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# BOOLEAN EMBEDDINGS AND HIDDEN VARIABLES

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*Summary.* In this paper, two known methods of embedding a quantum logic or more generally an orthocomplemented poset in a Boolean algebra are considered. Some simplifications in the proofs are introduced. It is also shown on an elementary example that the existence or non-existence of hidden variables depends on the definition of hidden variables and on the space states which can be realized on the considered system.

*Keywords:* orthocomplemented posets, hidden variables.

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Since the foundation of quantum theory, there has been a search for the description of quantum theory as a theory of the classical system where a part of the coordinates is unknown and the behaviour is governed by the laws of probability.

Therefore it is necessary to extend the description of a system with additional variables. If we take as the basis of the description the set of possible questions (yes — no experiments), then it is necessary to extend this set to a Boolean alebra which serves as a basis for the description of a classical system.

In the following, we shall give the survey of two methods which were developed for the solution of this task. Some simplifications, a slight generalization, and some corrections are added.

## 1.

The first method of extending the set of yes — no experiments (briefly the logic) $P$ to a Boolean algebra is in [ZS]. It is possible to extend this method to a more general case: the set $P$ can be only a poset with an orthocomplementation — the weak modularity is unnecessary.

Our assumption is therefore the following:

$P$ is a poset with the least element 0, the greatest element 1, and an involution $x \to x'$:

$$x'' = x\,, \quad x \leqq y \Rightarrow y' \leqq x' \quad \text{and} \quad x \wedge x' = 0\,, \quad x \vee x' = 1\,.$$

Let $C$ be a subset which has the following properties:

C1) it is a Boolean algebra with regard to the ordering induced by $P$ and to the functions of $P$,

C2) if $x, y \in C$, $z \in P$, then $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$.

If $P$ is an orthomodular poset with existence of sup for orthogonal elements, then the properties C1, C2 give that $C$ is a subset of center.

If $Q$ is another poset with the same properties as $P$, then a function $h: P \to Q$ is called a homomorphism if

(1) $x \leqq y \Rightarrow h(x) \leqq h(y)$, $h(0) = 0$,

(2) $h(x') = (h(x))'$,

(3) if $x, y \in C$ and $h(x) \vee h(y)$ exists, then $h(x \vee y) = h(x) \vee h(y)$.

Remark. It would be better to speak of a homomorphism of $(P, C)$ in $Q$. If $P$ is a Boolean algebra, then we shall suppose that $C = P$.

An ideal in $P$ (more correctly in $(P, C)$) is a non-void set $I$ with the properties

(4) $x \in I$ and $y \leqq x \Rightarrow y \in I$,

(5) $I \cap I' = \emptyset$,

(6) if $x, y \in C \cap I$, then $x \vee y \in I$.

We shall suppose that our poset $P$ has more than one element and so $0 \neq 1$. The kernel $h^{-1}(0)$ of a homomorphism $h: P \to Q$ is an ideal.

**Lemma 1.1.** *Every ideal is contained in a maximal ideal.*

This follows from Zorn's lemma.

**Lemma 1.2.** *If $x' \leqq x$, then $x = 1$ (or if $x \neq 1$, then $x' \leqq x$ does not hold).*

Proof. If $x' \leqq x$, then $1 = x \vee x' = x$.

**Lemma 1.3.** *If $I$ is an ideal and $x' \notin I$, then there is an ideal which contains $I$ and $x$.*

Proof. Let $J = \mathrm{E}[y \mid y \leqq x]$, $K = \mathrm{E}[z \mid z \leqq u \vee v, \ u \in I \cap C, \ v \in J \cap C]$, $I_1 = I \cup J \cup K$.

If $\eta \leqq \xi \in I$ ($J, K$ resp.), then $\eta \in I$ ($J, K$ resp.) and so (4) is satisfied.

Now to (6). Let $\xi, \eta \in I_1 \cap C$. Then there are substantially six possibilities for the distribution of $\xi, \eta$ in $I \cap C$, $J \cap C$, $K \cap C$. It is an easy consequence of C1, C2 and (4) that $\xi \vee \eta \in I_1 \cap C$ in all cases.

Now the proof of (5). Let $\xi, \xi' \in I_1$. $\xi, \xi' \in I \cup J$ is impossible. So we have, for example, $\xi \in K$, hence $\xi \leqq u \vee v$, $u \in I \cap C$, $v \in J \cap C$. We have $u \vee v \in K \cap C$, and $u' \wedge v' \in C$. Because $u' \wedge v' \leqq \xi' \in I_1$, so $u' \wedge v' \in I_1$ by (4). It is $u \vee v \vee \vee (u' \wedge v') = 1 \in I_1$ by (6). Hence $\eta_1 \vee \eta_2 = 1$ with $\eta_1 \in I \cap C$, $\eta_2 \in J \cap C$ (because 1 does not belong to $I \cup J$), $\eta_2 = \eta_2' \wedge \eta_1 \leqq \eta_1$. But $\eta_2' \geqq x'$ and so $x' \in I$ — a contradiction.

**Lemma 1.4.** *If $y \leqq x$ does not hold, then there is an ideal which contains $x$ and does not contain $y$.*

Proof. In Lemma 1.3, we choose $I = \mathrm{E}[z \mid z \leqq x]$. There is an ideal which contains $I$ and $y'$, and hence it does not contain $y$.

**Lemma 1.4'.** *If $x \neq y$, then there is an ideal which contains exactly one of the elements $x$, $y$. Hence there is a maximal ideal with this property.*

**Lemma 1.5.** *An ideal $I$ is maximal iff $I \cup I' = P$.*

**Lemma 1.6.** *The kernel of a homomorphism $P \to \mathbb{Z}_2$ is a maximal ideal. For $x \neq 0$ there is a homomorphism $\varphi\colon P \to \mathbb{Z}_2$ such that $\varphi(x) = 1$.*

Proof. Because $h(x') = (h(x))'$, the kernel is an ideal and by Lemma 1.5 it is a maximal ideal. If we set $h(x) = 0$ for $x \in I$ ($I$ a maximal ideal) and $h(x) = 1$ otherwise, then $h$ is a homomorphism $P \to \mathbb{Z}_2$.

**Theorem 1.1.** *Let $P$ be given. There is a Boolean algebra $A$ and a homomorphism $\alpha\colon P \to A$ that for a homomorphism $\beta\colon P \to B$, $B$ a Boolean algebra, there is a unique factorization $\beta = \chi \circ \alpha$, $\chi$ is a homomorphism $A \to B$. If $\alpha(x) \leqq \alpha(y)$, then $x \leqq y$ and so $\alpha$ is injective.*

Proof. For $x \in P$, let $E_x$ be the set of maximal ideals which do not contain $x$. Let $A$ be the Boolean algebra generated by all $E_x$ in the set of all maximal ideals. Let $\alpha$ be defined by $\alpha(x) = E_x$. Then $\alpha$ satisfies (1) and (2). If $x, y \in C$, then $x \vee y \geqq x, y$, and so $E_{x \vee y} \supseteq E_x \cup E_y$. If a maximal ideal $I$ is in $E_{x \vee y}$, then it does not contain $x \vee y$, hence it cannot contain $x, y$ simultaneously. It is, therefore, in $E_x$ or $E_y$, and so $\alpha(x \vee y) = E_x \cup E_y$.

$\alpha$ is, therefore, a homomorphism from $P$ to $A$.

Let $B$ be another Boolean algebra and $\beta\colon P \to B$ a given homomorphism.

Every element of $A$ is of the form $\xi = \bigcup_i \bigcap_k \alpha(x_{ij})$, $x_{ij} \in P$. We set $\chi(\xi) = \bigvee_i \bigwedge_k \beta(x_{ij})$.

If

$$(*) \qquad \bigcup_i \bigcap_j \alpha(x_{ij}) = \bigcup_k \bigcap_l \alpha(y_{kl})$$

and $\xi = \bigvee_i \bigwedge_j \beta(x_{ij}) \neq \bigvee_k \bigwedge_l \beta(y_{kl}) = \eta$, then there is a homomorphism $h\colon B \to \mathbb{Z}_2$ such that, for example, $h(\xi) = 0$, $h(\eta) = 1$.

Because $h$ is a homomorphism, we have $\bigvee_i \bigwedge_j h(\beta(x_{ij})) = 0$. If $I$ is the kernel of the hompmorphism $h \circ \beta\colon P \to \mathbb{Z}_2$, then for every $i$ there is such $j(i)$ that $h(\beta(x_{ij(i)})) = 0$. Hence $x_{ij(i)} \in I$, and $I \notin \alpha(x_{ij(i)})$. $I$ does not belong to the right-hand side in $(*)$. Therefore, for every $k$ there is $l(k)$ such that $y_{kl(k)} \in I$, and hence $h(\bigvee_k \bigwedge_l (\beta(y_{kl}))) = 0$ – a contradiction. We can set $\chi(\bigcup_i \bigcap_j \alpha(x_{ij})) = \bigvee_i \bigwedge_j \beta(x_{ij})$, and $\chi$ is uniquely determined on all $A$. The rest follows from Lemma 1.4.

Remarks.

1. If $\alpha$ is considered as a functor from the category $P$ to the category $A$, then $\alpha$ is full and it is faithful because there are no two distinct morphisms from $x$ to $y$ in $P$.

2. We have supposed that $x \to x'$ is an orthocomplementation. If $C = (0, 1)$, then the assumption $x \vee x' = 1$ is used only for the proof of Lemma 1.2. Hence we can suppose that the involution $x \to x'$ satisfies this lemma only.

However, this is necessary. If $x' \leqq x \neq 1$, then there is no maximal ideal not containing $x'$ which contains $x$.

## 2.

We shall now describe the second method which is given in [M]. The method depends on a subset $S$ of the given orthocoomplemented set $P$.

Let $P$ be a poset with orthocomplementation $x \to x'$ and let $S$ be a subset of $P$. For a set $M \subset S$, let $M_r$ be the set of minimal elements from $M$. We set $M * N = (M \cup N)_r$ for $M, N \subset S$. Let $\mathfrak{S}$ be the family of finite subsets $M \subset S$ which contain minimal elements only. $\mathfrak{S}$ with the multiplication $*$ is an abelian idempotent semigroup whose unit element is $\emptyset$.

Let $A_1(S)$ (or $A_1$ when $S$ is fixed) be the space of functions from $\mathfrak{S}$ to $\mathbb{Z}_2$ that have finite support. For $M \subset S$ let $M$ be also the element from $A_1$ whose support is exactly $M$.

The multiplication $*$ can be extended from $\mathfrak{S}$ to $A_1$ by the distributive law. The unit $1_A$ for this multiplication is the function which has the value 1 ($\in \mathbb{Z}_2$) for $\emptyset \in \mathfrak{S}$ and the value 0 otherwise.

With this multiplication, $A_1$ is a Boolean ring and we can introduce the lattice operations there:

$$x \leqq y \Leftrightarrow x * y = x\,, \quad x \wedge y = x * y\,,$$

$$x \vee y = x + y + (x * y)\,,$$

and the orthogonel complement $x' = 1 + x$.

We have the following lemmas.

**Lemma 2.1.** *If $M, N_1, N_2, \ldots \in \mathfrak{S}$ and $M \leqq \sum N_k$, then $M \leqq N_1$ for some $i$.*

Proof. We can suppose that all the right-hand side terms are distinct. $M \leqq \sum N_k$ means $M = \sum M * N_k$ and hence on the right-hand side there is a term, say $M * N_i$, equal to $M$. But $M * N_i = M$ yields $M \leqq N_i$.

**Lemma 2.2.** *If $M, N_1, N_2, \ldots \in \mathfrak{S}$ and $M \leqq \bigvee N_k$, then $M \leqq N_i$, for some $i$.*

Proof. $\bigvee N_k = \sum_k N_k + \sum_{k_1 < k_2} N_{k_1} * N_{k_2} + \ldots$ and so $M \leqq N_{k_1} * N_{k_2} \ldots$ for some indices $k_1 < k_2 < \ldots$. Therefore $M \leqq N_{k_1}, N_{k_2}, \ldots$.

**Lemma 2.3.** *If $\bigvee M_k \leqq \bigvee N_l$, $M_k, N_l \in \mathfrak{S}$, then for every $M_k$ there is such $N_{l(k)}$ that $M_k \leqq N_{l(k)}$.*

Proof. We have $M_k \leqq \bigvee N_l$ for every $k$.

Up to now, we have used only the fact that $P$ is a poset. Now, we shall reconsider the situation in Section **1**, i.e., $P$ is an orthocomplemented poset with the greatest element $1_P$ (or 1) and the least element $0_P$ (or 0) and with the orthocomplementation $x \to x'$ as in **1**. We set $S' = \mathrm{E}[x \mid x' \in S]$.

If $S$ contains $1_P$, then there is the function $(1_P)$ in $A_1$ and this function is covered by the function $\emptyset$. If we omit this function (that is, we consider only non-void sets in $\mathfrak{S}$), then the function $(1_P)$ becomes the unit element in $A_1$.

A similar situation occurs for the least element $0_P$; the function $(0_P) \in A_1$ covers the zero-function $\in A_1$ and is the least element in $A_1$ after omitting this function. This would be inconvenient because we need the zero-function and so we shall suppose that $0_P \notin S$.

In the following, we shall suppose that $1_P \in S$ (in the terminology of [M]: $S$ is unitary).

As we have selected a subset $S$ in $P$, we must modify the notion of homorphism.

A function $h$ defined on $S \cup S'$ to a Boolean algebra $B$ will be called a homomorphism iff

(1) $x, y \in S$ and $x \leqq y \Rightarrow h(x) \leqq h(y)$,

(2) $x, x' \in S \Rightarrow h(x') = (h(x))'$.

There is no loss of generality if we suppose that $h(1_P) = 1_B$.

If we set $\alpha_1(M) = M$ for $M \in \mathfrak{S}$, then $\alpha_1$ is an order-preserving function from $S$ to $A_1$. We can extend $\alpha_1$ by setting $\alpha_1(x') = (1_P) + (x)$ if $x \in S$ and $x' \notin S$.

We shall denote by $I$ the ideal in $A_1$ that is generated by all expressions $(1_P) + + (x) + (x')$ for $x, x' \in S$. Let $\pi$ be the canonical projection $A_1 \to A_1/I = A_2$ and $\alpha_2 = \pi \circ \alpha_1$.

**Lemma 2.4.** *$\alpha_2$ is defined for $x \in S \cup S'$ and has the following properties*

(3) *$x, y \in S$ and $x \leqq y \Rightarrow \alpha_2(x) \leqq \alpha_2(y)$,*

(4) *$x \in S \Rightarrow \alpha_2(x') = (\alpha_2(x))'$.*

Proof. (3) follows from the fact that $x = x * y$ in $A_1$ is preserved in $A_2$.

If $x \in S$ and $x' \notin S$, then (4) is valid due to the definition of $\alpha_1$.

If $x, x' \in S$, then (4) follows from the definition of $I$.

From now on, we shall suppose that $S$ is a base, that is, $S \cup S' = P$. Hence $\alpha_1, \alpha_2$ are defined on the whole $P$.

**Lemma 2.5.** *If $S_1, S_2$ are two bases and $S_1 \subset S_2$, then $A_2(S_2)$ is a homomorphic image of $A_2(S_1)$.*

Proof. There are homomorphisms $A_1(S_1) \to A_1(S_2) \to A_2(S_2)$. The first is given by the inclusion $S_1 \subset S_2$. If $x, x' \in S_1$, then $x, x' \in S_2$ and hence every generator $(1_P) + (x) + (x')$ of $I(S_1)$ is in $I(S_2)$ and hence we have a homomorphism $A_2(S_1) \to$

$\to A_2(S_2)$. On the other hand, every element of $A_1(S_2)$ can be transformed to an element of $A_1(S_2)$ mod $I(S_2)$ and so $A_1(S_1) \to A_2(S_2)$ is onto.

The implication (3) can be reversed.

**Lemma 2.6.** *If $\alpha_2(x) \leqq \alpha_2(y)$ for $x, y \in S$, then $x \leqq y$.*

Proof. If $x \leqq y$ does not hold, then there is a maximal ideal $I$ in $P$ which contains $y$ and does not contain $x$ (see Lemma 1.4). Let $\varphi$ be the homomorphism $P \to \mathbb{Z}_2$ with the kernel I.

If $\xi \in \mathfrak{S}$ and $\xi = (x_1) * \ldots * (x_k)$, then we set $\varphi(\xi) = \varphi(x_1) \ldots \varphi(x_k)$.

If $\xi \leqq \eta$, $\xi, \eta \in \mathfrak{S}$, then $\varphi(\xi) = \varphi(\xi) . \varphi(\eta)$: if $\varphi(\xi) = 1$, then $\varphi(\eta) = 1$ too, and if $\varphi(\xi) = 0$ then both sides vanish. Hence $\varphi(\xi * \eta) = \varphi(\xi) . \varphi(\eta)$. Therefore $\varphi$ can be extended on $A_1(S)$. If $\xi, \xi' \in S$, then $\varphi((1_P) + (\xi) + (\xi')) = 1 + \varphi(\xi) + \varphi(\xi') = 0$, and so $\varphi$ is defined on $A_2 = A_1/I$. We have $\varphi(\alpha_2(x)) = 1$ and $\varphi(\alpha_2(y)) = 0$, and so $\alpha_2(y) \geqq \alpha_2(x)$ cannot hold.

We shall show that Lemma 2.1 is not valid in $A_2(S)$. $P$ is the poset in Fig. 1.

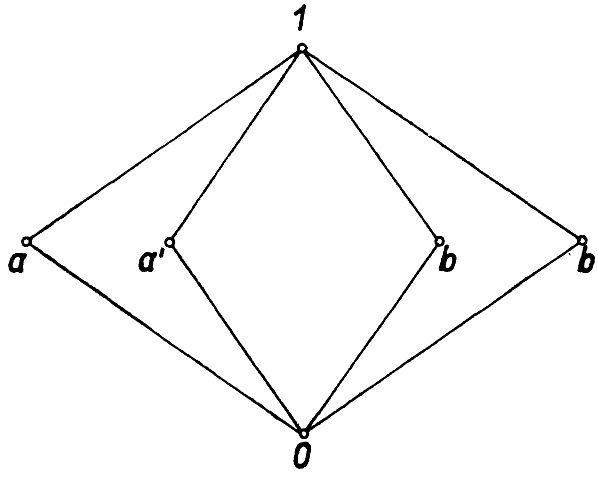


Fig. 1

We choose $S = (a, b, b', 1_P)$. In $A_2(S)$ we have $(b) + (b') = (1_P)$ and therefore $(a) \leqq (b) + (b')$. Let $\varphi$ be $P \to \mathbb{Z}_2$ such that $\varphi(1_P) = \varphi(a) = \varphi(b') = 1$, $\varphi(a') = \varphi(b) = \varphi(0_P) = 0$. $\varphi$ is a homomorphism with $\varphi(a) > \varphi(b)$ but not $(a) \leqq (b)$. Similarly, $(a) \leqq (b')$ does not hold.

**Theorem 2.1.** *Let $B$ be a Boolean algebra and $h$: $P \to B$ a homomorphism. Then there is a unique homomorphism $\chi$: $A_2(S) \to B$ such that $h = \chi \circ \alpha_2$.*

Proof. We set $\chi((x)) = h(x)$ for $x \in S$. If $M = (x_1) * \ldots * (x_k)$, we set $\chi(M) = h(x_1) \wedge \ldots \wedge h(x_k)$. $\chi$ is defined on $\mathfrak{S}$ and if $(x) = (y) * (z)$, then $\chi((x)) = h(x) = h(y) \wedge h(z) = \chi((y) * (z))$. We can extend $\chi$ to $A_1(S)$ by linearity.

If $x, x' \in S$, then $h((1_P) + (x) + (x')) = 1_B + h(x) + (h(x'))' = 0$, and therefore $\chi$ is 0 on $I$. $\chi$ is defined on $A_2(S) = A_1/I$ and is unique.

**Corollary.** *If $S = P$ (or $P - (0)$), then $A_2(S)$ from Theorem 2.1 is isomorphic with $A$ from Theorem 1.1.*

This is because the algebras $A_2(S)$ and $A$ have the same property of universality.

Another proof of Lemma 2.6. Let $S$ be a base, $x, y \in S$ and $\alpha_2(x) \leqq \alpha_2(y)$ in $A_2(S)$. Now, $A_2(P - (0))$ is a homomorphic image of $A_2(S)$ and therefore $\alpha_2(x) \leqq \alpha_2(y)$ in $A_2(P - (0))$. Theorem 1.1 yields that $x \leqq y$.

A special case occurs when $S \cap S' = \emptyset$; then $A_2 = A_1$ and Lemma 2.3 is valid. In this case, the homomorphism $\alpha_2(= \alpha_1)$ has the weakest property (1). This is compensated by the possibility of extending the monotone functions on $P$ to a measure on $A_2$. This is demonstrated in [M] (for states on $P$, but see Remark 15 in [M]). It seems to us that there is a gap in the proof and so we shall give the proof in detail.

**Theorem 2.2.** *Let $S$ be a base, $S \cap S' = \emptyset$ and let $\mu$ be a non-negative function on $P$ which is monotonic and $\mu(1_P) = 1$. Then there exists a measure on $A_2$ which is an extension of $\mu|_S$.*

The proof is based on the following theorem of Horn and Tarski:

If $M$ is a subset in Boolean algebra $B$, $1_B \in M$, and if $\mu$ is a non-negative function on $M$ with $\mu(1_B) = 1$, then there is a measure on $B$ which is an extension of $\mu$ if the following condition is satisfied:

if $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_n)$ are two sequences of elements from $M$ and if

$$(5) \qquad \bigvee_{(i_1,\ldots,i_k)} (\bigwedge(a_{i_1}, \ldots, a_{i_k})) \leqq \bigvee_{(j_1,\ldots,j_k)} (\bigwedge(b_{j_1}, \ldots, b_{j_k}))$$

for every $k = 1, \ldots, m$, then

$$\sum_{i=1}^{m} \mu(a_i) \leqq \sum_{j=1}^{n} \mu(b_j)\,.$$

Remark. This condition is also necessary, but we shall need only this sufficiency.

Proof. We shall apply this theorem to $M = \alpha_2(S) \subset A_2$; $a_1, \ldots, b_1, \ldots$ will be the element of the form $\alpha_2(a_1), \ldots, \alpha_2(b_1), \ldots$ and we shall omit $\alpha_2$.

If (5) is satisfied, then for every sequence $a_{i_1}, \ldots, a_{i_k}$ there is such a sequence $b_{j_1}, \ldots, b_{j_k}$ that every $b_{j_s}$ is $\geqq$ an element $a_{i_r} \in (a_{i_1}, \ldots, a_{i_k})$. This is Lemma 2.3 applied to $M_i = (a_{i_1}, \ldots, a_{i_k})$, $N_j = (b_{j_1}, \ldots, b_{j_k})$.

For $k = 1$, there is a non-void set $B_i \subset (b_1, \ldots, b_n)$ for every $i$ which contains those elements $b_1, \ldots$ which are $\geqq a_i$. We shall show that for every $k$ elements $a_{i_1}, \ldots, a_{i_k}$ the set $B_{i_1} \cup \ldots \cup B_{i_k}$ has at least $k$ elements.

For example, if we take $(a_1, \ldots, a_k)$, there are such $k$ elements $(b_{j_1}, \ldots, b_{j_k})$ that every $b_{j_s}$ is $\geqq$ an element $\in (a_1, \ldots, a_k)$. Therefore $B_1 \cup \ldots \cup B_k$ contains $b_{j_1}, \ldots, b_{j_k}$ and so $B_1 \cup \ldots \cup B_k$ has cardinality at least $k$.

By a known theorem [HV], there is an injection $\varphi: (a_1, \ldots) \to (b_1, \ldots)$ such that $\varphi(a_i) = b_{j(i)} \in B_i$.

We divide the sum $\sum \mu(b_j)$ in two parts: the first over $b_{j(i)}$ and the second over the other $b_j$. From the monotonicity we have

$$\sum_i \mu(b_{j(i)}) \geqq \sum_i \mu(a_i)$$

and the second part $\geqq 0$ as $\mu$ is non-negative. The assumption of the Horn-Tarski theorem is satisfied and so $\mu$ has an extension to a measure on $A_2$.

Remark. 1) In [M] the theorem of [HT] is applied for $k = 1$ and $k = m$ only. However, this does not suffice as the following example shows.

We have $m = n = 3$ and the elements $(a_1, a_2, a_3)$, $(b_1, b_2, b_3)$ are ordered as shown in Fig. 2.

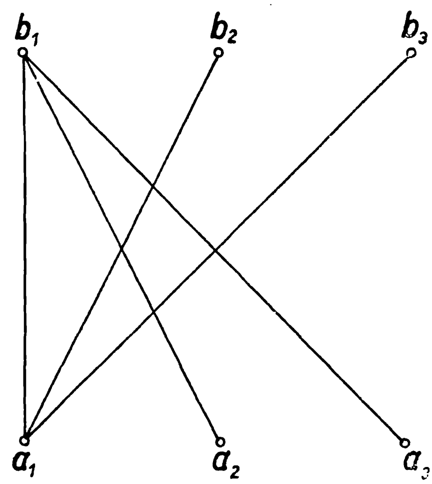


Fig. 2

If the function $\mu$ is such that

$$\mu(b_1), \mu(a_2), \mu(a_3) \quad \text{are great}$$

and

$$\mu(a_1), \mu(b_1), \mu(b_2) \quad \text{are small},$$

then we cannot conclude that $\sum \mu(a_i) \leqq \sum \mu(b_j)$, in despite of the fact that (5) is satisfied for $k = 1, 3$.

2) In [M], such a Hilbert space $H(S)$ is constructed that every $x \in P$ is represented by a one-dimensional projector. The construction depends on the base $S$ and the space $H(S)$ is rather the space of Hilbert-Schmidt operators of the usual (von Neumann) formulation. This is similar to Gudder's construction in [G].

It would be desirable to have, in $H(S)$, a multiplication so that $H(S)$ becomes a Hilbert algebra. Then it may be possible to find von Neumann's formulation of [M].

## 3.

The problem of hidden variables for a system is to extend the description so it becomes a description of a classical system.

For von Neumann's description with a Hilbert space, such an extension is impossible. For the description through a quantum logic, this problem was solved by Jauch and Piron [JP]. They excluded the introduction of hidden variables, too. But there is a discussion by Bohm and Bubb [BB] which was followed by a remark of Gudder.

The problem with hidden variables is that it is not exactly stated what should be done and von Neumann's proof is rejected as not solving problem proper.

We shall present a simple example and show the situations which can occur.

The basis for the description will be a set of yes-no experiments $L$ which is a poset with orthocomplementation as was $P$ in 1. But the description must contain also the set of states. We must make a difference between the set of all states — functions from $L$ to $\langle 0, 1 \rangle$ — and the set of states which can be observed. The latter set can be smaller. This set of realizable states $S$ must fulfill some conditions connected with the order in $L$. The strongest condition is that the set $S$ of possible states is strictly order determining: $(s(a) = 1 \Rightarrow s(b) = 1 \text{ for all } s \in S) \Rightarrow a \leqq b$. This implies that $S$ is order determining [P] (or full in [ZS]), hence $S$ is separating and $L$ is weakly modular. Further, there is a state $s \in S$ such that $s(a) = 1$ if $a \neq 0$. We shall suppose that $S$ is full, and that there exists a state $s$ with $s(a) = 1$ for $a \neq 0$, and that $S$ is convex.

The pair $(L, S)$ will be called a system.

The existence of hidden variables means:

1) there is a Boolean algebra $B$ and a homomorphism $\varphi: L \to B$,

2) every state $s \in S$ can be extended from $\varphi(L)$ to a probability on $B$,

3) the algebra $B$ and the extended states form a system.

**Lemma.** *Let $B$ be a Boolean algebra and $S$ a set of states on $B$. $(B, S)$ is a system iff for every $a \neq 0$ there is a state such that $s(a) = 1$.*

Proof. We have to show that $S$ is full. Let $a, b \in B$ and not $a \leqq b$. Then $x = a \wedge b' \neq 0$. There exists a state $\sigma \in S$ such that $\sigma(x) = 1$, and we have $\sigma(a) = 1$, $\sigma(b) = 0$. Therefore $s(a) \leqq s(b)$ does not hold for $s \in S$.

We have seen that there exists a Boolean algebra $A$ and a homomorphism $\alpha: L \to A$, and that $\alpha$ is mono. By [ZS], every state $s$ which can be extended to $B$ can be extended to $A$. The solution is given by the theorem of Horn and Tarski [HT].

**Theorem.** *If $a_1, \ldots, a_m, b_1, \ldots, b_n$ are elements from $L$, $s \in S$ and*

$$(*) \qquad \bigcup_{(i_1,\ldots,i_k)} (\alpha(a_{i_1}) \cap \ldots \cap \alpha(a_{i_k})) \subseteq \bigcup_{(j_1,\ldots,j_k)} (\alpha(b_{j_1}) \cap \ldots \cap \alpha(b_{j_k}))$$

*for every $k = 1, \ldots, m$, then the n.s.c. for the existence of an extension to a state on $A$ is*

$$\sum_{i=1}^{m} s(a_i) \leqq \sum_{j=1}^{n} s(b_j) .$$

The condition $(*)$ means that at every homomorphism $L \to \mathbb{Z}_2$ which has the value 1 on $k$ elements $a_{i_1}, \ldots, a_{i_k}$ has the value 1 on some $k$ elements $b_{j_1}, \ldots, b_{j_k}$ as well.

This solution is only theoretical. It is possible to establish some consequences. For example:

If for elements $a_1, a_2, b_1, b_2$ every homomorphism $L \to \mathbb{Z}_2$ which has the value 1 on $a_1$ and $a_2$ has the value 1 on $b_1$ and $b_2$ and $s(a_1) + s(a_2) > s(b_1) + s(b_2)$, then $s$ cannot be extended.

The condition 3) in the definition of extension is necessary because some elements in $B$ would be non-identifiable without this condition.

Now, we shall consider the logic form Fig. 1. Every state $s$ on $L$ is determined by two numbers $s(a), s(b)$ which are from $\langle 0, 1 \rangle$. Because we suppose that $S$ is full, the numbers $s(a)$ $(s(b))$ fill the whole interval $\langle 0, 1 \rangle$.

The couples $(x, y) = (s(a), s(b))$ form a convex set in the unit square $Q$.

There are four homomorphism $L \to \mathbb{Z}_2$, viz $h_{11}, h_{12}, h_{21}, h_{22}$, corresponding to four pair $(a, b)$, $(a, b')$, $(a', b)$, $(a', b')$, such that $h_{11}(a) = h_{11}(b) = 1$ and so on.

So $\alpha: L \to A$ satisfies

$$\begin{aligned}
\alpha(a) &= (h_{11}, h_{12}),\\
\alpha(a') &= (h_{21}, h_{22}),\\
\alpha(b) &= (h_{11}, h_{21}),\\
\alpha(b') &= (h_{12}, h_{22}).
\end{aligned}$$

$A$ is the algebra of the subsets of the set $(h_{11}, h_{12}, h_{21}, h_{22})$.

Every state on $A$ is determined by four numbers $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}$. If this state should be an extension of the state $s$ on $L$, then

$$\begin{aligned}
x &= \mu_{11} + \mu_{12} ,\\
y &= \mu_{11} + \mu_{21} ,\\
1 - x &= \mu_{21} + \mu_{22} ,\\
1 - y &= \mu_{12} + \mu_{22} .
\end{aligned}$$

One solution is $\mu_{11} = xy$, $\mu_{12} = x(1 - y)$, $\mu_{21} = (1 - x)\, y$, $\mu_{22} = (1 - x)(1 - y)$. After subtracting this solution, we obtain a homogeneous system which has this solution

$$\mu_{12} = \mu_{21} = -\mu_{11}\ ; \quad \mu_{22} = \mu_{11}\,.$$

The complete solution is

$$\mu_{11} = xy + v\,, \quad \mu_{12} = x(1 - y) - v\,,$$
$$\mu_{21} = (1 - x)\, y - v\,, \quad \mu_{22} = (1 - x)(1 - y) + v$$

with $v$ restricted by the conditions

$$0 \leqq \mu_{11}, \mu_{12}, \ldots \leqq 1\,.$$

If all states are realizable then the solution is complete.

Another situation occurs when $S$ contains only states with $x + y \leqq 1$. Then $2\mu_{11} + \mu_{12} + \mu_{21} = 1 + (\mu_{11} - \mu_{22}) \leqq 1$. Hence $\mu_{22} \geqq \mu_{11}$ and the extended states have not the value 1 on $(h_{11})$, and so we do not have a system.

This case can be interpreted in the following way. In three-dimensional (real) space, we measure spin in two orthogonal directions. For this interpretation, there is no Boolean extension.

Nonetheless, we can have the following extension for the original system:

the algebra $B$ is generated by three elements $\varrho$, $\sigma$, $\tau$ and if $s$ is a state on $L$, then the prolongation $\bar{s}$ on $B$ is

$$\begin{aligned}
&\text{if} \quad s(a) = 1\,, \quad s(b) = 0\,, \quad \text{then} \quad \bar{s}(\varrho) = 1\,, \quad \bar{s}(\sigma) = \bar{s}(\tau) = 0\,,\\
&\text{if} \quad s(a) = 0\,, \quad s(b) = 1\,, \quad \text{then} \quad \bar{s}(\varrho) = 0\,, \quad \bar{s}(\sigma) = 1\,, \quad s(\tau) = 0\,,\\
&\text{if} \quad s(a) = s(b) = 0\,, \quad \text{then} \quad \bar{s}(\varrho) = \bar{s}(\sigma) = 0\,, \quad \bar{s}(\tau) = 1\,,
\end{aligned}$$

if $s$ is a linear combination of the preceding states, then $\bar{s}$ is the same combination.

The both cases differ by the presence or absence of further irreducible states.

Similar situation occurs when $S$ is the set of states with $x + y = 1$. But now the both cases have a Boolean extension.

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## Souhrn

## BOOLSKÉ VNOŘENÍ A SKRYTÉ PARAMETRY

VÁCLAV ALDA

V článku je dáno srovnání dvou metod vnoření orthokomplementární uspořádané množiny do Booleovy algebry. Dále na jednoduchém příkladu je ukázáno, že existence či neexistence skrytých parametrů závisí jednak na definici skrytých parametrů, jednak na prostoru fyzikálně realizovatelných stavů.

## Резюме

## БУЛЕВЫ ВЛОЖЕНИЯ И СКРЫТЫЕ ПАРАМЕТРЫ

VÁCLAV ALDA

В статье сравниваются два метода вложения ортокомплементарного упорядоченного множества в булеву алгебру. Кроме того на простом примере показывается, что существование или несуществование скрытых параметров зависит как от их определения, так и от пространства физически реализуемых состояний.

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