ON REPRESENTATIONS OF BAIRE FUNCTIONS IN A GIVEN FAMILY AS SUMS OF BAIRE DARBOUX FUNCTIONS WITH A COMMON SUMMAND

H. W. Pu, H. H. Pu, College Station

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Summary. The authors show that Mišík's result on representation of Baire α-functions ($\alpha > 1$) from a given family as sums of Baire Darboux functions with a common summand can be extended to the case $\alpha = 1$ provided the family considered is finite, and give a counterexample if the family is infinite.

Keywords: Baire functions, Baire Darboux functions.

1. In 1967 [5], Mišík proved the following theorem.

Theorem M. *If $\mathscr{A}$ is a countable family of Baire α functions and $\alpha > 1$, then there exists a Baire α function f such that $f + g$ has the Darboux property for every $g \in \mathscr{A}$.*

In other words, if $\mathscr{A}$ is a countable family of Baire α functions and $\alpha > 1$, then the functions in $\mathscr{A}$ can be represented as sums of two Darboux Baire α functions with a common summand. Naturally we want to know whether Theorem M is still true if $\alpha = 1$. This question has been raised by Ceder and Pearson [3]. In this paper, an example is given to show that a common summand cannot be expected for the case $\alpha = 1$ if $\mathscr{A}$ is infinite. Furthermore, we prove that if $\mathscr{A}$ is finite, then the conclusion of Theorem M remains valid even if $\alpha = 1$.

Throughout this paper, we shall use R to denote the real line, $\mathscr{B}_1$ the family of Baire 1 functions, $\mathscr{D}$ the family of Darboux functions and $\mathscr{D}\mathscr{B}_1$ the family $\mathscr{B}_1 \cap \mathscr{D}$.

2. In the proof of our theorem, a result from [2] proved by Bruckner, Ceder and Keston will be used. We state their lemma and some facts from its proof as a lemma here.

Lemma. *Let D be a first category set in R, (a, b) an open interval $(-\infty \leqq a < b \leqq +\infty)$, $0 < \lambda \leqq +\infty$. Then there exist an $h \in \mathscr{D}\mathscr{B}_1$ on (a, b) and a first category subset P of (a, b) such that $P \cap D = \emptyset$, the closure $\bar{P} = P \cup \{a, b\}$, $|h(x)| < \lambda$ for every $x \in (a, b)$, $\{x: h(x) \neq 0\} \subset P$ and*

$$\varliminf_{x\to a+} h(x) = \varliminf_{x\to b-} h(x) = -\lambda\,,$$
$$\varlimsup_{x\to a+} h(x) = \varlimsup_{x\to b-} h(x) = \lambda\,.$$

Moreover, let $x_0 = y_0$ be a fixed point in (a, b), let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be strictly monotone sequences such that $x_n \searrow a$ and $y_n \nearrow b$, $I_n = [x_n, x_{n-1}]$ and $J_n = [y_{n-1}, y_n]$ for $n = 1, 2, \ldots$. If $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\lambda_n \nearrow \lambda$, then h can be chosen such that

$$\sup h(I_n) = \sup h(J_n) = \lambda_n \quad \text{if } n \text{ is even}\,,$$
$$\inf h(I_n) = \inf h(J_n) = -\lambda_n \quad \text{if } n \text{ is odd}\,.$$

Also, we shall use the following criterions for a function in $\mathscr{B}_1$ to be Darboux. They were proved by Young, Sen and Massera (see [1], p. 9).

Let $h \in \mathscr{B}_1$. Then

(1) *$h \in \mathscr{D}$ if and only if for each x, there exist sequences $\{x_n'\}$ and $\{x_n''\}$ such that $x_n' \searrow x$, $x_n'' \nearrow x$, and*

$$\lim_{n\to\infty} h(x_n') = \lim_{n\to\infty} h(x_n'') = h(x)\,.$$

(2) *$h \in \mathscr{D}$ if and only if for each x, we have*

$$\varliminf_{t\to x+} h(t) \leqq h(x) \leqq \varlimsup_{t\to x+} h(t)$$

and

$$\varliminf_{t\to x-} h(t) \leqq h(x) \leqq \varlimsup_{t\to x-} h(t)\,.$$

3. First we give the example mentioned in § 1. Let g be defined as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational}\,, \\ 1 & \text{if } x = 0, \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ is a nonzero rational number in reduced form with } q > 0\,. \end{cases}$$

Let $g_n = ng$ for $n = 1, 2, \ldots$. Clearly $\lim_{t\to x} g_n(t) = 0$ for every n and every x, and $\mathscr{A} = \{g_n\}_n$ is a countable family of Baire 1 functions. Suppose that f is a function such that $f + g_n$ is Darboux for every n. We now show that $f \notin \mathscr{B}_1$.

Since $f + g_n \in \mathscr{D}$, we have, for every x,

$$\varliminf_{t\to x} (f + g_n)(t) \leqq f(x) + g_n(x) \leqq \varlimsup_{t\to x} (f + g_n)(t)\,.$$

In particular, since $\lim_{t\to x} g_n(t) = 0$, we have, for $x = p/q$,

$$\overline{\lim_{t\to x}} f(t) = \overline{\lim_{t\to x}} (f + g_n)(t) \geqq f(x) + \frac{n}{q}.$$

This holds for every n. Thus $\overline{\lim_{t\to x}} f(t) = +\infty$ for every x. It follows that f is not continuous at any point. Consequently, $f \notin \mathscr{B}_1$.

Theorem. *Let $\mathscr{A}$ be a finite family of Baire 1 functions. Then there exists a Baire 1 function f such that $f + g$ is Darboux for every $g \in \mathscr{A}$.*

Proof. We use $\omega(g, x)$ to denote the oscillation of a function g at a point x. For each positive integer i, let $D_i(g) = \{x: \omega(g, x) \geqq 2^{-i}\}$ and $D_i = \bigcup\{D_i(g): g \in \mathscr{A}\}$. Since $\mathscr{A}$ is finite and $\mathscr{A} \subset \mathscr{B}_1$, each D_i is a nowhere dense closed set. It follows that $D = \bigcup_{i=1}^{\infty} D_i$ is a first category set.

Similar to the proof of Proposition 1 in [2], we shall use induction to construct a series of functions and prove that the sum is the desired function f. Since we need to modify their construction and we do not use the theorem appearing on p. 294 of Kuratowski [4] that is used in [2], we present the construction here.

The construction involves a sequence of open residual sets $\{G_k\}_{k=1}^{\infty}$. Each G_k has components $\{(a_{kj}, b_{kj})\}_j$ (j runs from 1 to ∞ or to a certain integer depending on k). Let $\lambda_1 = +\infty$ and $\lambda_k = 2^{-(k-2)}$ if $k \geqq 2$. We take D as above, $(a, b) = (a_{kj}, b_{kj})$, $\lambda = \lambda_k$. By Lemma, there exist $h_{kj} \in \mathscr{D}\mathscr{B}_1$ on (a_{kj}, b_{kj}) and a first category set P_{kj} in (a_{kj}, b_{kj}) such that

(i) $P_{kj} \cap D = \emptyset$,

(ii) $\bar{P}_{kj} = P_{kj} \cup \{a_{kj}, b_{kj}\}$,

(iii) $|h_{kj}(x)| < \lambda_k$ for every $x \in (a_{kj}, b_{kj})$,

(iv) $\{x: h_{kj}(x) \neq 0\} \subset P_{kj}$,

(v) $\underline{\lim}_{x\to a_{kj}+} h_{kj}(x) = \underline{\lim}_{x\to b_{kj}-} h_{kj}(x) = -\lambda_k$, and
$\overline{\lim}_{x\to a_{kj}+} h_{kj}(x) = \overline{\lim}_{x\to b_{kj}-} h_{kj}(x) = \lambda_k$.

For the case $k = 1$, we require more from each h_{1j}. This will be made clear later.

For each k, we define h_k on R by

$$h_k(x) = h_{kj}(x) \quad \text{if} \quad x \in (a_{kj}, b_{kj}) \quad \text{for some } j,$$
$$= 0 \quad \text{if} \quad x \notin G_k,$$

and set $P_k = \bigcup_{i=1}^{k} \bigcup_j P_{ij}$. Clearly $h_k \in \mathscr{B}_1$ and P_k is a first category set disjoint from D. Moreover, by (ii),

(ii+) $$\overline{\bigcup_j P_{kj}} \subset \left(\bigcup_j P_{kj}\right) \cup (R - G_k) \quad \text{for each } k.$$

Also, since each G_k is an open residual set, the sets $\{a_{kj}\}_j$ and $\{b_{kj}\}_j$ are dense in $R - G_k$. Using (v), we can easily show

$$\text{(v+)} \qquad \varliminf_{t \to x+} h_k(t) = \varliminf_{t \to x-} h_k(t) = -\lambda_k \quad \text{and}$$
$$\varlimsup_{t \to x+} h_k(t) = \varlimsup_{t \to x-} h_k(t) = \lambda_k \quad \text{at each} \quad x \in R - G_k\,.$$

Let $G_1 = R - D_1$ and a component (a_{1j}, b_{1j}) be fixed. Let the intervals (a_{1j}, b_{1j}), I_{jn}, J_{jn} $(n = 1, 2, \dots)$ correspond to (a, b), I_n, J_n in Lemma. For each n, $(I_{jn} \cup J_{jn}) \cap \cap D_1 = \emptyset$, and hence $\omega(g, x) < \frac{1}{2}$ for every $x \in I_{jn} \cup J_{jn}$ and every $g \in \mathscr{A}$. Since each $I_{jn} \cup J_{jn}$ is a compact set, there exists $M_{jn} > 0$ such that $|g(x)| < M_{jn}$ for every $x \in I_{jn} \cup J_{jn}$ and every $g \in \mathscr{A}$. With no loss of generality, we assume that $M_{j1} \leqq \leqq M_{j2} \leqq \dots$. Let $\lambda_1 = +\infty$, $\lambda_{jn} = 2M_{jn} + n$ correspond to λ and λ_n in Lemma. Then h_{1j} can be chosen to satisfy the conditions (i)–(v) (for $k = 1$) and

$$\text{(vi)} \qquad \sup h_{1j}(I_{jn}) = \sup h_{1j}(J_{jn}) = \lambda_{jn} \quad \text{if } n \text{ is even}\,,$$
$$\inf h_{1j}(I_{jn}) = \inf h_{1j}(J_{jn}) = -\lambda_{jn} \quad \text{if } n \text{ is odd}\,.$$

By (ii+), $\bar{P}_1 \subset P_1 \cup (R - G_1) = P_1 \cup D_1$ and hence $D_1 \cup P_1 = D_1 \cup \bar{P}_1$ is closed.

We now proceed with the induction step. Assume that for some $k \geqq 1$, we have constructed an open residual set G_k, the associated functions h_{kj} (j runs through the enumeration of the components of G_k) and h_k, the associated first category sets P_{kj} and P_k such that $D_k \cup P_k$ is closed. Clearly $D_{k+1} \cup P_k$ is a closed first category set. We take $G_{k+1} = R - (D_{k+1} \cup P_k)$. The associated functions and sets are as described above. To complete the induction, we need to show that $D_{k+1} \cup P_{k+1}$ is closed. By (ii+) and the choice of G_{k+1},

$$\overline{\bigcup_j P_{k+1,j}} \subset \left(\bigcup_j P_{k+1,j}\right) \cup (D_{k+1} \cup P_k) = D_{k+1} \cup P_{k+1}\,.$$

Since $D_{k+1} \cup P_k$ is closed, $D_{k+1} \cup P_k = \overline{D_{k+1} \cup P_k} = D_{k+1} \cup \bar{P}_k$. Consequently,

$$D_{k+1} \cup P_{k+1} \supset D_{k+1} \cup \bar{P}_k \cup \overline{\bigcup_j P_{k+1,j}} = D_{k+1} \cup \overline{P_{k+1}}\,.$$

This implies that $D_{k+1} \cup P_{k+1}$ is closed. Thus we have constructed the series

$$\sum_{k=1}^{\infty} h_k(x)$$

by induction.

It can be easily seen from the definition of h_k and (iii) that this series converges uniformly on R. Therefore we can define a function f on R by letting

$$f(x) = \sum_{k=1}^{\infty} h_k(x)$$

and claim that $f \in \mathscr{B}_1$.

Now we show that $f \in \mathscr{D}$. This will be used later. From the construction, we see that the sets P_{kj} are mutually disjoint. Thus, owing to (iv), we have

$$f(x) = h_{kj}(x) \quad \text{if} \quad x \in P_{kj} \quad \text{for some } k \text{ and some } j\,,$$
$$= 0 \qquad \text{if} \quad x \notin \bigcup_{k=1}^{\infty} \bigcup_{j} P_{kj}\,.$$

Since $\bigcup_{k=1}^{\infty} \bigcup_{j} P_{kj}$ is a first category set, $\{x : f(x) = 0\}$ is dense in R. For x such that $f(x) = 0$, there are clearly sequences $\{x'_n\}$ and $\{x''_n\}$ such that $x'_n \searrow x$, $x''_n \nearrow x$ and

$$(1_f) \qquad \lim_{n\to\infty} f(x'_n) = \lim_{n\to\infty} f(x''_n) = f(x)\,.$$

If x is given such that $f(x) \neq 0$, then $x \in P_{kj}$ for some k and some j. Since $h_{kj} \in \mathscr{D}\mathscr{B}_1$ on (a_{kj}, b_{kj}), by (1), there exist sequences $\{x'_n\}$ and $\{x''_n\}$ such that $x'_n \searrow x$, $x''_n \nearrow x$ and

$$\lim_{n\to\infty} h_{kj}(x'_n) = \lim_{n\to\infty} h_{kj}(x''_n) = h_{kj}(x)\,.$$

Now $h_{kj}(x) = f(x) \neq 0$. We may assume that $h_{kj}(x'_n) \neq 0 \neq h_{kj}(x''_n)$ for every n. Then, in view of (iv), the sequences $\{x'_n\}$ and $\{x'_n\}$ are in P_{kj} and hence $f(x'_n) = h_{kj}(x'_n)$ and $f(x''_n) = h_{kj}(x''_n)$ for every n. Thus (1_f) also holds for this case and, by (1), $f \in \mathscr{D}$.

It remains to show that $f + g \in \mathscr{D}$ for every $g \in \mathscr{A}$. Let $g \in \mathscr{A}$ and $x \in R$ be given. We want to establish the inequalities in (2) with h replaced by $f + g$. We shall prove the inequalities in which $t \to x+$ is involved. The others can be proved analogously. There are two cases.

Case 1: $x \notin D$, g is continuous at x and hence

$$\varliminf_{t\to x+} (f + g)(t) = \varliminf_{t\to x+} f(t) + g(x)\,,$$
$$\varlimsup_{t\to x+} (f + g)(t) = \varlimsup_{t\to x+} f(t) + g(x)\,.$$

From this and the fact that $f \in \mathscr{D}\mathscr{B}_1$, the desired inequalities follow. That is,

$$(2_{f+g}) \qquad \varliminf_{t\to x+} (f + g)(t) \leqq f(x) + g(x) \leqq \varlimsup_{t\to x+} (f + g)(t)\,.$$

Case 2: $x \in D$, there is a first integer n_0 such that $x \in D_{n_0}$.

If $n_0 > 1$, then $x \notin D_{n_0-1}$ and $\omega(g, x) < 2^{-(n_0-1)}$. This implies

$$g(x) - \frac{1}{2^{n_0-1}} \leqq \varliminf_{t\to x+} g(t) \leqq \varlimsup_{t\to x+} g(t) \leqq g(x) + \frac{1}{2^{n_0-1}}\,.$$

Also, $x \in D_{n_0} \subset R - G_{n_0}$. By (v+), there are sequences $\{x_n\}$ and $\{y_n\}$ decreasing to x such that

$$\lim_{n\to\infty} h_{n_0}(x_n) = -\lambda_{n_0} = -\frac{1}{2^{n_0-2}}$$

and

$$\lim_{n\to\infty} h_{n_0}(y_n) = \lambda_{n_0} = \frac{1}{2^{n_0-2}}.$$

Clearly we can assume that $h_{n_0}(x_n) \neq 0 \neq h_{n_0}(y_n)$ for every n. Thus x_n and y_n are in the set $\bigcup_j P_{n_0 j}$ and $f(x_n) = h_{n_0}(x_n)$, $f(y_n) = h_{n_0}(y_n)$ for every n. The above equalities imply that

$$\varliminf_{t\to x+} f(t) \leqq -\frac{1}{2^{n_0-2}} \quad \text{and} \quad \varlimsup_{t\to x+} f(t) \geqq \frac{1}{2^{n_0-2}}.$$

Now

$$\varliminf_{t\to x+} (f+g)(t) \leqq \varliminf_{t\to x+} f(t) + \varlimsup_{t\to x+} g(t)$$

$$\leqq -\frac{1}{2^{n_0-2}} + g(x) + \frac{1}{2^{n_0-1}} < g(x),$$

$$\varlimsup_{t\to x+} (f+g)(t) \geqq \varlimsup_{t\to x+} f(t) + \varliminf_{t\to x+} g(t)$$

$$\geqq \frac{1}{2^{n_0-2}} + g(x) - \frac{1}{2^{n_0-1}} > g(x).$$

By (i), $x \notin \bigcup_{k=1}^{\infty} \bigcup_j P_{kj}$ and hence $f(x) = 0$. The inequalities (2_{f+g}) follow.

If $n_0 = 1$, then $x \in D_1 = R - G_1$. It should be noted that for each j (such that (a_{1j}, b_{1j}) is a component of G_1), $\lambda_{jn} > M_{jn} + n$. By (vi) and the way we have defined h_1, there exists $t_{jn} \in I_{jn}$ such that

$$h_1(t_{jn}) > \quad M_{jn} + n \quad \text{if } n \text{ is even},$$

$$h_1(t_{jn}) < -M_{jn} - n \quad \text{if } n \text{ is odd}.$$

Clearly $t_{jn} \in P_{1j}$ for each j, each n, and hence h_1 in the above inequalities can be replaced by f. Since $D_1 = R - G_1$ is a nowhere dense closed set, there exists a sequence $\{a_{1j_n}\}_{n=1}$ such that $a_{1j_1} \geqq a_{1j_2} \geqq \dots$ and $\lim_{n\to\infty} a_{1j_n} = x$. (If $x = a_{1j_0}$ for some j_0, then $j_1 = j_2 = \dots = j_0$.) Let $x_n = t_{j_n n}$, where $t_{j_n n}$ are as chosen above. Then $|g(x_n)| < M_{j_n n}$ and hence

$$f(x_n) + g(x_n) > \quad n \quad \text{if } n \text{ is even},$$

$$f(x_n) + g(x_n) < -n \quad \text{if } n \text{ is odd}.$$

Consequently, $\varliminf_{t\to x+} (f+g)(t) = -\infty$ and $\varlimsup_{t\to x+} (f+g)(t) = +\infty$. Again, (2_{f+g}) follows. The proof is completed.

Remark. In the above construction, the sets P_{kj} can be chosen null in the sense of Lebesgue. Then the function f equals zero except on a first category set of Lebesgue measure zero.

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Souhrn

REPRESENTACE BAIROVÝCH FUNKCÍ Z DANÉ MNOŽINY VE TVARU SOUČTŮ BAIRE-DARBOUXOVÝCH FUNKCÍ SE SPOLEČNÝM ČLENEM

H. W. Pu, H. H. Pu

Autoři dokazují, že Mišíkův výsledek o reprezentaci Bairových α-funkcí ($\alpha > 1$) z dané množiny ve tvaru součtu Baire-Darbouxových funkcí se společným členem může být rozšířen na případ $\alpha = 1$, jestliže uvažovaná množina je konečná, a udávají protipříklad, je-li tato množina nekonečná.

Резюме

ПРЕДСТАВЛЕНИЕ ФУНКЦИЙ БЭРА ИЗ ДАННОГО МНОЖЕСТВА В ВИДЕ СУММЫ ФУНКЦИЙ БЭРА-ДАРБУ С ОБЩИМ ЧЛЕНОМ

H. W. Pu, H. H. Pu

Авторы доказывают, что результат Мишика о представлении α-функций Бэра ($\alpha > 1$) из данного множества в виде суммы функций Бэра-Дарбу с общим членом можно распространить на случай $\alpha = 1$, если рассматриваемое множество конечно, и приводят контрпример в противоположном случае.

Authors' address: Department of Mathematics, Texas A & M University, College Station, Texas, U.S.A.