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INVARIANCE OF THE FREDHOLM RADIUS OF AN OPERATOR IN POTENTIAL THEORY

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Summary. One of the classical methods of solving the Dirichlet problem in R^n is the method of integral equations. Using this method for a non-smooth regions it is useful to know the Fredholm radium of an integral operator playing a role in the method. It is shown in the paper that in the case of a Jordan domain in the plane the Fredholm radius of that operator does not change under the conformal mapping of the boundary.

Keywords: integral equations, integral operators, Fredholm radius.

Classification AMS: 31A25, 35J05.

INTRODUCTION

We shall deal with the plane R^2. The real plane R^2 will be identified with the complex plane C, that is, a point $[x, y] \in R^2$ will be identified with the point $z \in C$, $z = x + iy$; similarly we shall write $[\xi, \eta] = \zeta = \xi + i\eta$, etc. If f is a real function defined on a subset of R^2, we may consider f as a real function of the complex variable z, but usually as a real function of two real variables $[x, y]$. The partial derivatives of f with respect to the real variables x, y will be denoted by $\partial_x f$, $\partial_y f$, respectively. Further, we shall write $\operatorname{grad} f = [\partial_x f, \partial_y f]$. If u is a real function of two complex variables $[z, \zeta]$, $z = x + iy$, $\zeta = \xi + i\eta$, one can write $u(z, \zeta) = u(x, y, \xi, \eta)$. By $\operatorname{grad}_z u$ ($\operatorname{grad}_\zeta u$) we mean the gradient of u with respect to z (with respect to ζ, respectively), that is,

$$\operatorname{grad}_z u = [\partial_x u, \partial_y u] \quad (\operatorname{grad}_\zeta u = [\partial_\xi u, \partial_\eta u]) .$$

Let K be a simple closed (i.e. Jordan) oriented curve in R^2. In the following we denote by $G(K)$ the interior of K (that is, the bounded complementary domain of K) and by ι_K the constant value of the index of a point with respect to K on $G(K)$ (that is, $\iota_K = 1$ if K is positively (counterclockwise) oriented and $\iota_K = -1$ if K is negatively (clockwise) orinted). Further, we shall always suppose that K is rectifiable.

Let $\mathscr{C}(K)$ stand for the Banach space of all real continuous functions on K (equipped with the supremum norm which is denoted by $\|\ldots\|$). For any $f \in \mathscr{C}(K)$, $z \in R^2$, $z \notin K$ let us put

$$(0.1) \qquad W_K f(z) = \frac{1}{\pi} \operatorname{Im} \int_K \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

(Im ... denotes the imaginary part of the complex integral on the right-hand side of (0.1)). $W_K f$ is called the (logarithmic) double layer potential of the density f on K. For $f \in \mathscr{C}(K)$, $\zeta \in K$ let us define $\overline{W}_K f(\zeta)$ by putting

$$(0.2) \qquad \overline{W}_K f(\zeta) = \lim_{\substack{z \to \zeta \\ z \in G(K)}} W_K f(z) - \iota_K f(\zeta),$$

provided the limit on the right-hand side of (0.2) exists and is finite. If for any $f \in \mathscr{C}(K)$ the potential $W_K f$ is uniformly continuous on $G(K)$, then $\overline{W}_K f$ is defined on K and, moreover, $\overline{W}_K f \in \mathscr{C}(K)$ for every $f \in \mathscr{C}(K)$. The operator $\overline{W}_K$ ($\overline{W}_K: f \mapsto \overline{W}_K f$, $\overline{W}_K$: $\mathscr{C}(K) \to \mathscr{C}(K)$) is then a linear bounded operator on $\mathscr{C}(K)$ (see, for instance, [6]).

The operator $\overline{W}_K$ plays a role in solving the Dirichlet problem on $G(K)$ by means of integral equations. In this connection it is useful to know what is the value of the Fredholm radius of $\overline{W}_K$. The Fredholm radius of the oprator $\overline{W}_K$ on $\mathscr{C}(K)$ is defined as the reciprocal value of $\omega \overline{W}_K$, where

$$\omega \overline{W}_K = \inf_A \| \overline{W}_K - A \|,$$

A ranging over all compact (linear) operators acting on $\mathscr{C}(K)$.

Let $D \subset R^2$ be an open set such that $K \subset D$. Suppose that Ψ is a conformal mapping defined on D, that is $\Psi: D \to R^2$, Ψ is one-to-one, and Ψ as a complex-valued function of the complex variable is holomorphic on D. Let us put

$$\hat{K} = \Psi(K).$$

Then, of course, $\hat{K}$ is a simple closed curve in R^2, and if K is rectifiable then $\hat{K}$ is rectifiable as well. Thus the double layer potential $W_{\hat{K}} \hat{f}(z)$ for $\hat{f} \in \mathscr{C}(\hat{K})$, $z \notin \hat{K}$ can be defined analogously to (0.1). Symbols $G(\hat{K})$, $\iota_{\hat{K}}$ will have the same meaning for $\hat{K}$ as the symbols $G(K)$, ι_K for K (we shall assume that $\hat{K}$ is oriented). Further, we can define the term $\overline{W}_{\hat{K}} \hat{f}(\zeta)$ ($\zeta \in \hat{K}$) analogously to (0.2). If for every $\hat{f} \in \mathscr{C}(\hat{K})$ the potential $W_{\hat{K}} \hat{f}$ is uniformly continuous on $G(\hat{K})$, then (similarly to the preceding) $\overline{W}_{\hat{K}} \hat{f} \in \mathscr{C}(\hat{K})$ for each $\hat{f} \in \mathscr{C}(\hat{K})$ and the operator $\overline{W}_{\hat{K}}$ ($\overline{W}_{\hat{K}}: \hat{f} \mapsto \overline{W}_{\hat{K}} \hat{f}$, $\overline{W}_{\hat{K}}: \mathscr{C}(\hat{K}) \to \mathscr{C}(\hat{K})$) is a linear bounded operator acting on $\mathscr{C}(\hat{K})$.

The aim of this paper is to prove the following assertion:

If for each $f \in \mathscr{C}(K)$ the potential $W_K f$ is uniformly continuous on $G(K)$ then the potential $W_{\hat{K}} \hat{f}$ is uniformly continuous on $G(\hat{K})$ for each $\hat{f} \in \mathscr{C}(\hat{K})$. If this condition is fulfilled then the Fredholm radii of the operators $\overline{W}_K$, $\overline{W}_{\hat{K}}$ coincide.

I. AUXILIARY ASSERTIONS

In this part two simple lemmas needed in the sequel will be proved.

Throughout this section $D \subset R^2$ will be a fixed open set and Ψ a conformal mapping on D (holomorphic and one-to-one). Let us denote

$$\hat{D} = \Psi(D).$$

If h is a function defined on a subset of $\hat{D}$ (of D) then we use the term $h * \Psi$ ($h * \Psi^{-1}$) for the composition of the functions h, Ψ (h, Ψ^{-1}), that is, $h * \Psi(z) = h(\Psi(z))$ ($h * \Psi^{-1}(z) = h(\Psi^{-1}(z))$, respectively).

For $r = 1, 2$ we denote by $\mathscr{H}_r$ the r-dimensional Hausdorff measure on R^2. $\mathscr{H}_r$ is supposed to be normalized in such a way that $\mathscr{H}_2$ coincides on R^2 with the outer 2-dimensional Lebesgue measure while $\mathscr{H}_1$ coincides on the lines in R^2 with the outer linear (1-dimensional) Lebesgue measure on those lines.

1.1. Lemma. *Let $M \subset D$ be open and let g, h be (real) functions defined on $\Psi(M)$ possessing continuous first partial derivatives there. Then the integral*

$$\int_M \operatorname{grad}[g * \Psi] \operatorname{grad}[h * \Psi] \, d\mathscr{H}_2 \tag{1.1}$$

exists if and only if the integral

$$\int_{\Psi(M)} \operatorname{grad} g \operatorname{grad} h \, d\mathscr{H}_2 \tag{1.2}$$

exists. If these integrals exist then

$$\int_M \operatorname{grad}[g * \Psi] \operatorname{grad}[h * \Psi] \, d\mathscr{H}_2 = \int_{\Psi(M)} \operatorname{grad} g \operatorname{grad} h \, d\mathscr{H}_2 . \tag{1.3}$$

Proof. The assertion is nothing else than the theorem on integration by substituion applied to this special case.

Suppose that $\Psi = \Psi_1 + i\Psi_2$ (Ψ_1, Ψ_2 are real functions). For a while let $[x, y]$ stand for the variables on D and $[u, v]$ for the variables on $\hat{D}$; $[u, v] = \Psi(x, y) = [\Psi_1(x, y), \Psi_2(x, y)]$. Then we have

$$\operatorname{grad}[g * \Psi] = [\partial_u g \, \partial x \Psi_1 + \partial_v g \, \partial_x \Psi_2, \, \partial_u g \, \partial_y \Psi_1 + \partial_v g \, \partial_y \Psi_2] ;$$

similarly for $\operatorname{grad}[h * \Psi]$. As Ψ is holomorphic and consequently,

$$\partial_x \Psi_1 = \partial_y \Psi_2 , \quad \partial_y \Psi_1 = -\partial_x \Psi_2 ,$$

we easily obtain that

$$\operatorname{grad}[g * \Psi] \operatorname{grad}[h * \Psi] = [(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2] [\partial_u g \, \partial_u h + \partial_v g \, \partial_v h] =$$
$$= [(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2] \operatorname{grad} g \operatorname{grad} h .$$

More precisely, we can write

$$\{\operatorname{grad}[g * \Psi] \operatorname{grad}[h * \Psi]\} (x, y) =$$
$$= [(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2] (x, y) \{\operatorname{grad} g \operatorname{grad} h\} (\Psi_1(x, y), \Psi_2(x, y)) .$$

Using again the Cauchy-Riemann conditions we find that the Jacobian J_Ψ of Ψ (considered as a real mapping of D to R^2) has the form

$$J_\Psi(x, y) = [(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2](x, y)$$

($[x, y] \in D$). Since Ψ is one-to-one, we have

$$(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2 \neq 0$$

on D and the Jacobian $J_{\Psi^{-1}}$ of Ψ^{-1} on $\hat{D}$ can be expressed in the form

$$J_{\Psi^{-1}}(u, v) = [J_\Psi(\Psi^{-1}(u, v))]^{-1} = \{[(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2](\Psi^{-1}(u, v))\}^{-1}.$$

The theorem on integration by substitution immediately yields

$$\int_M \operatorname{grad}[g * \Psi] \operatorname{grad}[h * \Psi] \, d\mathscr{H}_2 =$$

$$= \int_M [(\partial_x \Psi_1)^2 + (\partial_y \Psi_1)^2](x, y) \{\operatorname{grad} g \operatorname{grad} h\} (\Psi(x, y)) \, d\mathscr{H}_2(x, y) =$$

$$= \int_{\Psi(M)} \{\operatorname{grad} g \operatorname{grad} h\} (u, v) [(\partial_x \Psi_1)^2 +$$

$$+ (\partial_y \Psi_1)^2](\Psi^{-1}(u, v)) \, |J_{\Psi^{-1}}(u, v)| \, d\mathscr{H}_2(u, v) = \int_{\Psi(M)} \operatorname{grad} g \operatorname{grad} h \, d\mathscr{H}_2,$$

that is, the equality (1.3) holds. At the same time we obtain from the substitution theorem that the integral (1.1) exists if and only if the integral (1.2) does.

1.2. Notation. For $z \in R^2$ let h_z stand for the function defined on R^2 such that $h_z(z) = +\infty$ and

$$h_z(\zeta) = \frac{1}{\pi} \log \frac{1}{|\zeta - z|} \tag{1.4}$$

for $\zeta \in R^2 - \{z\}$.

1.3. Lemma. *There is a function $u(z, \zeta)$ defined on $D \times D$ such that $u(z, \zeta)$ is harmonic on D in the variable ζ for each $z \in D$ and $u(z, \zeta)$ is harmonic on D in the variable z for each $\zeta \in D$ and such that for every $z, \zeta \in D$*

$$h_{\Psi(z)}(\Psi(\zeta)) = h_z(\zeta) + u(z, \zeta). \tag{1.5}$$

Further, $\operatorname{grad}_\zeta u(z, \zeta)$ is locally bounded on $D \times D$ as a function of two variables, $[z, \zeta]$

Proof. Fix $z \in D$ and for $\zeta \in D$, $\zeta \neq z$ put

$$u(z, \zeta) = h_{\Psi(z)}(\Psi(\zeta)) - h_z(\Psi). \tag{1.6}$$

Then $u(z, \zeta)$ is harmonic in ζ on $D - \{z\}$ since $h_z(\zeta)$ is harmonic in ζ on $R^2 - \{z\}$ and Ψ is holomorphic and one-to-one on D. It is easy to see that the limit (z fixed)

$$\lim_{\zeta \to z} u(z, \zeta) = \lim_{\zeta \to z} \left(\frac{1}{\pi} \log \frac{1}{|\Psi(z) - \Psi(\zeta)|} - \frac{1}{\pi} \log \frac{1}{|z - \zeta|} \right) =$$
$$= \frac{1}{\pi} \lim_{\zeta \to z} \log \left| \frac{z - \zeta}{\Psi(z) - \Psi(\zeta)} \right| = \frac{1}{\pi} \log \frac{1}{|\Psi'(z)|}$$

is finite since $\Psi'(z) \neq 0$ (Ψ is one-to-one on D). But singletons are removable singularities for bounded harmonic functions, from which it follows that $u(z, \zeta)$ is harmonic in ζ on D if we put

$$u(z, z) = \frac{1}{\pi} \log \left(|\Psi'(z)|^{-1} \right).$$

As $h_z(\zeta) = h_\zeta(z)$ then also $u(z, \zeta) = u(\zeta, z)$ and for each $\zeta \in D$ the function $u(z, \zeta)$ is harmonic in z on D.

If $z = [x, y]$, $\zeta = [\xi, \eta]$, one can write $u(z, \zeta) = u(x, y, \xi, \eta)$. It is seen from (1.6) that u as a function of four variables is continuously differentiable on $M = D \times D - \{[z, z];\ z \in D\}$; especially $\operatorname{grad}_\zeta u = [\partial_\xi u, \partial_\eta u]$ is locally bounded on M as a function of two variables $[z, \zeta]$. Now it suffices to show that for each $z_0 \in D$ there are $\delta > 0$, $k \in R^1$ such that for any $z \in D$, $\zeta \in D$ with $|z - z_0| < \delta$, $|\zeta - z_0| < \delta$ the inequalities

$$|\partial_\xi u(z, \zeta)| \leqq k, \quad |\partial_\eta u(z, \zeta)| \leqq k$$

are valid.

Given $z_0 \in D$ choose $r > 0$ such that

$$\operatorname{cl}(\Omega_r(z_0)) \subset D;$$

here and in the following $\Omega_r(z_0)$ stands for the open disc with centre z_0 and radius r, that is

$$\Omega_r(z_0) = \{\zeta \in R^2;\ |\zeta - z_0| < r\}$$

(cl (...) denotes the closure of a set).

Since $\partial_\xi u$ is harmonic on D in the variable ζ, for $z \in D$, $\zeta \in \Omega_r(z_0)$ we have

$$|\partial_\xi u(z, \zeta)| \leqq \sup \{|\partial_\xi u(z, \tilde{\zeta})|;\ \tilde{\zeta} \in \partial\Omega_r(z_0)\} \tag{1.7}$$

(∂ ... denotes the boundary of a given set). Clearly

$$\operatorname{cl}(\Omega_{1/2r}(z_0)) \times \partial\Omega_r(z_0) \subset M$$

and this set is compact. In virtue of the local boundedness of $\partial_\xi u$ on M there is a constant $k \in R^1$ such that

$$|\partial_\xi u(z, \tilde{\zeta})| \leqq k$$

for every $z \in \mathrm{cl}\,(\Omega_{1/2r}(z_0))$, $\xi \in \partial\Omega_r(z_0)$. Now it follows from (1.7) that

$$|\partial_\xi u(z, \zeta)| \leqq k$$

for any $z \in \Omega_{1/2r}(z_0)$, $\zeta \in \Omega_r(z_0)$. Similarly for $\partial_\eta u$.

II. THE FREDHOLM RADIUS

First let us recall the notion of the cyclic variation which will play the central role in the following. At the same time we shall introduce the notation which we shall need.

The term path (or curve) in the plane is taken to mean a continuous mapping $\varphi: \langle a, b\rangle \to R^2$, where $\langle a, b\rangle$ is a compact interval. A simple closed path (Jordan curve) is a path $\varphi: \langle a, b\rangle \to R^2$ such that $\varphi(a) = \varphi(b)$ and $\varphi(t_1) \neq \varphi(t_2)$ for any $t_1, t_2 \in \langle a, b\rangle$, $|t_1 - t_2| < b - a$. The variation of the vector-valued (complex-valued) function φ on an interval $I \subset \langle a, b\rangle$ is denoted by $\mathrm{var}\,[\varphi; I]$ (in the same way the variation of a scalar (real) function is denoted). (See for instance [9] for the definition and properties of the variation of a vector function, the curvilinear integral, etc.) The path φ is of finite length if $\mathrm{var}\,[\varphi; \langle a, b\rangle] < \infty$.

From now on let $\langle a, b\rangle$ be a fixed compact interval, $\varphi: \langle a, b\rangle \to R^2$ a fixed simple path. Putting

$$K = \varphi(\langle a, b\rangle)$$

we shall talk also about the curve K. As in the introduction let $G(K)$ stand for the interior of K and let $\iota_K\ (= \pm 1)$ be the constant value which the index of a point with respect to K takes on $G(K)$.

For $z \in R^2$ let ϑ_z be a single-valued continuous branch of $\arg\,[\varphi - z]$ on $\langle a, b\rangle - \varphi^{-1}(z)$. For $0 < r \leqq +\infty$ we denote by $\gamma_{z,r}$ the family of all components of the set

$$\{t \in \langle a, b\rangle;\ 0 < |\varphi(t) - z| < r\}\,.$$

For $\alpha \in R^1$ the number (finite or $+\infty$) of points in

$$\{t \in \langle a, b\rangle;\ \varphi(t) - z = |\varphi(t) - z|\, e^{i\alpha},\ 0 < |\varphi(t) - z| < r\}$$

is denoted by $n_r^K(\alpha, z)$. The following assertion is valid (see [6], [9]):

For any $z \in R^2$, $r > 0$ the function $n_r^K(\alpha, z)$ of the variable $\alpha \in R^1$ is Lebesgue measurable. If we define

$$v_r^K(z) = \int_0^{2\pi} n_r^K(\alpha, z)\,\mathrm{d}\alpha\,, \tag{2.1}$$

then

$$v_r^K(z) = \sum_{I \in \gamma_{z,r}} \mathrm{var}\,[\vartheta_z; I]\,.$$

Further, we denote shortly

$$\gamma_z = \gamma_{z,\infty}\,, \quad v^K(z) = v^K_\infty(z)\,.$$

The term $v^K(z)$ is called the cyclic variation of the curve K at the point z.

Let us note that the cylic variation can be also defined by means of the notion of the so-called hits on a set (the cyclic variation is then defined for much more general sets than for domains bounded by a Jordan curve). Let $G \subset R^2$ be a Borel set, $z \in R^2$. For $\alpha \in R^1$ put

$$H(\alpha, z) = \{z + te^{i\alpha};\ t > 0\}\,.$$

A point $\zeta \in H(\alpha, z)$ is termed a hit of $H(\alpha, z)$ on G if for every $r > 0$

$$\mathscr{H}_1[H(\alpha, z) \cap G \cap \Omega_r(\zeta)] > 0$$

and at the same time

$$\mathscr{H}_1[(H(\alpha, z) - G) \cap \Omega_r(\zeta)] > 0\,.$$

For $0 < r \leqq +\infty$, $\alpha \in R^1$ denote by $N^G_r(\alpha, z)$ the total number of hits of $H(\alpha, z)$ on G lying in $\Omega_r(z)$. Then $N^G_r(\alpha, z)$ is a Lebesgue measurable function of $\alpha \in R^1$ (see [4], [5]). Define

$$V^G_r(z) = \int_0^{2\pi} N^G_r(\alpha, z)\,\mathrm{d}\alpha\,.$$

It is easily seen that in the case $G = G(K)$ (or $G = R^2 - \mathrm{cl}\,(G(K))$) $N^G_r(\alpha, z) \leqq$ $\leqq n^K_r(\alpha, z)$ and it may happen that $N^G_r(\alpha, z) \neq n^K_r(\alpha, z)$. However, we have in this case (see [2])

$$V^G_r(z) = v^K_r(z)\,.$$

The notion of hits in the definition of the cyclic variation is used, for example, in [4], [5].

Let $z \in R^2$ be such that $v^K(z) < \infty$. Then for $f \in \mathscr{C}(K)$ the (logarithmic) double layer potential $W_K f(z)$ at z is defined by the equality

$$W_K f(z) = \frac{1}{\pi} \sum_{I \in \gamma_z} \int_I f(\varphi(t))\,\mathrm{d}\vartheta_z(t)\,. \tag{2.2}$$

If $\mathrm{var}\,[\varphi; \langle a, b\rangle] < \infty$ (that is, K is of finite length) then $v^K(z) < \infty$ for every $z \in R^2 - K$ and for such z we have

$$W_K f(z) = \frac{1}{\pi}\,\mathrm{Im} \int_K \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \tag{2.3}$$

(cf. (0.1)). It is known (see [6], [9], or [4], [5]) that $W_K f$ is uniformly continuous on $G(K)$ (or on $R^2 - \mathrm{cl}\,(G(K))$) for each $f \in \mathscr{C}(K)$ if and only if

$$\sup_{\zeta \in K} v^K(\zeta) < \infty\,.$$

Let us briefly recall the notion of the perimeter of a set and some properties of sets with finite perimeter we shall need later.

$\mathcal{D}$ will stand for the family of all real infinitely differentiable functions with compact support in R^2. The support of $g \in \mathcal{D}$ is denoted by spt g and the supremum norm of g is denoted by $\|g\|$. If $G \subset R^2$ is a Borel set then we put

$$\mathscr{P}(G) = \sup_w \int_G \operatorname{div} w \, d\mathscr{H}_2 ,$$

where $w = [w_1, w_2]$ ranges over all vector-valued functions with components $w_1, w_2 \in \mathcal{D}$ such that $\|w\|^2 = w_1^2 + w_2^2 \leqq 1$. $\mathscr{P}(G)$ is called the perimeter of G. If $G = G(K)$ (or $G = R^2 - \operatorname{cl}(G(K))$) then (see [11])

$$\mathscr{P}(G) = \operatorname{var}[\varphi; \langle a, b\rangle] = \mathscr{H}_1(K) .$$

The term $\boldsymbol{n}^K(z)$ is used in the following to denote the exterior normal in Federer's sense of $G(K)$ at $z \in R^2$ (for the definition of the normal in Federer's sense see, for example, [4]). The following assertion is valid (divergence theorem):

Suppose that $\mathscr{P}(G(K)) < \infty$. If $w = [w_1, w_2]$, where w_1, w_2 are continuously differentiable functions on a neighbourhood of $\operatorname{cl}(G(K))$, then

$$\int_K w(\zeta) \, \boldsymbol{n}^K(\zeta) \, d\mathscr{H}_1(\zeta) = \int_{G(K)} \operatorname{div} w(z) \, d\mathscr{H}_2(z) .$$

We shall use this assertion in the following situation. Let $g \in \mathcal{D}$ and let u be a harmonic function on a neighbourhood of

$$\operatorname{cl}(G(K)) \cap \operatorname{spt} g .$$

Then clearly

$$\operatorname{div}(g \operatorname{grad} u) = \operatorname{grad} g \operatorname{grad} u$$

(on R^2) and by divergence theorem we obtain that (under the assumption $\mathscr{P}(G(K)) < < \infty$)

$$\text{(2.4)} \qquad \int_{G(K)} \operatorname{grad} g \operatorname{grad} u \, d\mathscr{H}_2 = \int_K g(\zeta) \, \boldsymbol{n}^K(\zeta) \operatorname{grad} u(\zeta) \, d\mathscr{H}_1(\zeta) .$$

If $z \in R^2$, $g \in \mathcal{D}$, $z \notin \operatorname{spt} g$ then, in particular,

$$\int_{G(K)} \operatorname{grad} g \operatorname{grad} h_z \, d\mathscr{H}_2 = \int_K g(\zeta) \, \boldsymbol{n}^K(\zeta) \operatorname{grad}_\zeta h_z(\zeta) \, d\mathscr{H}_1(\zeta) .$$

Recall that for $z \in R^2$ the quantity $v^K(z)$ can be expressed in the (provided $\mathscr{P}(G(K)) < < \infty$)

$$v^K(z) = \pi \int_K |\boldsymbol{n}^K(\zeta) \operatorname{grad}_\zeta h_z(\zeta)| \, d\mathscr{H}_1(\zeta) .$$

In the following let $D \subset R^2$ be a fixed open set such that $K \subset D$, Ψ a conformal mapping (that is, holomorphic and one-to-one) on D. We shall suppose that D is connected (we may consider only that component of D which contains K). As in the introduction, let

$$\hat{K} = \Psi(K) .$$

Then $\hat{K}$ is also a Jordan curve. We shall suppose that $\hat{\varphi}$ is a parametrization of $\hat{K}$ defined again on the interval $\langle a, b \rangle$ (one can put, for example, $\hat{\varphi}(t) = \Psi(\varphi(t))$). The terms $G(\hat{K})$, $v^{\hat{K}}$, $W_{\hat{K}} \hat{f}(z)$ ($\hat{f} \in \mathscr{C}(\hat{K})$), etc. have the same meaning for $\hat{K}$ as the terms $G(K)$, $v^K(z)$, $W_K f(z)$ ($f \in \mathscr{C}(K)$), etc. have for K.

2.1. Proposition. *If*

$$\operatorname{var} [\varphi; \langle a, b \rangle] < \infty \tag{2.5}$$

then also

$$\operatorname{var} [\hat{\varphi}; \langle a, b \rangle] < \infty , \tag{2.6}$$

and for every $z \in K$ the implication

$$v^K(z) < \infty \Rightarrow v^{\hat{K}}(\Psi(z)) < \infty \tag{2.7}$$

is valid. Suppose that

$$\sup_{\zeta \in K} v^K(\zeta) < \infty . \tag{2.8}$$

Then also

$$\sup_{\zeta \in \hat{K}} v^{\hat{K}}(\zeta) < \infty , \tag{2.9}$$

and the equality

$$\limsup_{r \to 0+} \sup_{\zeta \in \hat{K}} v_r^{\hat{K}}(\zeta) = \limsup_{r \to 0+} \sup_{\zeta \in K} v_r^K(\zeta) \tag{2.10}$$

holds.

Proof. Since D is connected then either $\Psi(G(K) \cap D) \subset G(\hat{K})$ or $\Psi(G(K) \cap D) \subset \subset R^2 - \operatorname{cl}(G(\hat{K}))$. Put $\hat{G} = G(\hat{K})$ in the case $\Psi(G(K) \cap D) \subset G(\hat{K})$ and $\hat{G} = R^2 - \operatorname{cl}(G(\hat{K}))$ in the other case.

It follows from [4], Corollary 1.11 that for $z \in R^2$, $r > 0$,

$$v_r^K(z) = \pi \sup \left\{ \int_{G(K)} \operatorname{grad} g \operatorname{grad} h_z \, \mathrm{d}\mathscr{H}_2 ; \ g \in \mathscr{D}, \ \|g\| \leqq 1, \ \operatorname{spt} g \subset \Omega_r(z) - \{z\} \right\}$$

and similarly

$$v_r^{\hat{K}} = \pi \sup \left\{ \int_{\hat{G}} \operatorname{grad} g \operatorname{grad} h_z \, \mathrm{d}\mathscr{H}_2 ; g \in \mathscr{D}, \ \|g\| \leqq 1, \ \operatorname{spt} g \subset \Omega_r(z) - \{z\} \right\} .$$

For any $g_1, g_2 \in \mathscr{D}$ such that $g_1 = g_2$ on K, $g_1(z) = g_2(z)$ ($z \in R^2$ fixed) we have

$$\int_{G(K)} \operatorname{grad} g_1 \operatorname{grad} h_z \, \mathrm{d}\mathscr{H}_2 = \int_{G(K)} \operatorname{grad} g_2 \operatorname{grad} h_z \, \mathrm{d}\mathscr{H}_2$$

(see [4], lemma 2.1); similarly for the integrals over $\hat{G}$. Let us denote

$$\mathscr{D}_\Psi = \{g \in \mathscr{D};\ \operatorname{spt} g \subset D\}\,, \quad \mathscr{D}_\Psi^\wedge = \{g \in \mathscr{D};\ \operatorname{spt} g \subset \Psi(D)\}$$

and

$$S = G(K) \cap D\,, \quad \hat{S} = \Psi(S) = \hat{G} \cap \Psi(D)\,.$$

Then it follows from the preceding that for every $z \in R^2$, $r > 0$,

$$(2.11) \qquad v_r^K(z) =$$

$$= \pi \sup \left\{ \int_S \operatorname{grad} g \operatorname{grad} h_z \, d\mathscr{H}_2;\ g \in \mathscr{D}_\Psi,\ \|g\| \leqq 1,\ \operatorname{spt} g \subset \Omega_r(z) - \{z\} \right\},$$

$$(2.12) \qquad v_r^{\hat{K}}(z) =$$

$$= \pi \sup \left\{ \int_{\hat{S}} \operatorname{grad} g \operatorname{grad} h_z \, d\mathscr{H}_2;\ g \in \mathscr{D}_\Psi^\wedge,\ \|g\| \leqq 1,\ \operatorname{spt} g \subset \Omega_r(z) - \{z\} \right\}.$$

If $z \in D$ then in particular,

$$(2.13) \qquad v^{\hat{K}}(\Psi(z)) =$$

$$= \pi \sup \left\{ \int_{\hat{S}} \operatorname{grad} g \operatorname{grad} h_{\Psi(z)} \, d\mathscr{H}_2;\ g \in \mathscr{D}_\Psi^\wedge,\ \|g\| \leqq 1,\ \Psi(z) \notin \operatorname{spt} g \right\}.$$

Let u be the function from Lemma 1.3. Using Lemma 1.1 we obtain for $z \in D$, $g \in \mathscr{D}_\Psi^\wedge$ with $\Psi(z) \notin \operatorname{spt} g$,

$$(2.14) \qquad \int_{\hat{S}} \operatorname{grad} g \operatorname{grad} h_{\Psi(z)} \, d\mathscr{H}_2 = \int_S \operatorname{grad} (g * \Psi) \operatorname{grad} \left(h_{\Psi(z)} * \Psi\right) d\mathscr{H}_2 =$$

$$= \int_S \operatorname{grad} (g * \Psi) \operatorname{grad} h_z \, d\mathscr{H}_2 + \int_S \operatorname{grad} (g * \Psi)(\zeta) \operatorname{grad}_\zeta u(z, \zeta) \, d\mathscr{H}_2(\zeta)\,.$$

If we put

$$(2.15) \qquad c_z = \sup_{\zeta \in K} |\operatorname{grad}_\zeta u(z, \zeta)|$$

then, of course, $c_z < \infty$. Suppose that the condition (2.5) is fulfilled. If $g \in \mathscr{D}_\Psi^\wedge$ then clearly $g * \Psi \in \mathscr{D}_\Psi$ (defining $g * \Psi(z) = 0$ for $z \notin D$) and if, in addition, $\|g\| \leqq 1$, $\Psi(z) \notin \operatorname{spt} g$ then $\|g * \Psi\| \leqq 1$, $z \notin \operatorname{spt}(g * \Psi)$. Now it is seen (using the equality (2.4)) that

(2.16)

$$\sup \left\{ \int_S \operatorname{grad} (g * \Psi)(\zeta) \operatorname{grad}_\zeta u(z, \zeta) \, d\mathscr{H}_2(\zeta);\ g \in \mathscr{D}_\Psi^\wedge,\ \|g\| \leqq 1,\ \Psi(z) \notin \operatorname{spt} g \right\} =$$

$$= \sup \left\{ \int_K g * \Psi(\zeta)\, n^K(\zeta) \operatorname{grad}_\zeta u(z, \zeta) \, d\mathscr{H}_1(\zeta);\ g \in \mathscr{D}_\Psi^\wedge,\ \|g\| \leqq 1,\ \Psi(z) \notin \operatorname{spt} g \right\} \leqq$$

$$\leqq c_z \operatorname{var} [\varphi; \langle a, b \rangle]\,.$$

Further,

$$\sup \left\{ \int_S \operatorname{grad}(g * \Psi) \operatorname{grad} h_z \, d\mathscr{H}_2;\ g \in \mathscr{D}_{\Psi}^{\hat{}},\ \|g\| \leqq 1,\ \Psi(z) \notin \operatorname{spt} g \right\} =$$

$$= \sup \left\{ \int_S \operatorname{grad} g \operatorname{grad} h_z \, d\mathscr{H}_2;\ g \in \mathscr{D}_{\Psi},\ \|g\| \leqq 1,\ z \notin \operatorname{spt} g \right\} = \frac{1}{\pi} v^K(z).$$

From the last equality and from (2.16), (2.14), (2.13) we obtain

$$v^{\hat{K}}(\Psi(z)) \leqq v^K(z) + \pi c_z \operatorname{var}[\varphi; \langle a, b \rangle] \tag{2.17}$$

($z \in D$). Thus the implication (2.7) is valid (even for any $z \in D$). If $z \in D - K$ then (provided (2.5)) $v^K(z) < \infty$ (see, for instance, [6], [4], [9]) and it is now seen that $v^{\hat{K}}(z) < \infty$ for every $z \in \Psi(D)$, $z \notin \hat{K}$. However, by [4], Theorem 2.12 this means that (2.6) is fulfilled (note that the fact that (2.5) implies (2.6) can be proved much more simply by using only the definition of variation and the smoothness of the mapping Ψ).

As $\operatorname{grad}_\zeta u(z, \zeta)$ is locally bounded on $D \times D$ as a function of two variables $[z, \zeta]$ (Lemma 1.3), we have

$$c = \sup_{z \in K} c_z < \infty \tag{2.18}$$

(c_z is defined by (2.15)). Now it follows immediately from (2.17) that

$$\sup_{\zeta \in \hat{K}} v^{\hat{K}}(\zeta) \leqq \sup_{\zeta \in K} v^K(\zeta) + \pi c \operatorname{var}[\varphi; \langle a, b \rangle], \tag{2.19}$$

so that (2.8) implies (2.9).

Now it suffices to prove the equality (2.10). Since $\Psi'(z) \neq 0$ for $z \in D$, $K \subset D$ is compact, one can easily find that for each $r > 0$ there are $r_1, r_2 > 0$ such that for every $z \in K$,

$$\Omega_{r_1}(z) \subset \Psi^{-1}(\Omega_r(\Psi(z))) \subset \Omega_{r_2}(z),$$

where r_2 can be chosen in such a way that $r_2 \to 0+$ for $r \to 0+$. If (2.8) is fulfilled then by [4], Corollary 2.17 there is a $k \in R^1$ such that

$$\mathscr{H}_1(\Omega_r(z) \cap K) \leqq kr$$

for every $z \in K$, $r > 0$. This implies

$$\lim_{r \to 0+} \sup_{z \in K} \int_{K \cap \Omega_r(z)} |\boldsymbol{n}^K(\zeta) \operatorname{grad}_\zeta u(z, \zeta)| \, d\mathscr{H}_1(\zeta) = 0 \tag{2.20}$$

since

$$\int_{K \cap \Omega_r(z)} |\boldsymbol{n}^K(\zeta) \operatorname{grad}_\zeta u(z\ \zeta)| \, d\mathscr{H}_1(\zeta) \leqq c\, \mathscr{H}_1(\Omega_r(z) \cap K)$$

by the preceding.

Let $z \in K$. If $g \in \mathscr{D}_{\Psi}^{\wedge}$ $\|g\| \leqq 1$, $\operatorname{spt} g \subset \Omega_r(\Psi(z)) - \{\Psi(z)\}$ then $g * \Psi \in \mathscr{D}_{\Psi}$, $\|g * \Psi\| \leqq 1$, $\operatorname{spt}(g * \Psi) \subset \Omega_{r_2}(z) - \{z\}$. This together with (2.11), (2.12), (2.14) and (2.4) yields

$$(2.21) \qquad v_r^{\hat{K}}(\Psi(z)) \leqq v_{r_2}^K(z) + \pi \int_{K \cap \Omega r_2(z)} |n^K(\zeta) \operatorname{grad}_\zeta u(z, \zeta)| \, d\mathscr{H}_1(\zeta).$$

If $g \in \mathscr{D}_{\Psi}^{\wedge}$ is such that $\|g * \Psi\| \leqq 1$, $\operatorname{spt}(g * \Psi) \subset \Omega_{r_1}(z) - \{z\}$ then $\operatorname{spt} g \subset \subset \Omega_r(\Psi(z)) - \{\Psi(z)\}$. Combining this result again with (2.11), (2.12), (2.14) and (2.4) we conclude that

$$(2.22) \qquad v_{r_1}^K(z) - \int_{K \cap \Omega r_1(z)} |n^K(\zeta) \operatorname{grad}_\zeta u(z, \zeta)| \, d\mathscr{H}_1(\zeta) \leqq v_r^{\hat{K}}(\Psi(z)).$$

Since $r_1 \to 0+$, $r_2 \to 0+$ for $r \to 0+$, the equality (2.10) now follows immediately from (2.21), (2.22) and (2.20).

Now we are in position to prove the following assertion. Note only that the terms $\overline{W}_K f$, $\overline{W}_{\hat{K}} \hat{f}$ have the same meaning as in the introduction, that is, $\overline{W}_K f(\zeta)$ for $\zeta \in K$ is defined by (0.2) and analogously for $\overline{W}_{\hat{K}} \hat{f}(\zeta)$ $(\zeta \in \hat{K})$.

2.2. Theorem. *Suppose that* var $[\varphi; \langle a, b \rangle] < \infty$. *Then $W_{\hat{K}} \hat{f}$ is uniformly continuous on $G(\hat{K})$ for each $\hat{f} \in \mathscr{C}(\hat{K})$ if and only if $W_K f$ is uniformly continuous on $G(K)$ for each $f \in \mathscr{C}(K)$. If this condition is fulfilled then $\overline{W}_K$: $\mathscr{C}(K) \to \mathscr{C}(K)$ $(\overline{W}_K : f \mapsto \overline{W}_K f)$ and $\overline{W}_{\hat{K}}$: $\mathscr{C}(\hat{K}) \to C(\hat{K})$ $(\overline{W}_{\hat{K}} : \hat{f} \mapsto \overline{W}_{\hat{K}} \hat{f})$ are bounded linear operators and the Fredholm radii of $\overline{W}_K$ and $\overline{W}_K$ coincide.*

Proof. It suffices to note that $W_K f$ is uniformly continuous on $G(K)$ for each $f \in \mathscr{C}(K)$ if and only if (2.8) is fulfilled, and $W_{\hat{K}} \hat{f}$ is uniformly continuous on $G(\hat{K})$ for each $\hat{f} \in \mathscr{C}(\hat{K})$ if and only if (2.9) takes place (see, for instance, [6], [4], [9]). By Proposition 2.1 the condition (2.9) follows from (2.8). The converse implication is evident by the symmetry (one can replace K by $\hat{K}$ and Ψ by Ψ^{-1}).

Further, it suffices to note that by [7] (see also [4], [5], [10]), under the condition (2.8) the operator $\overline{W}_K$ on $\mathscr{C}(K)$ is bounded (and linear) and the Fredholm radius of $\overline{W}_K$ is equal to the reciprocal value of

$$\omega \overline{W}_K = \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in K} v_r^K(\zeta),$$

and that under the condition (2.9) the operator $\overline{W}_{\hat{K}}$ on $\mathscr{C}(\hat{K})$ is bounded (and linear) and the Fredholm radius of $\overline{W}_{\hat{K}}$ is equal to the reciprocal value of

$$\omega \overline{W}_{\hat{K}} = \frac{1}{\pi} \lim_{r \to 0+} \sup_{\zeta \in \hat{K}} v_r^{\hat{K}}(\zeta).$$

Thus the equality between the Fredholm radii of $\overline{W}_K$ and $\overline{W}_{\hat{K}}$ follows from (2.10).

2.3. Remark. Let us consider the simple case when the curve K consists of simple arcs of the class $\mathscr{C}^2$ and has only some corner points. As Ψ is conformal, $\hat{K}$ has the same property. It is known that then $\overline{W}_K$, $\overline{W}_{\hat{K}}$ are bounded linear operators and the equivalence relation between the conditions on the uniform continuity of W_K, $W_{\hat{K}}$ (on $G(K)$, $G(\hat{K})$, respectively) is clear in this case. But it is also clear in this case that the Fredholm radii of $\overline{W}_K$, $\overline{W}_{\hat{K}}$ coincide since then the Fredholm radius is determined by the angles between the right-hand and left-hand half-tangents at the corner points (see, for example, [12], [7]), and the conformal mapping does not vary the angles. Now it is further seen that in order to ensure the invariance of the Fredholm radius it is natural to require the conformality of Ψ.

A natural question arises whether the conformality of Ψ is necessary to ensure the equivalence of the conditions (2.8), (2.9). J. Král has formulated the following conjecture:

Let the condition (2.5) be fulfilled and let Ψ be a diffeomorphism of the class $\mathscr{C}^2$ (defined on an open set which contains K). Then the conditions (2.8), (2.9) are equivalent to each other.

2.4. Remark. An analogue of Theorem 2.2 can be proved also in the case of a multiply connected region with a boundary consisting of finitely many Jordan curves. We shall not formulate it precisely here but let us sketch one of the possible versions.

Let $G \subset R^2$ be a bounded region, B the boundary of G and suppose that

$$B = K_0 \cup K_1 \cup \ldots \cup K_n ,$$

where $K_0, K_1, \ldots, K_n$ are Jordan curves, $K_i \cap K_j = \emptyset$ for $i \neq j$. Let $D \subset R^2$ be an open set such that $B \subset D$ (D is not supposed to be connected), let $\Psi: D \to R^2$ be holomorphic on D (as a complex function of the complex variable) and suppose that Ψ is one-to-one on each component of D. Further, suppose that $\Psi(K_i) \cap \cap \Psi(K_j) = \emptyset$ for $i \neq j$ and that there is a bounded region $\hat{G} \subset R^2$ with a boundary $\hat{B}$ such that

$$\hat{B} = \Psi(K_0) \cup \Psi(K_1) \cup \ldots \cup \Psi(K_n) .$$

(Note that the boundedness of G, $\hat{G}$ is not necessary in the assertion; similarly one can take $R^2 - \mathrm{cl}\,(G(K))$ or $R^2 - \mathrm{cl}\,(G(\hat{K}))$ instead of $G(K)$ or $G(\hat{K})$ in the preceding.)

Suppose that the curves $K_0, K_1, \ldots, K_n$ are of finite lengths. Similarly to the definition of W_K for $f \in \mathscr{C}(K)$ one can define the double layer potentials $W_B f$ for $f \in \mathscr{C}(B)$ and $W_{\hat{B}} \hat{f}$ for $\hat{f} \in \mathscr{C}(\hat{B})$ (some natural conditions concerning the orientation of boundary curves should be imposed). Then, similarly to the preceding, the operator $\overline{W}_B$ on $\mathscr{C}(B)$ (or $\overline{W}_{\hat{B}}$ on $\mathscr{C}(\hat{B})$) can be defined provided $W_B \hat{f}$ ($W_{\hat{B}} \hat{f}$) is uniformly continuous on G for each $f \in \mathscr{C}(B)$ (on $\hat{G}$ for each $\hat{f} \in \mathscr{C}(\hat{B})$, respectively). Since $W_B f$ is uniformly continuous on G for each $f \in \mathscr{C}(B)$ if and only if (see [6])

$$\max_{i=0,1,\ldots,n} \left\{ \sup_{\zeta \in K_i} v^{K_i}(\zeta) \right\} < \infty$$

(and similarly for $W_B\hat{f}$ on $\hat{G}$), Proposition 2.1 yields that $W_B\hat{f}$ is uniformly continuous on $\hat{G}$ for each $\hat{f} \in \mathscr{C}(\hat{B})$ if and only if for each $f \in \mathscr{C}(B)$ the same holds for $W_B f$ on G. If this condition is fulfilled then $\overline{W}_B$, $\overline{W}_{\hat{B}}$ are bounded linear operators (on $\mathscr{C}(B)$ and $\mathscr{C}(\hat{B})$, respectively) with coinciding Fredholm radii. The last assertion follows from (2.10) and from the fact that the Fredholm radius of $\overline{W}_B$ (of $\overline{W}_{\hat{B}}$) is equal to the reciprocal value of (see [7])

$$\frac{1}{\pi} \max_{i=0,1,\ldots,n} \left\{ \lim_{r \to 0+} \sup_{\zeta \in K_i} v_r^{K_i}(\zeta) \right\}$$

(to the reciprocal value of

$$\frac{1}{\pi} \max_{i=0,1,\ldots,n} \left\{ \lim_{r \to 0+} \sup_{\zeta \in \Psi(K_i)} v_r^{\Psi(K_i)}(\zeta) \right\},$$

respectively).

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Souhrn

INVARIANCE FREDHOLMOVA POLOMĚRU OPERÁTORU V TEORII POTENCIÁLU

MIROSLAV DONT, EVA DONTOVÁ

Mezi klasické metody řešení Dirichletovy úlohy v R^n patří metoda integrálních rovnic. V souvislosti s užitím této metody pro nehladké oblasti je účelné znát Fredholmův poloměr integrálního

operátoru, se kterým se v této metodě pracuje. V článku se ukazuje, že v případě rovinné Jordanovy oblasti se Fredholmův poloměr tohoto operátru nemění při konformním zobrazení hranice dané oblasti.

Резюме

ИНВАРИАНТНОСТЬ РАДИУСА ФРЕДГОЛЬМА ОПЕРАТОРА В ТЕОРИИ ПОТЕНЦИАЛА

MIROSLAV DONT, EVA DONTOVÁ

Метод интегральных уравнений — это один из классических методов решения проблемы Дирихле. При использовании этого метода для областей с нерегулярными границами является полезным изучать радиус Фредгольма интегрального оператора входящего в этот метод. В статье показано, что в случае плоской области Жордана радиус Фредгольма этого оператора не меняется при конформном отображении границы этой области.

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