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SOME NEW RESULTS ABOUT THE SHORTNESS EXPONENT IN POLYHEDRAL GRAPHS

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Summary. The shortness exponent $\sigma(\Gamma)$ of a family Γ of graphs G is defined as $\sigma(\Gamma) = \liminf_{G\in\Gamma} (\log h(G))/(\log n(G))$, where $n(G)$ or $h(G)$ denotes the number of vertices of G or the maximum number of vertices of G belonging to a circuit, respectively.

The paper deals with shortness exponents of families of regural graphs of polyhedra having the smallest number of types of faces and a shortness exponent <1.

Keywords: Shortness exponent, polyhedral graphs.

We denote by $n(G)$ the number of vertices and by $h(G)$ the number of vertices in a maximum circuit of a graph G. In the following, we deal with families Γ of graphs containing non-Hamiltonian members G, that means $h(G) < n(G)$ and, moreover, for any $\varepsilon > 0$ there exists a $G \in \Gamma$, such that $h(G)/n(G) < \varepsilon$.

A suitable concept for estimating the length of a maximum circuit is the shortness exponent $\sigma(\Gamma)$ of a family Γ of graphs G:

$$\sigma(\Gamma) := \liminf_{G\in\Gamma} \frac{\log h(G)}{\log n(G)}.$$

In this paper, we only study families of polyhedral graphs. Let us denote by Γ_r, $r \in \{3, 4, 5\}$ the family of all regular polyhedral graphs of degree r. As is well-known, a graph G is polyhedral iff G is planar and three-connected. Let us denote by $\Gamma_r(p_1, \ldots, p_m)$ the subfamily of Γ_r containing at most m types of faces, namely, p_1-gons, p_2-gons, …, p_m-gons.

J. Zaks [4] searched for the minimum $m(r)$ such that there exist $m = m(r)$ integers $p_1, p_2, \ldots, p_m$ with the property that $\sigma(\Gamma_r(p_1, p_2, \ldots, p_m)) < 1$. P. J. Owens [3] proved that for $r \in \{4, 5\}$ the inequality $m(r) \leqq 3$ holds. We can prove

Theorem 1. $m(r) = 2$ *for* $r \in \{3, 4, 5\}$.

If $\Gamma_r(p(r), q(r)) \neq \emptyset$ and $p(r) < q(r)$ then $p(3) \in \{3, 4, 5\}$, $p(4) = p(5) = 3$ (see [2]). We can prove

Theorem 2.

$$\sigma(\Gamma_3(4, K)) \leqq \frac{\log 44}{\log 45} \quad \textit{for any odd} \quad K \geqq 21,$$

$$\sigma(\Gamma_3(5, K)) \leqq \frac{\log 36}{\log 37} \quad \textit{for any} \quad K \geqq 79 \quad \textit{and} \quad K \not\equiv 0 \pmod 5,$$

$$\sigma(\Gamma_4(3, K)) \leqq \frac{\log 16}{\log 17} \quad \textit{for any} \quad K \geqq 58 \quad \textit{and} \quad K \not\equiv 0 \pmod 3,$$

$$\sigma(\Gamma_5(3, K)) \leqq \frac{\log 16}{\log 17} \quad \textit{for any} \quad K \geqq 43 \quad \textit{and} \quad K \not\equiv 0 \pmod 3.$$

In this paper we will only prove the first inequality. (The proofs of the remaining estimates of the shortness exponents indicated in Theorem 2 can be seen from [2].) To this aim we construct a sequence $\{G_i\}$ of 3-regular polyhedral graphs containing only 4-gons and K-gons and satisfying

$$\lim_{i\to\infty} \frac{\log h(G_i)}{\log n(G_i)} \leqq \frac{\log 44}{\log 45}.$$

Let G_0 be the graph of a cube with exact one distinguished (black) vertex.

G_{l+1} arises from G_l by replacing each of the black vertices by a suitable figure Z still to be constructed.

Definition. Let i, j, m be integers and G be a graph with following properties:

(i) G is planar, 3-connected and 3-regular.

(ii) G contains a vertex P incident with an $(i + 1)$-gon, a $(j + 1)$-gon and an $(m + 1)$-gon. All the other faces of G are 4-gons or K-gons.

(iii) G contains a vertex Q $(Q \neq P)$ incident with three 4-gons. Exactly one of these vertices will be distinguished. We call it black.

A ***figure*** (i, j, m) is the object arising from G by splitting up the vertex P into 3 half-edges (in Fig. 1 G is the graph of a cube and we obtain by splitting up an arbitrary vertex a figure (3, 3, 3)).

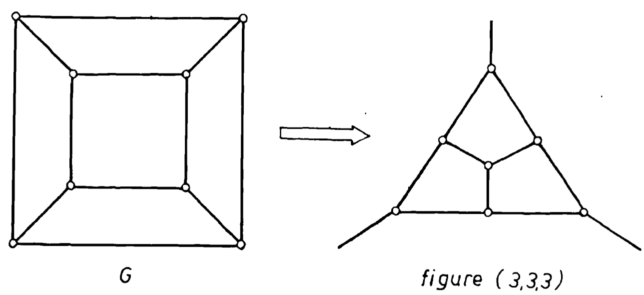

Fig. 1,

Let us now construct the figure Z. This construction proceeds in two steps:

Step 1: We start with the figure E arising from the well-known non-Hamiltonian

Grinberg ian graph (see [5]) splitting off some vertex (remark: E has the property that a path through E which connects any two arbitrary halfedges leaves out at least one vertex).

Step 2: Each of the vertices of E is to be replaced by a suitable figure (i, j, m) and we obtain the figure Z shown in Fig. 2. As one can easily see, we need eleven different

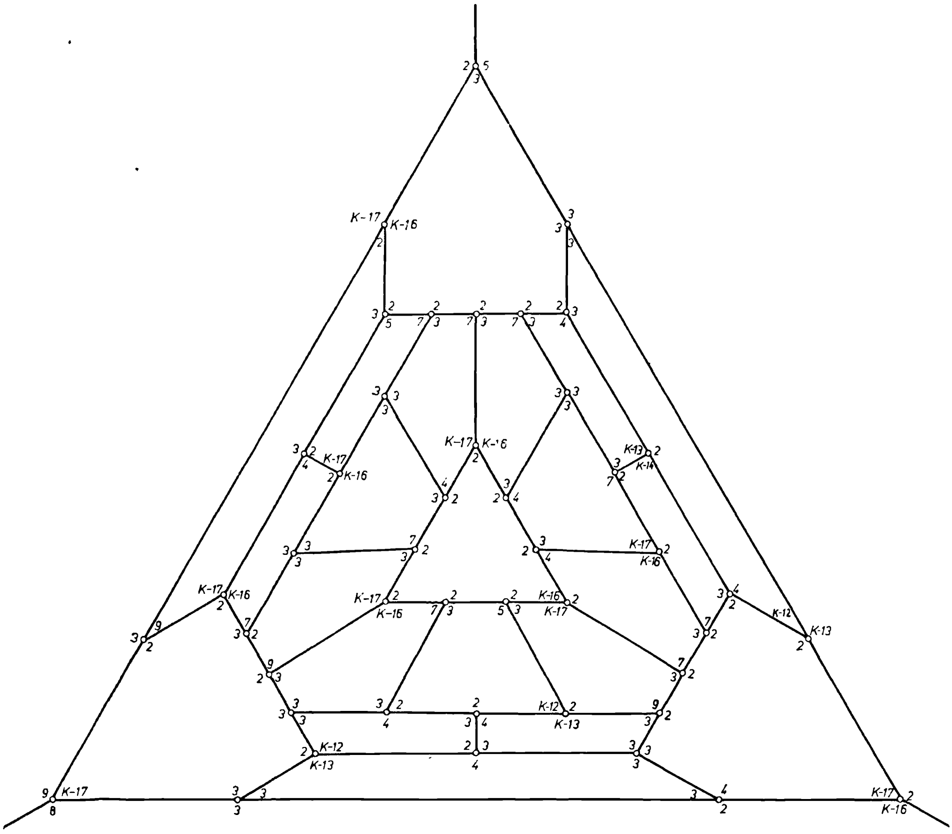

Fig. 2,

figures (i, j, m), namely $(3, 3, 3)$, $(2, 3, 4)$, $(2, K-16, K-17)$, $(2, 3, 9)$, $(2, 3, 8)$, $(2, 3, 7)$, $(2, 3, 10)$, $(8, 9, K-17)$, $(2, 3, 5)$, $(2, K-12, K-13)$, $(2, K-13, K-14)$.

Fig. 3 shows a figure $(3, 3, 3)$.

For constructing the eleven figures (i, j, m) needed let us introduce some operations taking into consideration $2 \leqq i, j, m \leqq K-2$ (We use the abbreviation: (i, j, m) $A(u, v, w)$, if we obtain a figure (u, v, w) by applying the operation A to the figure (i, j, m)):

α: $(i, j, m)\, \alpha (K-i, j+2, m+2)$, see Fig. 4 for $K = 13$.

Bold-face italics (e.g. $(\boldsymbol{i}, \boldsymbol{K} - \boldsymbol{i})$ means that the operation is performed at the exterior

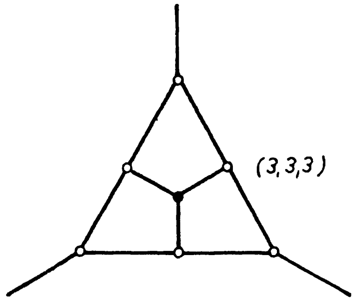

Fig. 3,

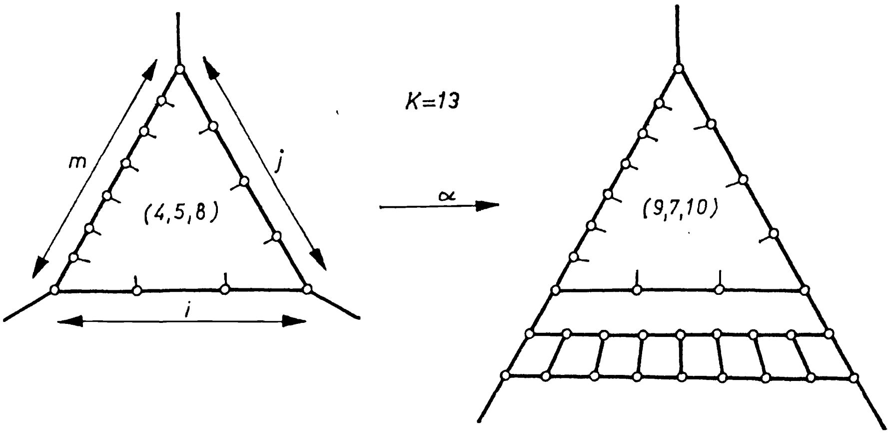

Fig. 4,

face the border of which contains i vertices.

β: $(\boldsymbol{i}, j, m)\, \beta(\boldsymbol{i}, j + 4, m + 4) := (\boldsymbol{i}, j, m)\, \alpha(\boldsymbol{K} - \boldsymbol{i}, j + 2, m + 2)$
$\alpha(i, j + 4, m + 4)$.

γ: $(i, j, m)\, \gamma(i + 8, j + 8, m + 8) := (\boldsymbol{i}, j, m)\, \beta(i, \boldsymbol{j} + \boldsymbol{4}, m + 4)$
$\beta(i + 4, j + 4, \boldsymbol{m} + \boldsymbol{8})\, \beta(i + 8, j + 8, m + 8)$.

δ: $(\boldsymbol{K} - \boldsymbol{2}, j, m)\, \delta(2, j + 1, m + 1)$, see Fig. 5 for $K = 13$.

ε: $(\boldsymbol{2}, j, m)\, \varepsilon(2, j + 1, m + 1)$, see Fig. 6.

φ_i: $(2, 3 + i, 3 + i + j)\, \varphi_i(2, 3, j + 3) :=$
$:= (\boldsymbol{2}, 3 + i, 3 + i + j)\, \varepsilon(\boldsymbol{2}, 3 + i + 1, 3 + i + j + 1)$
$\varepsilon \ldots \varepsilon(2, K - j - 2, \boldsymbol{K} - \boldsymbol{2})\, \delta$
$(3, K - j - 1, \boldsymbol{2})\, \varepsilon(4, K - j, \boldsymbol{2})$
$\varepsilon \ldots \varepsilon(j + 2, \boldsymbol{K} - \boldsymbol{2}, 2)\, \delta$
$j + 3, 2, 3)$.

ψ: $(2, 3, \boldsymbol{j} + \mathbf{11})\,\psi(2, 3, j + 3) := (2, 3, \boldsymbol{j} + \mathbf{11})\,\beta(\mathbf{6}, 7, j + 11)$
$\alpha(\boldsymbol{K} - 6, 9, \boldsymbol{j} + \mathbf{13})\,\beta(\boldsymbol{K} - \mathbf{2}, 13, j + 13)$
$\delta(\mathbf{2}, 14, j + 14)\,\varphi_{11}(2, 3, j + 3)$.

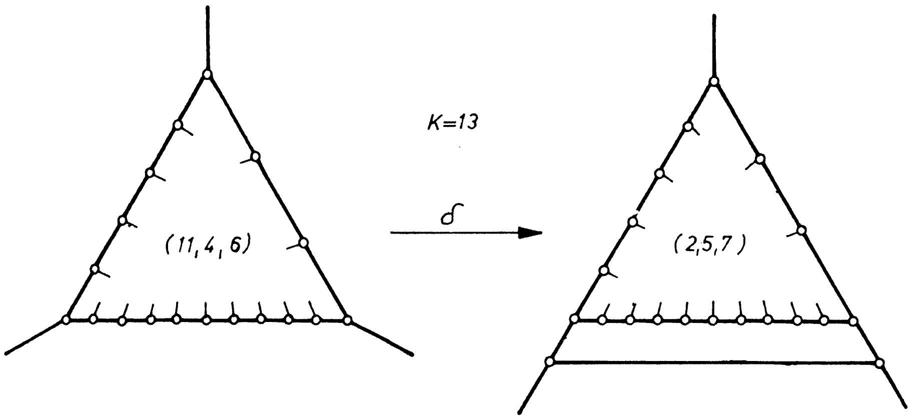

Fig. 5,

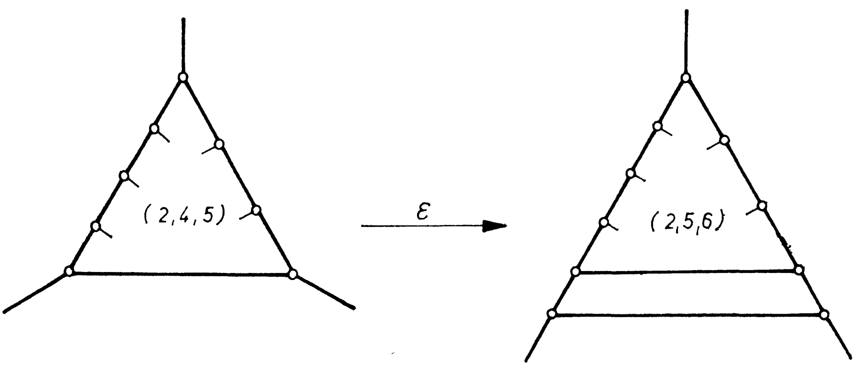

Fig. 6,

Given a figure (2, 3, 5) we get the other seven figures (i, j, m) in the following way.

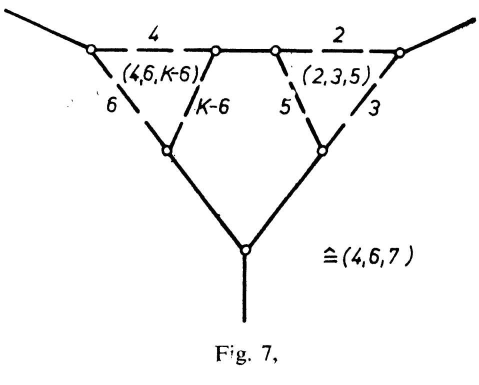

Fig. 7,

– $(\mathbf{2}, 3, 5)\ \varepsilon(2, 4, \mathbf{6})\ \alpha(4, 6, K-6)$ and according to Fig. 7, $(4, \mathbf{6}, 7)\ \alpha$
$(6, K-6, \mathbf{9})\ \beta(10, \boldsymbol{K}-\mathbf{2}, 9)\ \delta(11, \mathbf{2}, 10)\ \varepsilon^{K-13}(\boldsymbol{K}-\mathbf{2}, 2, K-3)\ \delta(2, 3, \boldsymbol{K}-\mathbf{2})$
$\delta(3, 4, 2)$.

– $(3, 4, \mathbf{2})\ \varepsilon^{K-20}(K-17, K-16, 2)$, analogously $(K-12, K-13, 2)$ and $(K-13, K-14, 2)$.
– $(2, 3, \mathbf{5})\ \beta(\mathbf{6}, 7, 5)\ \alpha(K-6, 9, \mathbf{7})\ \beta(\boldsymbol{K}-\mathbf{2}, 13, 7)\ \delta(\mathbf{2}, 14, 8)\ \varphi_5(2, 9, 3)$.
– $(2, 3, \mathbf{5})\ \beta(\mathbf{6}, 7, 5)\ \alpha(K-6, 9, 7)$ and according to Fig. 8, $(4, \mathbf{11}, 8)\ \alpha$

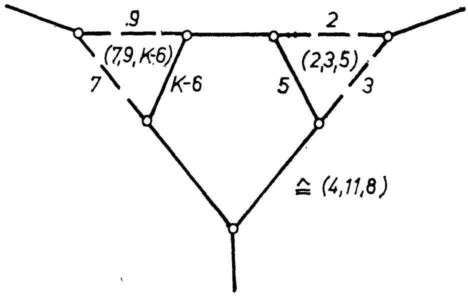

Fig. 8,

$(\mathbf{6}, K-11, 10)\ \alpha(K-6, K-9, \mathbf{12})\ \beta(\boldsymbol{K}-\mathbf{2}, K-5, 12)\ \delta(\mathbf{2}, K-4, 13)$
$\varepsilon^2(2, \boldsymbol{K}-\mathbf{2}, 15)\ \delta(3, 2, \mathbf{16})\ \psi(3, 2, 8)$.
– $(\mathbf{2}, 3, 8)\ \varepsilon^2(2, 5, \mathbf{10})\ \alpha(4, 7, K-10)$ } see Fig. 9
$(\mathbf{2}, 3, 5)\ \varepsilon^4(2, 7, 9)$ }
$(8,\mathbf{6},8)\ \alpha(\mathbf{10}, K-6, 10)\ \beta(10, \boldsymbol{K}-\mathbf{2}, 14)\ \delta(11, \mathbf{2}, 15)\ \varphi_8(3, 2, 7)$.

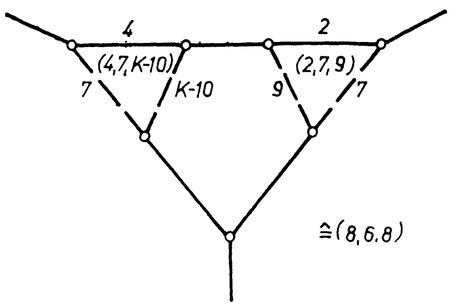

Fig. 9,

– $(\mathbf{2}, 3, 4)\ \varepsilon^2(2, 5, \mathbf{6})\ \alpha(\mathbf{4}, 7, K-6)\ \beta(4, 11, \boldsymbol{K}-\mathbf{2})\ \delta(5, 12, \mathbf{2})\ \varphi_2(3, 10, 2)$.
– $(2, \mathbf{3}, 9)\ \beta(\mathbf{6}, 3, 13)\ \beta(6, 7, \mathbf{17})\ \alpha(8, 9, K-17)$.
Now, let us construct a $(2, 3, 5)$. We have to distinguish four cases.

Case 1, $K = 8N + 7,\ N \geqq 2$.
$(3, 3, 3)\ \gamma^{N-1}(\mathbf{8}\boldsymbol{N}-\mathbf{5}, 8N-5, 8N-5)\ \beta(8N-5, \mathbf{8}\boldsymbol{N}-\mathbf{1}, 8N-1)$
$\alpha(8N-3, \mathbf{8}, 8N+1)\ \beta(8N+1, 8, 8\boldsymbol{N}+\mathbf{5})\ \delta(8N+2, 9, \mathbf{2})\ \varepsilon^3(\mathbf{8}\boldsymbol{N}+5, 12, 2)$
$\delta(2, \mathbf{13}, 3)\ \psi(2, 5, 3)$.

Case 2, $K = 8N + 5$, $N \geqq 2$.
$(3, 3, 3)\ \gamma^{N-1}(\mathbf{8N - 5}, 8N - 5, 8N - 5)\ \beta(8N - 5, \mathbf{8N - 1}, 8N - 1)$
$\beta(8N - 1, 8N - 1, \mathbf{8N + 3})\ \delta(8N, 8N, \mathbf{2})\ \varepsilon(\mathbf{8N + 1}, 8N + 1, 2)\ \alpha(4, \mathbf{8N + 3}, 4)$
$\delta(5,2, \mathbf{5})\ \alpha(7, 4, K - 5)$, on the one hand according to Fig. 10, $(8, \mathbf{11}, K - 4)$

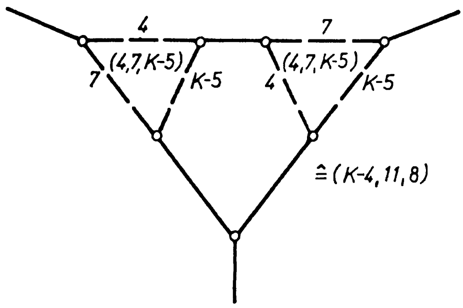

Fig. 10,

$\alpha(10, K - 11, K - 2)\ \delta(\mathbf{11}, K - 10, 2)\ \beta^2(11, \mathbf{K - 2}, 10)\ \delta(12, \mathbf{2}, 11)\ \varphi_8(4, 2, 3)$;
on the other hand, $(7, \mathbf{4}, K - 5)\ \alpha(9, K - 4, K - 3)$ and according to Fig. 11,

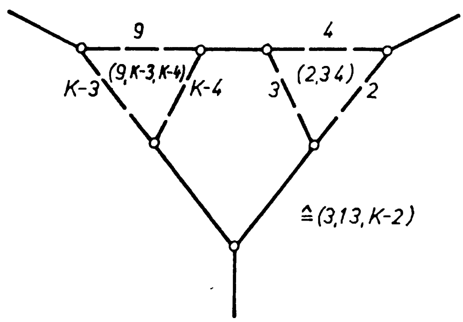

Fig. 11,

$(3, 13, K - \mathbf{2})\ \delta(4, 14, \mathbf{2})\ \varphi_1(3, \mathbf{13}, 2)\ \psi(3, 5, 2)$.

Case 3, $K = 8N + 3$, $N \geqq 3$.
$(3, 3, 3)\ \gamma^{N-1}(\mathbf{8N - 5}, 8N - 5, 8N - 5)\ \alpha(8, \mathbf{8N - 3}, 8N - 3)$
$\beta(12, 8N - 3, \mathbf{8N + 1})\ \delta(13, 8N - 2, \mathbf{2})\ \varepsilon^3(16, \mathbf{8N} + 1, 2)\ \delta(\mathbf{17}, 2, 3)$
$\psi(\mathbf{9}, 2, 3)\ \alpha'(K - 9, 4, \mathbf{5})\ \alpha(K - 7, \mathbf{6}, K - 5)\ \alpha'(K - 5, K - 6, \mathbf{K - 3})$
$\alpha(\mathbf{K - 3}, K - 4, 3)\ \alpha(3, \mathbf{K - 2}, 5)\ \delta(4, 2, 6)\ \varphi_1(3, 2, 5)$.

Case 4, $K = 8N + 1$, $N \geqq 3$.
$(3, 3, 3)\ \gamma^{N-1}(\mathbf{8N - 5}, 8N - 5, 8N - 5)\ \alpha(6, \mathbf{8N - 3}, 8N - 3)\ \alpha(8, 4, \mathbf{8N - 1})$
$\delta(*)(9, 5, \mathbf{2})\ \varepsilon(10, \mathbf{6}, 2)\ \alpha(12, K - 6, 4)$ together with $(*)$ according to Fig. 12 yields
$(5, \mathbf{10}, 14)\ \alpha(7, K - 10, \mathbf{16})\ \beta^2(15, \mathbf{K - 2}, 16)\ \delta(16, \mathbf{2}, 17)\ \varphi_{13}(3, 2, 4)\ (**)$.

From $(*)$ we get $(9, 5, \mathbf{2})\ \varphi_2(7, \mathbf{3}, 2)\ \alpha(9, K-3, 4)$ together with $(**)$ according to Fig. 13 yields a $(5, 7, \mathbf{10})\ \alpha(\mathbf{7}, 9, K-10)\ \beta(7, \mathbf{13}, K-6)\ \beta(11, 13, \boldsymbol{K}-2)$ $\delta(12, 14, \mathbf{2})\ \varphi_9(3, 5, 2)$.

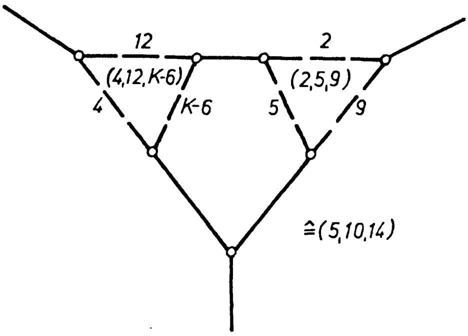

Fig. 12,

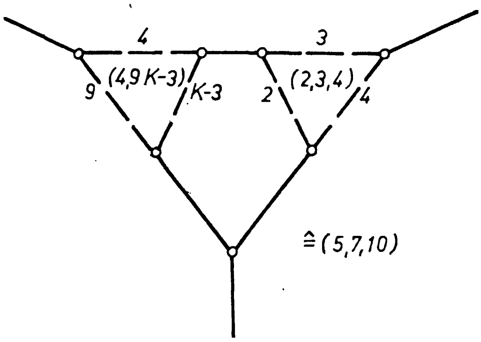

Fig. 13

In all cases we have found a figure $(2, 3, 5)$, that means for all odd $K \geqq 21$ we have found the necessary eleven types of (i, j, m) listed in Step 2.

Constructing the figures (i, j, m) needed we have always started with a figure $(3, 3, 3)$, that means, each figure (i, j, m) contains at least one vertex incident with three 4-gons. Hence the figure Z contains exactly 45 black vertices and an arbitrary path trough Z connecting any two halfedges avoids at least one black vertex. Obviously Z is a figure $(K-3, K-3, K-3)$.

By induction, if G_l contains only 4-gons and K-gons (which is really true for $l = 0$), then after replacing all black vertices of G_l by a figure Z, the resulting graph G_{l+1} has only 4-gons and K-gons as well.

The estimate $\log 44/\log 45$ for the shortness exponent may be established in an analogous way as in [1]. Obviously, the number of vertices increases by the factor 45, but the number of vertices in the longest circuit only by the factor 44 if we pass from G_l to G_{l+1}.

The following problems remain unsolved:

1. What holds in the open cases of Theorem 2, $K \equiv 0 \pmod 5$ for $\Gamma_3(5, K)$ and $K \equiv 0 \pmod 3$ for $\Gamma_4(3, K)$ and $\Gamma_5(3, K)$?

2. Conjecture: $\sigma(\Gamma_3(3, K) = 1$ for all $K \leqq 11$ (obviously, $\Gamma_3(3, K) = \emptyset$ for $K \geqq 12$.

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Souhrn

NOVÉ VÝSLEDKY O „SHORTNESS EXPONENTU" POLYEDRICKÝCH GRAFŮ

JOCHEN HARANT, HANSJOACHIM WALTER

Exponent $\sigma(\Gamma)$ („shortness exponent") soustavy Γ grafů G je definován jako $\sigma(\Gamma) = \liminf_{G \in \Gamma} (\log h(G))/(\log n(K))$, kde $n(G)$ resp. $h(G)$ znamená počet vrcholů G resp. maximální počet vrcholů G, patřících cyklu. V práci se studuje exponent $G(\Gamma)$ soustav regulárních polyedrických grafů, které mají nejmenší počet typů stěn, a pro něž platí $\sigma(\Gamma) < 1$.

Резюме

НОВЫЕ РЕЗУЛЬТАТЫ О „ПОКАЗАТЕЛЕ КОРОТКОСТИ" ПОЛИЭДРИЧЕСКИХ ГРАФОВ

ЙОХЕН ГАРАНТ, ХАНСЙОАХИМ ВАЛТЕР

Показатель $\sigma(\Gamma)$ („показатель короткости") системы Γ графов G определяется формулой $\sigma(\Gamma) = \liminf_{G \in \Gamma} (\log h(G)/(\log n(G))$, где $n(G)$ и $h(G)$ обозначают соответсвенно число вершин графа G и максимальное число вершин графа G, принадлежащих циклу. В работе изучается показатель $G(\Gamma)$ систем регулярных полиэдрических графов, имеющих наименьшее число типов граней и удовлетворяющих $\sigma(\Gamma) < 1$.

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