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Časopis pro pěstování matematiky, Vol. 112 (1987), No. 2, 162--172

Persistent URL: `http://dml.cz/dmlcz/118304`

ON CERTAIN ITERATIVE SEQUENCES*)

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Dedicated to John Oosterhout

(Received January 17, 1984)

Summary. The author studies small functions (introduced by V. Pták). Among others, he proves the following theorem: a continuous function is small, iff it is majorized by a monotone small function.

Keywords: Small function, rate of convergence, iterative sequence.

1. INTRODUCTION

In [1]Pták introduces the notion of a *small function* (= "convergence rate") as ω: $[0, \tau] \to [0, \tau]$ such that the series

$$t + \omega(t) + \omega(\omega(t)) + \ldots =: \sum_0^\infty \omega_n(t) =: \sigma(t) \tag{1.1}$$

converges for each t. (We take the notation $[0, \tau]$ as including the possibility of $\mathbb{R}^+ := [0, \infty)$. The notation $\omega_n(\cdot)$ is defined by setting

$$\omega_0(t) := t\,; \quad \text{recursively (for } n = 1, 2, \ldots\text{):} \quad \omega_n(t) := \omega(\omega_{n-1}(t)) \tag{1.2}$$

for $t \in [0, \tau]$.) An example is $\omega(t) := \theta t$ $(0 \leqq \theta < 1)$ – with the obvious relation of (1.1) to the rate of convergence given by the Contractive Mapping Theorem, for which $\sigma(t) := \sum_n \omega_n(t)$ provides the bound on distance to the fixed point in terms of the initial step length t.

For the applications discussed in [1], all the functions $\omega(\cdot)$ considered were both monotone (i.e., nondecreasing) and continuous but these properties were not made part of the definition of a small function. We note (see Observation 4, below) that any function majorized by a monotone small function is itself a small function and a principal result of this note is a converse of this: if a small function is continuous, then it must necessarily be majorized by a monotone small function. Other principal results are:

▻ If ω is a continuous small function, then convergence in (1.1) is uniform on bounded intervals so $\sigma(t) := \sum \omega_n(t)$ depends continuously on t.

*) This research was partly supported under grant # AFOSR-82-0271.

▷ If $\omega(t) \sim [t - ct^{\alpha+1}]$ with $c > 0$, $\alpha > 0$, then ω is a small function for $0 \leqq \alpha < 1$ but not for $\alpha \geqq 1$.
▷ The set of continuous (respectively, monotone) small functions is closed under max, min and convex combinations.

By way of perspective, we note that small functions, with no supplementary properties, can be quite wild: one easily sees that

$$(1.3) \qquad \omega(t) := \begin{cases} t/2 & \text{for positive rational } t \\ \text{arbitrary (but positive rational)} & \text{otherwise} \end{cases}$$

gives a small function which can be constructed so its graph is dense in $[0, \tau] \times [0, \tau]$ (or in $\mathbb{R}^+ \times \mathbb{R}^+$).

2. SOME OBSERVATIONS

Observation 1. *Let $\phi: [0, \tau] \to [0, \hat{\tau}]$ be strictly increasing with $0 < \theta_- \leqq \phi(t)/t < \theta_+$ for t near 0. (Note that ϕ^{-1} is well-defined with similar estimate.) For $\omega: [0, \tau] \to [0, \tau]$, let $\hat{\omega}$ denote $\phi \circ \omega \circ \phi^{-1}: [0, \hat{\tau}] \to [0, \hat{\tau}]$. Then ω is a small function if and only if $\hat{\omega}$ is.*

Proof. One has, by induction, $\hat{\omega}_n(s) = \phi(\omega_n(t))$ for $s := \phi(t)$. If ω is a small function, then, given t, one must have $\omega_n(t) \to 0$ so, for large n, one has $\hat{\omega}_n(s) \leqq \theta_+ \omega_n(t)$ so $\sum \hat{\omega}_n(s) < \infty$. The argument for the converse is essentially identical. □

Observation 2. *Suppose $\omega_n(t) \to 0$ for each t; let $\hat{\omega}$ denote the restriction of ω to $[0, \varepsilon]$. Then ω is a small function if and only if $\hat{\omega}$ is. (I.e., smallness depends primarily on behavior near 0.)*

Proof. The 'only if' is immediate. Given t, one has $\omega_n(t) < \varepsilon$ for $n \geqq N = N(\varepsilon, t)$; set $\hat{t} := \omega_N(t)$. Clearly $\omega_n(t) = \hat{\omega}_{n-N}(\hat{t})$ for $n \geqq N$ so, assuming $\hat{\omega}$ is small,

$$\sum_0^\infty \omega_n(t) = \sum_0^{n-1} \omega_n(t) + \sum_0^\infty \hat{\omega}_n(\hat{t}) < \infty$$

and ω is also a small function. □

Observation 3. *Let $\hat{\omega}$ be any iterate of ω: say, $\hat{\omega} := \omega_K$. Then ω is a small function if and only if $\hat{\omega}$ is.*

Proof. Clearly $\hat{\omega}_n = \omega_{nK}$ so $\sum \hat{\omega}_n \leqq \sum \omega_n$. If, on the other hand, $\hat{\omega}$ is small, set $t_k := \omega_k(t)$ for $0 \leqq k < K$. One can always set $n = mK + k$ with $0 \leqq k < K$ so $\omega_n(t) = \hat{\omega}_m(t_k)$. Thus,

$$\sum_n \omega_n(t) = \sum_m \hat{\omega}_m(t_0) + \ldots + \sum_m \hat{\omega}_m(t_{K-1}) < \infty$$

so ω is then also small. □

Observation 4. *Let $f, \omega: [0, \tau] \to [0, \tau]$ with ω nondecreasing and $f \leqq \omega$ for each t. Then $f_n \leqq \omega_n$ for each n. If, also, ω is a small function, then so is f.*

Proof. One has $f_n \leqq \omega_n$ for each n since, inductively,

$$f_{n+1}(t) = f(f_n(t)) \leqq \omega(f_n(t)) \leqq \omega(\omega_n(t)) = \omega_{n+1}(t).$$

For small ω one then has $\sum f_n(t) \leqq \sum \omega_n(t) < \infty$ and f is small. □

Observation 5. *A small function ω cannot have a positive fixed point. Hence, if either* (a) *ω is monotone or* (b) *ω is continuous, then ω small implies*

$$\omega(t) < t \quad \textit{for each} \quad t > 0. \tag{2.1}$$

Proof. If $\omega(t_*) = t_*$, then $\omega_n(t_*) = t_*$ for each n so, if $t_* > 0$, one would have $\sum \omega_n(t_*) = \infty$, contradicting smallness. Suppose, next, one could have ω small with $\omega(t_*) > t_*$. For case (a), ω monotone, this gives, inductively,

$$\omega_{n+1}(t_*) = \omega(\omega_n(t_*)) \geqq \omega(\omega_{n-1}(t_*)) = \omega_n(t_*)$$

so $\omega_n(t_*) \geqq \ldots \geqq \omega_1(t_*) > t_* \geqq 0$ and $\sum \omega_n(t_*) \geqq t_* + \sum_1^\infty \omega_1(t_*) = \infty$, contradicting smallness. For case (b), ω continuous, note that $\omega_n(t_*) \to 0$ to have $\sum \omega_n(t_*) < < \infty$ so there must be a first n with $\omega_{n+1}(t_*) < \omega_n(t_*)$ – i.e., (noting $n \geqq 1$) one has

$$b < a \quad \text{with} \quad \omega(b) > \omega(a)$$

where $b := \omega_{n-1}(t_*)$ so $\omega(b) = \omega_n(t_*) =: a$ and $\omega(a) = \omega_{n+1}(t_*)$. This would give a fixed point $\hat{t} \in (a, b)$ for ω – but we have already shown that is impossible. Thus (2.1) holds in either case. □

Observation 6. *Let $\tau = \tau_0 > \tau_1 > \ldots$ with $\tau_n \to 0$ and define $\hat{\omega}: [0, \tau] \to [0, \tau]$ by*

$$\hat{\omega}(0) = 0 \quad \textit{and} \quad \hat{\omega}(t) := \tau_n \quad \textit{for} \quad \tau_n < t \leqq \tau_{n-1}. \tag{2.2}$$

If $\tilde{\omega}$ is similarlly defined using $\{\{\tilde{\tau}_m\}$ with $\{\tau_n\} \subset \{\tilde{\tau}_m\}$, then $\hat{\omega} \leqq \tilde{\omega}$ on $[0, \tau]$. If ω is a monotone function on $[0, \tau]$, $\tau_n := \omega_n(\tau)$. One then has

$$\tau_{n+1} \leqq \omega(t) \leqq \tau_n \quad \textit{for} \quad \tau_n \leqq t \leqq \tau_{n-1} \quad (n = 1, 2, \ldots) \tag{2.3}$$

so $\omega \leqq \hat{\omega}$ given by (2.2). *Thus, ω is a small function if and only if $\hat{\omega}$ is.*

Proof. The graph of $\hat{\omega}$, given by (2.2), corresponds to a 'staircase' with 'upper corners' $\{(\tau_n, \tau_n)\}$ and 'lower corners' $\{(\tau_n, \tau_{n+1})\}$. Introducing additional τ's merely introduces additional steps, so increasing the value of the function at the new 'treads' while leaving it unchanged on the remaining portions of the old ones; hence $\tilde{\omega} \leqq \hat{\omega}$. For $\{\tau_n := \omega_n(\tau)\}$ with ω monotone (We may assume $\tau_{n+1} < \tau_n$ for $n = = 0, 1, \ldots$) and $t \in [\tau_n, \tau_{n-1}]$, one has, from the definition (2.2) and monotonicity,

$$\tau_{n+1} = \hat{\omega}(\tau_n) = \omega(\tau_n) \leqq \omega(t) \leqq \omega(\tau_{n-1}) = \tau_n = \hat{\omega}(t),$$

giving (2.3) and $\omega \leqq \hat{\omega}$. Clearly ω will be small if $\hat{\omega}$ is by Observation 4 while $\hat{\omega}$ is small if (and only if) $\sum \tau_n < \infty$, i.e., if $\sum \omega_n(\tau) < \infty$. □

3. TWO THEOREMS

A principal result of this note is the following.

Theorem 1. *Let f: $[0, \tau] \to [0, \tau]$ be a continuous function. Define a monotone function $\tilde{f}$: $[0, \tau] \to [0, \tau]$ by setting*

$$\tilde{f}(t) := \max \{f(s) : 0 \leqq s \leqq t\} \quad \textit{for} \quad t \in [0, \tau]\,. \tag{3.1}$$

Then $\tilde{f}$ is continuous and is a small function if and only if f is.

That $\tilde{f}$ is monotone is immediate; that it is continuous (given that f is) is a standard exercise and will be omitted; the 'only if' part of the proof follows immediately from Observation 4. Before proceeding to (the rest of) the proof of Theorem 1 – which now consists of showing that $\tilde{f}$ is a small function, given that f is small – we need the following information about $\tilde{f}$, defined by (3.1).

Lemma 1. *Assume f is a small function. Then $\tilde{f}(t) < t$ for $t \in (0, \tau]$.*

Proof. By case (b) of Observation 5 we have $f(s) < s$ on $[0, t]$. The max in (3.1) is, indeed, attained since $[0, t]$ is compact and f continuous so, for some $s \in [0, t]$, one has $\tilde{f}(t) = f(s) < s \leqq t$. □

Lemma 2. *Assume f is a small function and define $\tilde{f}$ by (3.1). For any $t_0 \in [0, \tau]$, recursively set $t_n := \tilde{f}(t_{n-1})$. (Note that $t_0 > t_1 > \ldots$ by Lemma 1.) Then for any $m \geqq 1$ and any $s \in [t_{m+1}, t_m]$ there exists $\hat{s} \in [t_m, t_{m-1}]$ such that $s = s = f(\hat{s})$.*

Proof. One has $f(t_m) \leqq \tilde{f}(t_m) = t_{m+1}$. On the other hand, $\tilde{f}(t_{m-1}) = t_m$ so, for some $\bar{s} \leqq t_{m-1}$ one has $f(\bar{s}) = t_m$ by the definition (3.1); clearly $\bar{s} > t_m$ since $\tilde{f}(t_m) = t_{m+1} < t_m$. Thus, for any $s \in [t_{m+1}, t_m]$ one has $f(t_m) \leqq s \leqq f(\bar{s})$ so, by continuity, there exists $\hat{s} \in [t_m, \bar{s}] \subset [t_m, t_{m-1}]$ for which $f(\hat{s}) = s$. □

Proof of Theorem 1. We assume f is a small function and suppose $\tilde{f}$ were not small so, for some $t_0 \in [0, \tau]$, one would have $\sum \tilde{f}_n(t_0) = \infty$. (One has $\tilde{f}_n(t_0) =: t_n$ as in Lemma 2.)

For each m, arbitrarily choose $s_{m,m} \in [t_{m+1}, t_m]$ – say, $s_{m,m} := t_m$. By Lemma 2 one has existence of $s_{m,m-1} \in [t_m, t_{m-1}]$ such that $s_{m,m} = f(s_{m,m-1})$. Applying Lemma 2 again one has $s_{m,m-2} \in [t_{m-1}, t_{m-2}]$ such that $s_{m,m-1} = f(s_{m,m-2})$ so $s_{m,m} = f_2(s_{m,m-2})$. Repeating this procedure, one eventually obtains existence of $\hat{s}_m := s_{m,0} \in [t_1, t_0]$ such that

$$f_k(\hat{s}_m) = s_{m\,k} \in [t_{k+1}, t_k] \quad \text{for} \quad 0 \leqq k \leqq m\,. \tag{3.2}$$

By the compactness of $[t_1, t_0]$ one has, for a subsequence, $\hat{s}_m \to t_* \in [t_1, t_0]$ and by continuity (3.2) gives $f_k(t_*) \in [t_{k+1}, t_k]$ for every k. In particular, this gives $\tilde{f}_k(t_0) = t_k \leqq f_{k-1}(t_*)$ so

(3.3) $$\sum \tilde{f}_k(t_0) \leqq \sum f_{k-1}(t_*) < \infty$$

(using the smallness of f), contradicting the supposition that $\sum \tilde{f}_k(t_0) = \infty$. □

Corollary 1. *Let $\omega: [0, \tau] \to [0, \tau]$ be a continuous small function. Then the convergence in (1.1) is uniform (on bounded intervals) so $\sigma := \sum \omega_n$ is continuous.*

Proof. In view of Observation 5 it is clearly sufficient to consider only finite τ. Define $\tilde{\omega}$ from ω as in (3.1). Then, uniformly for $t \in [0, \tau]$, one has $\omega(t) \leqq \tilde{\omega}(t) \leqq \leqq \tilde{\omega}(\tau)$ so, by Observation 4, $0 \leqq \omega_n(t) \leqq \tilde{\omega}_n(\tau)$ and $\sum \omega_n(t) \leqq \sum \tilde{\omega}_n(\tau)$. □

Corollary 2. *The set of continuous small functions is convex; if f, g are continuous small functions, then so are $f \wedge g$ and $f \vee g$.*

Proof. With no loss of generality, given (2.1), we may take τ to be finite. Given continuous small functions $f, g: [0, \tau] \to [0, \tau]$, let $\tilde{f}, \tilde{g}$ be defined as in (3.1) and define $\hat{f}, \hat{g}$ from $\tilde{f}, \tilde{g}$ as in Observation 6. Set

$$u_n := \tilde{f}_n(\tau), \quad v_n := \tilde{g}_n(\tau)$$

and let $\{\tau_n\}$ be the set $\{u_n\} \cup \{v_n\}$, ordered so $\tau = \tau_0 > \tau_1 > \dots$. By the Theorem we have $\tilde{f}, \tilde{g}$ monotone small functions so, by Observation 6, also $\hat{f}, \hat{g}$ are small. Clearly

$$0 \leqq f(t) \leqq \tilde{f}(t) \leqq \hat{f}(t), \quad 0 \leqq g(t) \leqq \tilde{g}(t) \leqq \hat{g}(t)$$

for $t \in [0, \tau]$. Defining $\hat{\omega}$ by (2.2), one has $\hat{f} \leqq \hat{\omega}$ and $\hat{g} \leqq \hat{\omega}$ by Observation 6 since $\{u_n\} \subset \{\tau_n\}$ and $\{v_n\} \subset \{\tau_n\}$. Thus,

(3.4) $$\lambda f + (1 - \lambda) g, \quad f \wedge g, \quad f \vee g \leqq \hat{\omega} \quad (0 \leqq \lambda \leqq 1)$$

on $[0, \tau]$. Since $\tilde{f}, \tilde{g}$ are small so $\sum u_n < \infty$, $\sum v_n < \infty$, one has $\sum \tau_n < \sum u_n + \sum v_n < < \infty$ so, as in Observation 6, $\hat{\omega}$ is small. The desired results then follow by applying Observation 4 to (3.4). □

We next wish to consider the map: $\omega \mapsto \sigma$ defined by (1.1) and will obtain a limited form of lower semicontinuity for this in each of the contexts we have considered: continuous functions, topologized by uniform convergence, and (left continuous) monotone functions, topologized by a notion of 'uniform lower semiconvergence, $(\omega \prec_u \{\omega^j\})$. (*Definition*: We write $\omega \prec \{\omega^j\}$ if

(3.5) for each $\varepsilon > 0$ and each t there exists $\bar{j} = \bar{j}(\varepsilon, t)$ such that $\omega(t) \leqq \omega^j(t) + \varepsilon$ if $j \geqq \bar{j}$

and write $\omega \prec_u \{\omega^j\}$ if $\bar{j} = \bar{j}(\varepsilon)$, independent of t.) Before proving the theorem we need the following observations.

Lemma 3. (i) Let $f^j \to f$ uniformly with f continuous (hence, uniformly continuous for $[0, \tau]$ compact); let $g^j \to g$ pointwise (or uniformly). Then $f^j \circ g^j \to f \circ g$ pointwise (or uniformly).

(ii) Let $f \prec_u \{f^j\}$ with f monotone and left continuous and let $g \prec \{g^j\}$. Then $f \circ g \prec \{f^j \circ g^j\}$.

Proof. In either case,

$$f(g(t)) - f^j(g^j(t)) = [f(g(t)) - f(g^j(t))] + [f(g^j(t)) - f^j(g^j(t))] .$$

For (i) this, noting the uniform continuity of f, gives the result immediately. For (ii) one has $s = s^j := g^j(t) \geqq g(t) - \varepsilon'$ for large enough j and then $f(s) \geqq f(g(t) - \varepsilon') \geqq \geqq f(g(t)) - \varepsilon/2$ for $\varepsilon' = \varepsilon'(\varepsilon)$ small enough. At the same time $f(s) - f^j(s) \leqq \varepsilon/2$ for large j, uniformly in s since $f \prec_u \{f^j\}$. Thus, $f(g(t)) \leqq f^j(g^j(t)) + \varepsilon$ for large enough j. □

Corollary. (i) Let f^j, f be as in Lemma 3(i). Then the iterates converge uniformly: $f^j_n \to f_n$ for each n.

(ii) Let f^j, f be as in Lemma 3(ii). Then $f_n \prec \{f^j_n\}$ for each n.

Proof. By induction on n since $f_{n+1} = f \circ f_n$. □

Theorem 2. *Let f^j, f be as in Lemma 3(i) or 3(ii). Define σ^j, σ from f^j, f by (1.1) – not necessarily finite for each t. Then, for each t,*

$$\sigma(t) \leqq \bar{\sigma}(t) := \liminf \sigma^j(t) . \tag{3.6}$$

Proof. Given t, we assume $\bar{\sigma}(t) < \infty$ or there is nothing to prove. Extracting a subsequence if necessary, we may assume $\sigma^j(t) \to \bar{\varrho}(t)$. For $N = 1, 2, \ldots$, introduce

$$\sigma_N := \textstyle\sum_0^N f_n(t) , \quad \sigma^j_N := \sum_0^N f^j_n(t) \tag{3.7}$$

so that $\sigma_N \to \sigma(t)$ and $\sigma^j_N \to \sigma^j(t)$ as $N \to \infty$. For case (i) one has $f^j_n \to f_n$ for each n so $\sigma^j_N \to \sigma_N$. Since $\sigma^j_N \leqq \sigma^j(t) \to \bar{\sigma}(t)$, one has $\sigma^j_N \leqq \bar{\sigma}(t) + \varepsilon$ for each N and for $j \geqq j_N$. Thus $\sigma_N = \lim_j \sigma^j_N \leqq \bar{\sigma}(t) + \varepsilon$. Since this holds for each N, one has $\sigma(t) = = \lim_N \sigma_N \leqq \bar{\sigma}(t) + \varepsilon$ and, since $\varepsilon > 0$ is arbitrary, this gives (3.6). For case (ii) one has $f_n \prec \{f^j_n\}$ for each n so $\sigma_N \prec \{\sigma^j_N\}$. Since $\sigma^j_N \leqq \sigma^j(t) \to \bar{\sigma}$, this gives

$$\sigma_N \leqq \sigma^j_N + \varepsilon \leqq \sigma^j(t) + \varepsilon \leqq \bar{\sigma}(t) + 2\varepsilon$$

for large enough j whence $\sigma_N \leqq \bar{\sigma}(t) + 2\varepsilon$ for each N. Again, $\sigma(t) = \lim_N \sigma_N \leqq \leqq \sigma(t) + 2\varepsilon$ so (3.6) follows. □

Corollary. *If $\{f^j\}$ is a sequence of continuous small functions on $[0, \tau]$ with $\{\sigma^j\}$ bounded and if $f^j \to f$ uniformly on $[0, \tau]$, then f is also a small function.* □

If ω: $[0, \tau] \to [0, \tau]$ satisfies (2.1) and, for example, is continuous then Observation 2 applies: the essential criterion for smallness is the asymptotic behavior of $\omega(t)$ as $t \to 0$. In view of (2.1) we consider asymptotic behavior of the form:

$$\omega(t) = t[1 - ct^\alpha + O(t^\alpha)] \tag{4.1}$$

with $\alpha \geqq 0$ and $c > 0$. (The case $\alpha = 0$ reduces to $\omega(t) \sim \theta t$ with $\theta := 1 - c$. Presumably, here $c < 1$ so $0 < \theta < 1$ and, asymptotically, $\omega_n(t) \sim \theta^n t$ so $\sigma(t) := := \sum \omega_n(t) \sim t/(1 - \theta) < \infty$ as for the Contractive Mapping Theorem.) The interesting case is then $\alpha > 0$ and the principal result of this section is that (4.1) implies

$$\gamma_n := n^{1/\alpha}\, \omega_n(t) \to \bar{\gamma} := (\alpha c)^{-1/\alpha} \quad \text{as} \quad n \to \infty \tag{4.2}$$

for each $t > 0$.

Theorem 3. *Suppose ω: $[0, \tau] \to [0, \tau]$ is such that $\omega_n(t) \to 0$ for each t and* (4.1) *holds for t near* 0 *with $\alpha > 0$, $c > 0$. Then, for each t (assuming $\omega_n(t) \neq 0$) one has* (4.2).

Proof. We begin by considering the special case:

$$\omega(t) := t(1 - t^\alpha) \tag{4.3}$$

so $\bar{\gamma} := \beta^\beta$ where, for convenience, we have set $\beta = 1/\alpha$. Observe that $\gamma_n := n^\beta t_n$ with $t_n := \omega_n(t)$ gives $t_n = n^{-\beta}\gamma_n$ and

$$\gamma_{n+1} = (n + 1)^\beta\, t_{n+1} = (1 + 1/n)^\beta\, n^\beta \omega(n^{-\beta}\gamma_n) = (1 + 1/n)^\beta\, \gamma_n(1 - \gamma_n^\alpha/n)\,.$$

It is convenient to introduce

$$\phi(v) = \phi(v; \gamma) := (1 + v)^\beta\, (1 - \gamma^\alpha v) \quad (v \geqq 0)$$

so that

$$\gamma_{n+1} = \phi(1/n; \gamma_n)\, \gamma_n \quad n = 0, 1, \dots. \tag{4.4}$$

Since $\phi(0) = 1$ and

$$\phi'(v) = \beta(1 + v)^{\beta-1}\, [1 - \alpha\gamma^\alpha - (1 + \alpha)\, \gamma^\alpha v]\,,$$

one has

$$\begin{aligned} &\text{(i)} \quad \phi(v) < 1 \quad \text{if} \quad \gamma \geqq \bar{\gamma}\,, \quad v > 0\,, \\ &\text{(ii)} \quad \phi(v) > 1 \quad \text{if} \quad \gamma < \bar{\gamma}\,, \quad 0 < v \ \text{ small}\,. \end{aligned} \tag{4.5}$$

Step 1 [$\limsup \gamma_n \leqq \bar{\gamma}$]: From (4.4), (4.5i), if $\gamma_n \geqq \bar{\gamma}$ then $\gamma_{n+1} < \gamma_n$ while if $\gamma_n \leqq \bar{\gamma}$ then $\gamma_{n+1} < (1 + 1/n)^\beta\, \bar{\gamma}$; we need only eliminate the possibility: $\gamma_n > \gamma_{n+1} > \dots \dots > \hat{\gamma} > \bar{\gamma}$. Suppose $\gamma_n \downarrow \hat{\gamma} > \bar{\gamma}$ so, for large enough m and all $n \geqq m$, one has $\gamma_n \geqq \hat{\gamma}$ and

$$\phi(1/n; \gamma_n) \leqq \phi(1/n; \hat{\gamma}) \leqq 1 + (\beta - \hat{\gamma}^\alpha)/n$$

(some $\tilde{\gamma}$ between $\bar{\gamma}$ and $\hat{\gamma}$ so $(\beta - \tilde{\gamma}^\alpha) = -\delta < 0$). Then

$$\gamma_{m+k} = [\prod_m^{m+k-1} \phi(1/n; \gamma_n)] \gamma_m \leqq [\prod_m^{m+k-1} (1 - \delta/n)] \gamma_m .$$

Standard results on infinite products show this would give $\gamma_{m+k} \to 0$, a contradiction.

Step 2 [$\liminf \gamma_n \geqq \bar{\gamma}$]: For small $\varepsilon > 0$, if n is large enough (so $v := 1/n$ is small enough) then (4.4), (4.5ii) show that $\gamma_n \leqq \bar{\gamma} - \varepsilon$ implies $\gamma_{n+1} > \gamma_n$ while if $\gamma_n \geqq \bar{\gamma} - \varepsilon$ one has $\gamma_{n+1} > (1 - C\bar{\gamma}^\alpha/n)(\bar{\gamma} - \varepsilon)$; thus, we need only eliminate the possibility: $\gamma_n < \gamma_{n+1} < \ldots < \hat{\gamma} < \bar{\gamma} - \varepsilon$. As for Step 1, for some $\tilde{\gamma}$ between $\hat{\gamma}$ and $\bar{\gamma} - \varepsilon$ this would give

$$\phi(1/n; \gamma_n) \geqq \phi(1/n; \hat{\gamma}) \geqq 1 + \delta/n$$

with $\delta := (\beta - \tilde{\gamma}^\alpha) > 0$ for $n \geqq m$ large enough and a similar infinite product argument would then give the contradiction: $\gamma_n \to \infty$.

This shows the desired result for the special case (4.3). The result for $c \neq 1$, i.e., $\omega(s) := s(1 - cs^\alpha)$ follows from this by taking $t := c^\beta s$ so $\omega_n(s) = c^{-\beta} \omega_n(t)$ for each n. For the still more general (asymptotic) case (4.1), suppose

$$\omega(t) \leqq t[1 - ct^\alpha + o(t^\alpha)] \tag{4.1'}$$

near 0 so $\omega(t) \leqq t(1 - \hat{c}t^\alpha) =: \hat{\omega}(t)$ near (enough) to 0 for any $\hat{c} < c$. For small t one has $\hat{\omega}$ monotone so (cf., Observation 4) $\omega_n(t) \leqq \hat{\omega}_n(t)$ whence

$$\limsup n^\beta \omega_n(t) \leqq \lim n^\beta \hat{\omega}_n(t) = (\alpha\hat{c})^{-\beta}$$

and, since $\hat{c} < c$ is arbitrary, this gives $\limsup n^\beta \omega_n(t) \leqq (\alpha c)^{-\beta}$. On the other hand, a corresponding argument for ω satisfying (4.1) gives $\omega(t) \geqq t(1 - \hat{c}t^\alpha) =: \hat{\omega}(t)$ near 0 with $\hat{c} > c$ whence (parallel to Observation 4) $\omega_n(t) \geqq \hat{\omega}_n(t)$ so $\liminf n^\beta \omega_n(t) \geqq (\alpha c)^{-\beta}$ as above. □

Corollary. *Let* $\omega: [0, \tau] \to [0, \tau]$ *satisfy* (4.1) *with* $c > 0$ *but* $\alpha \geqq 1$. *Then* ω *cannot be a small function. Conversely, if* ω *is such that* $\omega_n(t) \to 0$ *for each* t *and satisfies* (4.1') *with* $c > 0$ *and* $0 < \alpha < 1$, *then* ω *is a small function.*

Proof. In the first case one has, at least for small t, $\omega_n(t) \sim \bar{\gamma}n^{-\beta}$ with $\beta \leqq 1$ so $\sum\omega_n(t)$ diverges. In the second case, setting $\hat{\omega}(t) := t(1 - \hat{c}t^\alpha)$ with $0 < \hat{c} < c$ as earlier, one has $\omega_n(t) \leqq \hat{\omega}_n(t) \sim \bar{\gamma}n^{-\beta}$ with $\beta > 1$ so $\sum\omega_n(t)$ converges. □

It is of possible interest also to consider the differentiability of $\sigma(\cdot)$ if $\omega(\cdot)$ is continuously differentiable. Differentiating (1.1) term by term gives, formally,

$$\sigma'(t_0) = \sum_{n=0}^{\infty} \omega_n'(t_0) = 1 + \sum_{n=1}^{\infty} (\prod_{k=0}^{n-1} \omega'(t_k)) \tag{4.6}$$

where, for each $t = t_0$ we have set $t_k := \omega_k(t_0)$. Suppose

$$\omega'(t) = 1 - (\alpha + 1)\, ct^\alpha \quad t \text{ near } 0 \tag{4.7}$$

which, integrating, gives (4.1′). Then, with $\bar{\gamma}$ as in (4.2), one has

$$t_k \sim \bar{\gamma} k^{-1/\alpha}, \quad \omega'(t_k) - 1 \sim -\left(\frac{\alpha+1}{\alpha}\right) k^{-1},$$

$$\omega_n'(t_0) = \prod_{k=0}^{n-1} \omega'(t_k) \approx \prod_{k=0}^{n-1} \left(1 - \frac{\alpha+1}{\alpha} k^{-1}\right) \approx \tag{4.8}$$

$$\approx C \exp\left[-\frac{\alpha+1}{\alpha} \sum_{k=0}^{n-1} k^{-1}\right] \approx C' n^{-(\alpha+1)/\alpha}$$

(where C, C' reflect the approximations involved, especially in the 'early' stages of the iteration – the approximation being asymptotically correct as $n \to \infty$). One then has convergence of (4.6) since $\alpha > 0$ gives $(\alpha + 1)/\alpha > 1$. This convergence is formally independent of c but, of course, one must have $c > 0$ in (4.7) to have convergence of (1.1) as in the Corollary to Theorem 3. The approximation $t_k \sim \bar{\gamma} k^{-1/\alpha}$ is not uniform in t_0. However, continuity of ω and a careful look at the proof above for Theorem 3 show that this is locally uniform away from 0. Thus, the approximation (4.7) is locally uniform and the convergence of (4.6) is uniform on compact subsets of $(0, \tau]$. This shows that σ is then continuously differentiable on $(0, \tau]$. We will not examine diffrentiability near 0.

We next provide counterexamples to some plausible conjectures.

Conjecture 1. For monotone functions, if $\omega^j \to \omega$ uniformly and, for some everywhere finite $\hat{\sigma}$ one has $\sigma^j \leqq \hat{\sigma}$ for each j, then ω is a small function.

Counterexample. Let $\omega^j(t) := \{0 \text{ on } [0,1/2);\ 1/2 - 1/j \text{ on } [1/2, 1]\}$ so $\sigma^j(t) = \{t \text{ on } [0, 1/2);\ t + 1/2 - 1/j \text{ on } [1/2, 1]\} \leqq \hat{\sigma}(t) = t + 1/2$. One has $\omega^j(t) \to \omega(t) := \{0 \text{ on } [0, 1/2);\ 1/2 \text{ on } [1/2, 1]\}$ uniformly but $\sigma(t) = \{t \text{ on } [0, 1/2);\ \infty \text{ on } [1/2, 1]\}$ since 1/2 is a fixed point of ω. (This emphasizes the requirement of left continuity in Theorem 2.) □

Conjecture 2. Let $\{\omega^j\}$ be a sequence of continuous small functions with $\omega^j \to \omega$ uniformly and $\sigma^j \to \bar{\sigma}$. Then $\sigma = \bar{\sigma}$. I.e., the map: $\omega \mapsto \sigma$ is continuous in this context.

Counterexample. By Theorem 2 one has $\sigma \leqq \bar{\sigma}$ in this context and by Corollary 1 to Theorem 1 one has σ continuous. Choose $\{\tau_n\}$ as in Observation 6 with $\sum \tau_n < \infty$ and define $\hat{\omega}$ by (2.2). Define ω by modifying this, shifting the "upper corners" slightly to the right – to $(\tau_n + \varepsilon_n, \tau_n)$ with suitably chosen positive $\{\varepsilon_n\}$ – and making $\hat{\omega}$ linear on each $[\tau_n, \tau_n + \varepsilon_n]$ so the risers of the staircase are steep but no longer vertical; we refer to this modification as (2.2′). Thus, ω is continuous; the numbers $\{\varepsilon_n\}$ need only be 'sufficiently small' for the rest of this construction to work. At the j-th stage, select a (finite) set $\{\hat{\tau}_1^j, \ldots, \hat{\tau}_{k(j)}^j\}$ in the interval $(\tau_{j+1} + \varepsilon_{j+1}, \tau_j)$ in such a way that

$$c_j := \textstyle\sum_1^{k(j)} \hat{\tau}_k^j \to c > 0 \quad \text{as} \quad j \to \infty\,.$$

This is clearly possible. Now define ω^j by (2.2') using $\{\tau_n\} \cup \{\hat{\tau}_k^j\}$ with the original $\{\varepsilon_n\}$ associated with $\{\tau_n\}$ and a new set of small $\{\hat{\varepsilon}_k^j\}$. Clearly $\omega^j(t) = \omega(t)$ except for $\tau_{j+1} + \varepsilon_{j+1} < t < \tau_j + \varepsilon_j$ and $0 < \omega^j(t) - \omega(t) \leqq \tau_j - \tau_{j+1}$ for all t; thus $\omega^j \to \omega$ uniformly. One then has $\sigma^j(t) = \sigma(t) + c_j$ for $t > \tau_j + \varepsilon_j$ so for each t one has $\sigma^j(t) \to \sigma(t) + c \neq \sigma(t)$ as $j \to \infty$. □

Conjecture 3. If ω is continuous and $\sigma(t)$ is finite for *almost all* t, then much of the analysis above applies.

Counterexample. First consider the example with $\omega = 0$ on $[0, 1/3]$, $\omega(2/3) = 2/3$, $\omega(1) = 0$ and ω linear on $[1/3, 2/3]$ and on $[2/3, 1]$. One has a fixed point $2/3$ so Observation 5 fails but $\sigma(t) < \infty$ for $t \neq 2/3$. Note that obtaining $\tilde{\omega}$ as in (3.1) gives $\tilde{\omega} = \omega$ on $[0, 2/3]$ and $\tilde{\omega} = 2/3$ on $[2/3, 1]$ so $\tilde{\sigma}(t) = \infty$ on $[2/3, 1]$.

A more interesting example is provided by taking $\omega(0) = \omega(1) = 0$ and $\omega = 1$ on $[1/3, 2/3]$ with ω linear on $[0, 1/3]$ and on $[2/3, 1]$. Not only is there a positive fixed point at $t = 3/4$, but one has $\omega(t) > t$ on $(0, 3/4)$. Note that $\omega_m(t) = 1$ gives $\omega_n(t) = 0$ for $n > m$ and so $\sigma(t) < \infty$. Recursively, we note that:

$$\omega_2(t) = 0 \quad \text{for} \quad t \in [1/3, 2/3]\,,$$

$$\omega\colon [1/9, 2/9],\ [7/9, 8/9] \to [1/3, 2/3] \quad \text{so} \quad \omega_3(t) = 0 \quad \text{for} \quad t \in [1/9, 2/9] \cup [7/9, 8/9]\,,$$

$$\omega_4(t) = 0 \quad \text{for} \quad t \in [1/27, 2/27] \cup [7/27, 8/27] \cup [19/27, 20/27] \cup [25/27, 26/27]\,,$$

etc. Thus, $\sigma(t) < \infty$ on the complement of the Cantor set, hence almost everywhere. (Clearly, from the above analysis, $\sigma(t) < \infty$ for some t in the Cantor set as well and it would be plausible to conjecture that for this example the set $\{t\colon \sigma(t) = \infty\}$ is actually countably infinite.) □

Remark 1. One might wish to replace (1.1) by the more general form:

$$\sigma(t) := \sum_{n=0}^{\infty} c_n\, \omega_n(t) \tag{4.9}$$

using non-negative sequences $\boldsymbol{c} := (c_0, c_1, \ldots)$ other than $(1, 1, \ldots)$. Then Observations 1, 2, 4, 5, 6 and Theorem 1 remain valid (with negligible modifications of the proofs) provided $\sum c_n$ diverges. The validity of Obesrvation 3 in this context would depend on the action of left hifts on $\boldsymbol{c}$.

A still more general notion would be to use

$$\sigma(t) := \textstyle\sum_0^\infty \psi_n(\omega_n(t)) \tag{4.10}$$

where $\psi_n\colon S \to \mathbb{R}^+$ is specified for $n = 0, 1, \ldots$ and, now, $\omega\colon S \to S$ with S an arbitrary set. In this generality there seems little useful structure but, for example, one interpretation might be:

$S := \{$bounded subsets of a metric space $X\}$,

$\omega\colon t \mapsto \{f(\xi)\colon \xi \in t\}$ for $t \in S$ where $f\colon X \to X$,

$\psi_n = \psi\colon t \mapsto \operatorname{diam}(t)$,

related to, e.g., the Contraction Mapping Principle. □

Remark 2. Of considerable interest would be a somewhat different generalization — keeping (1.1) as it is but now with t taking values in $[0, \tau]^m$ for some $m > 1$ so $\omega: [0, \tau]^m \to [0, \tau]^m$. The following example, however, shows that one must be careful in dealing with this situation: one need not have (2.1) but, in fact, can have a continuous small function for which $\omega(t_*) = 2t_*$ for some $t_* > 0$.

Example. Let $\tau = 1$, $m = 2$ so $t = (x, y)$ varies over the unit square and define ω as a composition of three maps:

Step 1: $(x, y) \mapsto (x, \min\{y, x, 1 - x\})$,

Step 2: $(x, y) \mapsto (\min\{x, y\}, y)$,

Step 3: The segment $[(0, 0), (1/4, 1/4)] \to [(0, 0), (1, 0)]$ linearly,
the segment $[(1/4, 1/4), (1/2, 1/2)] \to [(1, 0), (1, 1)]$ affinely.

Note that $\omega: [0, 1]^2 \to L := [(0, 0), (1, 0)] \cup [(1, 0), (1, 1)]$ and $\omega: L \to (0, 0)$ so $\omega_2(t) = (0, 0)$ for every t. However, for $t_* = (1/2, 1/2)$ one has $\omega(t_*) = (1, 1) = 2t_*$. $\square$

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Souhrn

O JISTÝCH ITERATIVNÍCH POSLOUPNOSTECH

THOMAS I. SEIDMAN

Autor studuje „malé funkce" (small functions) zavedené Vl. Ptákem. Mezi jinými dokazuje, že spojitá funkce je malá, právě když je majorizována monotonní malou funkcí.

Резюме

О НЕКОТОРЫХ ИТЕРАТИВНЫХ ПОСЛЕДОВАТЕЛЬНОСТЯХ

THOMAS I. SEIDMAN

В статье изучаются „малые функции", введённые Вл. Птаком, и кроме прочего доказывается, что непрерывна функция малая тогда и только тогда, когда она мажорируется монотонной малой функцией.

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