B are transmutative.*

Proof. By assumption, we can find two functions $\mathscr{E}_A, \mathscr{E}_B \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$ such that
(1) the function $\mathscr{E}_A$ is a resolution of the operator A,
(2) the function $\mathscr{E}_B$ is a resolution of the operator B.

(A) $\Rightarrow$ (B): We first fix a dense subset D of $\mathrm{cl}\,\mathsf{D}(A) \cap \mathrm{cl}\,\mathsf{D}(B)$ such that the condition (A) holds.

Since the operators A, B are densely defined by 3.1, we have

(3) the set D is a dense subset of E.

Let us now consider an arbitrary but fixed $d \in D$. For this $d \in D$, let us fix a projector $P \in \mathsf{L}(E)$ for which (A) (I)–(IV) hold.

By the closed graph theorem, we easily obtain from (A) (II), (III) that

(4) $AP, BP \in \mathsf{L}(E)$.

By 3.2, we obtain from (1), (2) and A(III) that

(5) the functions $\mathscr{E}_A P$, $\mathscr{E}_B P$ are resolutions of the operators AP, BP, respectively.

By [1] 2.8, we infer from (4) and (A) (IV) that

(6) the operators AP, BP are transmutative.

By [1] 5.5, it follows from (5) and (6) that

(7) $\mathscr{E}_A(X)\, P\, \mathscr{E}_B(Y)\, P = \mathscr{E}_B(Y)\, P\, \mathscr{E}_A(X)\, P$ for every X, $Y \in \mathrm{B(C)}$.

By [1] 5.4(b), it is clear from (A) (III) that

(8) $\mathscr{E}_A(X)\, P = P\, \mathscr{E}_A(X)$, $\mathscr{E}_B(Y)\, P = P\, \mathscr{E}_B(Y)$ for every X, $Y \in \mathrm{B(C)}$.

Now it is immediate from (7) and (8) that

(9) $\mathscr{E}_A(X)\, \mathscr{E}_B(Y)\, P^2 = \mathscr{E}_B(Y)\, \mathscr{E}_A(X)\, P^2$ for every X, $Y \in \mathrm{B(C)}$.

Since the operator P is a projector, we see from (9) that

(10) $\mathscr{E}_A(X)\, \mathscr{E}_B(Y)\, P = \mathscr{E}_B(Y)\, \mathscr{E}_A(X)\, P$ for every X, $Y \in \mathrm{B(C)}$.

On the other hand, by (A) (I) we have

(11) $Pd = d$.

Recalling that $d \in D$ was chosen fixed but arbitrary, we get from (10) and (11) that

(12) $\mathscr{E}_A(X)\, \mathscr{E}_B(Y)\, d = \mathscr{E}_B(Y)\, \mathscr{E}_A(X)\, d$ for every $d \in D$ and X, $Y \in \mathrm{B(C)}$.

By (3) and (12), we then have

(13) $\mathscr{E}_A(X)\, \mathscr{E}_B(Y) = \mathscr{E}_B(Y)\, \mathscr{E}_A(X)$ for every X, $Y \in \mathrm{B(C)}$.

The desired implication (A) $\Rightarrow$ (B) now follows from (1), (2) and (13) by virtue of [1] 5.5.

(B) $\Rightarrow$ (A): Since now the condition (B) is supposed to hold, we get from (1) and (2) by virtue of [1] 5.5 that

(14) $\mathscr{E}_A(X)\, \mathscr{E}_B(Y) = \mathscr{E}_B(Y)\, \mathscr{E}_A(X)$ for every X, $Y \in \mathrm{B(C)}$.

On the other hand, we get from (1) and (2) by virtue of 3.4 that

(15) $\mathscr{E}_A(X_1)\, \mathscr{E}_A(X_2) = \mathscr{E}_A(X_2)\, \mathscr{E}_A(X_1)$, $\mathscr{E}_B(Y_1)\, \mathscr{E}_B(Y_2) = \mathscr{E}_B(Y_2)\, \mathscr{E}_B(Y_1)$ for every $X_1, X_2, Y_1, Y_2 \in \mathrm{B(C)}$.

Let us denote

(16) $D = \{\mathscr{E}_A(X)\,\mathscr{E}_B(Y)\,x\colon x \in E, X, Y \in \mathsf{B}(\mathrm{C}),\ X, Y \text{ bounded sets}\}$.

It is clear from (16) that for every $d \in D$ we can fix sets $\tilde{X}$, $\tilde{Y}$ such that

(17) $\tilde{X}, \tilde{Y} \in \mathsf{B}(\mathrm{C}),\ \tilde{X}, \tilde{Y}$ bounded,

(18) $d \in \mathsf{R}(\mathscr{E}_A(\tilde{X})\,\mathscr{E}_B(\tilde{Y}))$.

We obtain from (1), (2) and (17) by 3.3 that

(19) $\mathsf{R}(\mathscr{E}_A(\tilde{X})) \subseteq \mathsf{D}(A),\ \mathsf{R}(\mathscr{E}_B(\tilde{Y})) \subseteq \mathsf{D}(B)$.

Further, from (1) and (2) by [1] 5.4 (a) we find that

(20) $\mathscr{E}_A(\tilde{X})\,A \subseteq A\,\mathscr{E}_A(\tilde{X}),\ \mathscr{E}_B(\tilde{Y})\,B \subseteq B\,\mathscr{E}_B(\tilde{Y})$.

Finally, from (1), (2) and (14) by [1] 5.4(c) we conclude that

(21) $\mathscr{E}_B(\tilde{Y})\,A \subseteq A\,\mathscr{E}_B(\tilde{Y}),\ \mathscr{E}_A(\tilde{X})\,B \subseteq B\,\mathscr{E}_A(\tilde{X})$.

Let us now put

(22) $P = \mathscr{E}_A(\tilde{X})\,\mathscr{E}_B(\tilde{Y})$.

It follows from (14), (15) and (17) that

(23) the operator P is a projector from $\mathsf{L}(E)$.

Further, it is obvious from (18) and (22) that

(24) $d \in \mathsf{R}(P)$.

By (19) and (22) we have

(25) $\mathsf{R}(P) \subseteq \mathsf{D}(A) \cap \mathsf{D}(B)$.

By (20), (21) and (22), we have

(26) $PA \subseteq AP,\ PB \subseteq BP$.

Since by 3.1, the operators A, B are closed, we get from (23) and (25) by virtue of the closed graph theorem that

(27) $AP, BP \in \mathsf{L}(E)$.

On the other hand, it follows from (1), (2), (23) and (26) by virtue of 3.2 that

(28) the functions $\mathscr{E}_A P$, $\mathscr{E}_B P$ are resolutions of the operators AP, BP respectively.

Further, it follows from (1), (2), (14), (15) and (22) that

(29) $\mathscr{E}_A(X)\,P = P\,\mathscr{E}_A(X),\ \mathscr{E}_B(Y)\,P = P\,\mathscr{E}_B(Y)$ for every $X, Y \in \mathsf{B}(\mathrm{C})$.

As an easy consequence of (14) and (29) we obtain

(30) $\mathscr{E}_A(X)\,P\,\mathscr{E}_B(Y)\,P = \mathscr{E}_B(Y)\,P\,\mathscr{E}_A(X)\,P$ for every $X, Y \in \mathsf{B}(\mathrm{C})$.

Using [1] 5.5, we get from (28) and (30) that

(31) the operators AP, BP are transmutative.

However, by [1] 2.8 it follows from (27) and (31) that

(32) $APBP = BPAP$.

Since $d \in D$ was arbitrary, we conclude from (23), (24), (25), (26) and (32) that

(33) for every $d \in D$ there exists a projector $P \in \mathsf{L}(E)$ for which the conditions (A) (I)–(IV) are fulfilled.

On the other hand, we get from (1), (2) and (16) by 3.5 that

(34) the set D is dense in E.

In view of 3.1, it follows from (34) that

(35) the set D is dense in cl $\mathsf{D}(A) \cap$ cl $\mathsf{D}(B)$.

Now the proof of the implication (B) $\Rightarrow$ (A) is in fact complete since the condition (A) follows from (33) and (35).

3.7. Theorem. *Let* $A, B \in \mathsf{L}^+(\mathsf{E})$. *If the operators,* A, B *are scalar, then the statement* 2.5(B) *implies the statement* 2.5(A).

Proof. By 3.6, we can find a dense subset D of cl $\mathsf{D}(A) \cap$ cl $\mathsf{D}(B)$ such that the condition 3.6(A) holds.

Consequently, for every $d \in D$ we can fix a projector $P_d \in \mathsf{L}(E)$ such that

(1) $d \in \mathsf{R}(P_d)$ for every $d \in D$,

(2) $\mathsf{R}(P_d) \subseteq \mathsf{D}(A) \cap \mathsf{D}(B)$ for every $d \in D$,

(3) $P_dA \subseteq AP_d$, $P_dB \subseteq BP_d$ for every $d \in D$,

(4) $AP_dBP_d = BP_dAP_d$ for every $d \in D$.

Let us now define

(5) $\Lambda = \bigcap_{d \in D} \mathsf{R}(P_d)$.

It is clear from (1) and (5) that

(6) $D \subseteq \Lambda$.

Since cl $\mathsf{D}(A) \cap$ cl $\mathsf{D}(B) = E$ by 3.1, we obtain from (6) that

(7) the set Λ is a dense subset of cl $\mathsf{D}(A) \cap$ cl $\mathsf{D}(B)$.

It follows from (2) that

(8) $\mathsf{D}(AP_d) = \mathsf{D}(BP_d) = E$ for every $d \in D$.

Further, by (3) we have

(9) $ABP_d \supseteq AP_dBP_d$, $BAP_d \supseteq BP_dAP_d$ for every $d \in D$.

Now we obtain from (8) and (9) that

(10) $\mathsf{D}(ABP_d) = \mathsf{D}(BAP_d) = E$ for every $d \in D$,

(11) $ABP_d = AP_dBP_d$, $BAP_d = BP_dAP_d$ for every $d \in D$.

Hence (4) and (11) give

(12) $ABP_d = BAP_d$ for every $d \in D$.

From (5), (10) and (12) we see that

(13) $\Lambda \subseteq \mathsf{D}(AB) \cap \mathsf{D}(BA)$,

(14) $ABx = BAx$ for every $x \in \Lambda$.

Now we prove that

(15) for every $x \in \Lambda$, there exists a closed linear subspace Γ of E such that $x \in \Gamma$, $\Gamma \subseteq \Lambda$, $A(\Gamma) \subseteq \Gamma$ and $B(\Gamma) \subseteq \Gamma$.

To that aim, let us fix an $x \in \Lambda$.
By (5), we can find a fixed $d \in D$ such that

(16) $x \in \mathsf{R}(P_d)$.

Let us now define

(17) $\Gamma = \mathsf{R}(P_d)$.

It is clear from (17) that

(18) the set Γ is a closed linear subspace of E.

By (5), (16) and (17) we have

(19) $x \in \Gamma$, $\Gamma \subseteq \Lambda$.

On the other hand, we see easily from (3) and (17) that

(20) $A(\Gamma) \subseteq \Gamma$, $B(\Gamma) \subseteq \Gamma$.

Since $x \in \Lambda$ was arbitrary, the points (18), (19) and (20) prove (15).

Summing up (7), (13), (14) and (15) we see that the desired property 2.5(A) is verified.

3.8. Theorem. *Let* $A, B \in \mathsf{L}^+(E)$. *If*
(α) *the operators* A, B *are scalar,*
(β) *the operators* A, B *are extensively resolventive,*
then the statements 2.5(A) *and* 2.5(B) *are equivalent.*

Proof. (A) $\Rightarrow$ (B): Immediately from (β) and 2.5.
(B) $\Rightarrow$ (A). Immediately from (α) and 3.7.

3.9. Examples. Normal operators in Hilbert spaces are scalar. Selfadjoint operators in Hilbert spaces are scalar extensively resolventive. Scalar operators with real spectra are also extensively resolventive.

3.10. Remark. It is interesting that the conditions (A) in 2.1 and 3.6 are formally very analogous.

3.11. Remark. The class of sclar extensively resolventive operators for which Theorem 3.8 holds may seem fairly restrictive but it is not so bad as shown in 3.9.

3.12. Remark. In view of [1] 5.5, the equivalence of the statements 2.5(A) and 2.5(B) following from Theorem 3.8 can be considered an answer to the demand of having a criterion of transmutativity directly in terms of the operators in question. Compare e.g. [3], VIII.5, p. 272. It is noteworthy that some simpler conditions than 2.5(A), which seem reasonable at the first sight, are shown insufficient (cf. 2.7 and 2.8).

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Souhrn

ZÁMĚNNOST NEOHRANIČENÝCH OPERÁTORŮ: SPECIÁLNÍ KRITERIA TRANSMUTATIVITY

MIROSLAV SOVA

Článek je pokračováním autorovy práce o záměnnosti neohraničených operátorů.

Резюме

ПЕРЕСТАНОВОЧНОСТЬ НЕОГРАНИЧЕННЫХ ОПЕРАТОРОВ: СПЕЦИАЛЬНЫЕ КРИТЕРИИ ПЕРЕСТАНОВОЧНОСТИ

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Статья является продолжением работы автора о перестановочности неограниченных операторов.

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