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ZPRÁVY

RECENT RESULTS OF NOVOSIBIRSK MATHEMATICIANS IN GRAPH THEORY

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Summary. The paper gives an overview of recent results obtained in graph theory by a group of Novosibirsk mathematicians (Aksionov, Borodin, Kostochka, Mel'nikov, Ponomarev, Taškinov). The following themes are dealt with: colouring, interval representations, topological imbeddings, Hadwiger number, Berge's conjecture on regular subgraphs of regular graphs, one problem on spanning trees.

1. INTERVALS AND COLOURINGS

Following [1], [2] let us consider graphs $G = (V, E)$ without loops and multiple edges. Assign to each vertex $v \in V(G)$ a nonnegative weight $h(v)$. The weight of the subset $S \subseteq V(G)$ will be defined naturally as $h(S) = \sum_{v \in S} h(v)$. Let us assume without loss of generality that the weights $h(v)$ are integers. The pair (G, h) will be called a *weighted graph* (WG). By an *interval representation* (IR) we shall mean such a mapping J of the set of the vertices of the WG into a set of intervals in the real axis that it assigns to each vertex $v \in V(G)$ an interval $J(v)$ of length $|J(v)| = h(v)$. We call an IR *chromatic* if the intervals assigned to adjacent vertices are disjoint, i.e. $(v, u) \in E(G) \Rightarrow J(v) \cap J(u) = \emptyset$. The *length* of an IR (G, h, J) is the number $L(G, h, J) = |\bigcup_{v \in V(G)} J(v)|$. If there are not conditions for the type of the IR then the least length of a given WG is obviously $\max_{v \in V(G)} h(v)$. But things are quite different for chromatic IR. Call the chromatic length of a WC (G, h) the number $\chi(G, h) = \min_J L(G, h, J)$, where the minimum is taken over all chromatic IR.

The problem to construct a chromatic IR may have various applications [8], e.g. connected to scheduling problems.

The clique length of a WG (G, h) shall be the number

$$\omega(G, h) = \max_K h(K),$$

where K ranges over all subsets of vertices that induce a clique in G. The following inequalities are obvious:

$$\omega(G, h) \leqq \chi(G, h) \leqq h(V(G)).$$

Proposition 1.1. [1]. *If* $h(v) = c$ *is constant for all* $v \in V(G)$ *then* $\chi(G, h) = c\,\chi(G)$, *where* $\chi(G)$ *is the chromatic number of the graph* G.

Proposition 1.2. [8]. $\chi(G, h) = \min_{G' \in A(G)} (\max_{P \subseteq G'} h(V(P)))$, *where* $A(G)$ *is the set of all acyclic orientations of the edges of* G, *and* $P \subseteq G'$ *is a directed path in the digraph* G'.

Proposition 1.3. [2]. $\chi(G, h) \leqq \Delta(G, h) \overset{\text{def}}{=} \max_{v \in V(G)} h(\bar{N}(v))$, *where by the neighborhood* $\bar{N}(v)$ *of the vertex* v *we mean the set of all vertices adjacent to* v *together with* v *itself*:

$$\bar{N}(v) = \{v\} \cup \{u/(v, u) \in E(G)\} .$$

By far not all known estimates for the chromatic number admit generalization to chromatic length. The following bound is well known: $\chi(G) \leqq \max_{G' \subseteq G} (\min_{v \in V(G')} [d(v) + 1])$. Define analogously to the right-hand side: $w(G, h) = \max_{G' \subseteq G} (\min_{v \in V(G')} h(\bar{N}(v)))$.

Proposition 1.4. [2] *For arbitrary* $k \geqq 0$ *there is* (G, h) *such that* $\chi(G, h) > w(G, h) + k$.

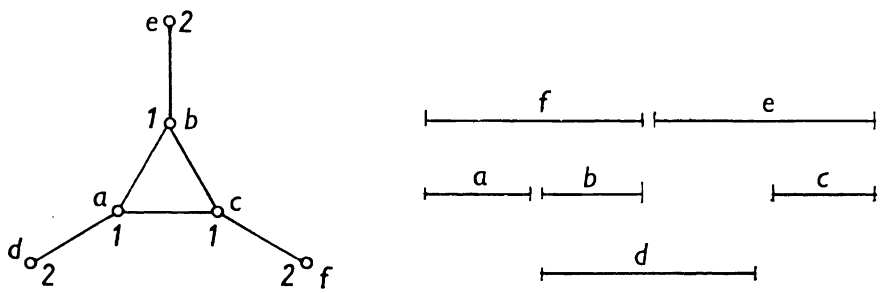


Fig. 1

Proposition 1.5. [2]. $\chi(C_{2k+1,h}) = \max\{ \max_{e \in E(C_{2k+1})} h(e), \min_{v \in V(C_{2k+1})} h(\bar{N}(v))\}$, *where* $h(e) = h(u) + h(v)$ *and* $e = (u, v)$. *If* K *is complete then* $\chi(K, h) = \Delta(K, h)$.

In view of this fact and of proposition 1.5, Aksinov assumes the following generalization of Brooks's theorem [6] to hold:

Conjecture 1.6. [2]. *Assume* G *to be connected and* $\chi(G, h) = \Delta(G, h)$, *then either* G *is complete or* G *is an odd cycle with* $h(v) = const$ *for all* $v \in V(G)$.

2. TOPOLOGICAL IMBEDDINGS AND COLOURINGS

Here I shall omit my old results [2] and concentrate on several new results of Borodin [3], [4].

Call a graph 1-*planar* if there exists its representation in the plane such that each edge intersects at most one other edge of the graph.

In [3], the following theorem is proved, verifying Ringel's hypothesis [15]:

Theorem 2.1. *Suppose the graph G is* 1-*planar, then for its chromatic number* $\chi(G) \leqq 6$.

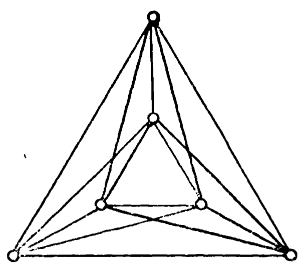


Fig. 2

The graph on Fig. 2 is K_6 and is obviously 1-planar, which shows that the theorem cannot be improved. The generalization of 1-planarity to 1-embedding into an arbitrary closed two dimensional surface F^N with Euler's characteristics N is straightforward, as well as the definition of the upper bound of the chromatic number of graphs admitting such a 1-embedding. Ringel [16] obtained such an upper bound of the chromatic number $\chi_1(N) \leqq [(9 + \sqrt{(81 - 32N)})/2]$ for $N \leqq 2$. He also showed it to be exact for Klein's bottle and for the torus ($N = 0$), for $N = 2$ its exactness follows from Theorem 2.1. Schumacher and Wegner showed that for the projective plane ($N = 1$) the bound is not sharp and $\chi_1(1) = 7$. However, further extension of these results meets substantial difficulties arising in connection with systematization of 1-embeddings of complete graphs into F^N. Unfortunately, the Ringel-Youngs theory of flow graphs and imbeddings connected with them admits no simple transfer to 1-embeddings.

Combined colourings appear rather often (see e.g. [19] the total chromatic number and the author's hypotheses [12]). In fact, in [3] the problem of vertex colouring of 1-planar graphs was reduced to the combined colouring of planar graphs having only 3- and 4-faces in such a way that two vertices adjacent to the same face are assigned different colours. The first to deal with combined colouring appears to have been Ringel [15] who conjectured the following result due to Borodin [3] which follows from Theorem 2.1.

Theorem 2.2. *For any planar graph there is a combined colouring of vertices and edges with* 6 *colours.*

Theorem 2.3. [3], [4] (without proof)

$$[3k/2]^+ \leqq \chi(k) \leqq 2k - 1\,,$$

where $\chi(k)$ *is the maximal chromatic number of planar graphs where all faces of degree* $d^*(F) \leqq k$ *have their vertices coloured in different colours.* ($[\cdot]^+$ denotes here the post office function.)

The pseudosphere (or pseudoplane) F_k^2 arises from the sphere by pairwise identifying $2k$ different points.

There are three different possible ways of imbedding a graph into a pseudosurface (in particular, into the pseudosphere):

1) through the "double" points of the pseudosurface the edges may not pass,
2) in the "double" points there may not lie vertices,
3) no conditions.

Theorem 2.4.
Case 1: [7], [5] $\chi^{(1)}(F_k^2) = \min\{k + 4,\ [(7 + \sqrt{(1 + 24k)})/2],\ 12\},\ k > 0.$
Case 2: [9] $\chi^{(2)}(F_k^2) = [(7 + \sqrt{(1 + 8k)})/2]$ *for* $k > 0$.
Case 3: [5] $\chi^{(3)}(F_k^2) = \min\{k + 4,\ [(7 + \sqrt{(1 + 24k)})/2],$
$[(11 + \sqrt{(73 + 8k)})/2]\}$ *for* $k > 0$.

For 1-embeddings into the pseudosphere Borodin proved (only for case 2):

Theorem 2.5. [4] $\chi_1^{(2)}(F_k^2) = \begin{cases} [(9 + \sqrt{(17 + 16k)})/2] & \text{for } 0 \leqq k \neq 4, \\ 8 \ \textit{for}\ k = 4. \end{cases}$

3. THE HADWIGER NUMBER $\eta(G)$

A. V. Kostochka disproved Zelinka's conjecture [20] that the inequality

$$\eta(G) + \eta(\bar{G}) \leqq n(G) + 1$$

is a sharp bound.

Theorem 3.1. [10]. *For an arbitrary simple graph of n vertices* ($n \geqq 5$) *the following sharp bounds hold*:

$$\eta(G) + \eta(\bar{G}) \leqq \left[\frac{6n}{5}\right], \quad \eta(G)\,.\,\eta(\bar{G}) \leqq \left[\frac{1}{4}\left(\left[\frac{6}{5}\,n\right]\right)^2\right].$$

Kostochka's paper [11] is devoted to classification of the behaviour of the minimal

Hadwiger number in the class $\mathscr{D}_k$ of graphs the average degree of which is not less than k. Denote

$$\eta(k) = \min_{G \in \mathscr{D}_k} \eta(G)\,, \quad w(k) = \min\{\eta(G)/\chi(G) \geqq k\}\,,$$

$$v(k) = \min\{\eta(G)/G \text{ is } k\text{-connected}\}\,,$$

$$\mathscr{E}_k = \left\{G/|V(G)| \geqq k,\ |E(G)| > k|V(G)| - \binom{k+1}{2}\right\}, \quad \eta_1(K) = \min_{G \in \mathscr{E}_k} \eta(G)\,.$$

Mader, Miller, Zelinka and Zykov looked into the behaviour of the function $\eta(k)$. The best results that could be achieved were the bounds

$$\frac{k}{8\log_2 k} < \eta(k) \leqq \frac{4k}{\sqrt{\log_2 k}}\,.$$

Theorem 3.2. [11] *For* $k \geqq 2$, $\eta(k) \geqq \dfrac{k}{270\sqrt{\log_2 k}}$.

Corollary 3.3. *For* $k \geqq 2$, $w(k) \geqq \dfrac{k}{540\sqrt{\log_2 k}}$.

Corollary 3.4. *Hadwiger's conjecture holds for almost all graphs (P. Erdös, B. Bollobás, P. Catlin).*

Corollary 3.5. *For k sufficiently large, Hadwiger's conjecture holds for almost all graphs with n vertices and kn edges.*

Corollary 3.6. $\displaystyle\min_{|V(G)|=n} (\eta(G) + \eta(\bar{G})) = O\left(\frac{n}{\sqrt{\log n}}\right)$.

Hence, we know the order of the lower bound for the sum $\eta(G) + \eta(\bar{G})$, but unfortunately an exact lower bound is not known.

Corollary 3.7. $v(k) = O\left(\dfrac{k}{\sqrt{\log k}}\right)$.

Theorem 3.8. [11]. $\eta_1(k) \geqq \dfrac{1}{27} \cdot \dfrac{k}{\log_2 k}$ *for* $k \geqq 2$.

4. REGULAR SUBGRAPHS OF REGULAR GRAPHS

Berge's conjecture states that any 4-regular graph has a 3-regular subgraph.

Theorem 4.1. [17], [18]. *Every 4-regular graph has a 3-regular subgraph.*

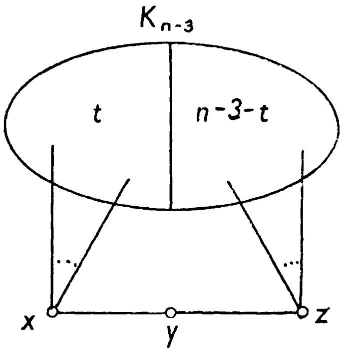
V. A. Taškinov studied in sufficient generality the problem under which conditions an r-regular graph has a ϱ-regular subgraph. His results are contained in a dissertation which is to be presented in the near future. Partial problems are answered in the following two theorems.

Theorem 4.2. [17]. *For any $r \geqq 3$ any r-regular graph has a 3-regular subgraph.*

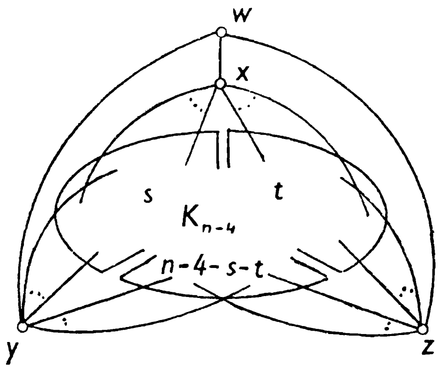
Theorem 4.3. [17] + [Dissertation]. *For any $r \geqq 5$ there is an r-regular graph which has no $(r - 1)$-regular subgraph.*

5. SPANNING TREES WITH LIMITED NUMBER OF END EDGES

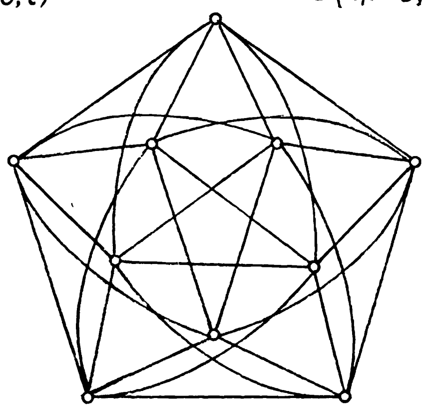
Vizing's problem [19] is: To find $\max |E(G)|/n(G) = n$ and any spanning tree of the graph G has no more than k end edges (i.e. edges adjacent to an end vertex).



$A(n, n-3, t)$



$B(n, n-3, s, t)$



$C(10, 7)$

Fig. 3

In the case of G connected, denote that maximum by $m(n, k)$, and in the case of an arbitrary graph G by $M(n, k)$.

Theorem 5.1. [13]. $m(n, k) = n + (k + 1)(k - 2)/2$ *for* $k \neq n - 2$, $2 \leqq k \leqq \leqq n - 1$,

$$m(n, k) = [n(n - 2)/2] \quad for \quad k = n - 2, \quad n \geqq 4,$$

$$m(n, k) = 1 \quad for \quad k = n = 2;$$

$$M(n, k) = \begin{cases} \max\left(n + \frac{1}{2}(k + 1)(k - 2), \left[\frac{n(n - 2)}{2}\right]\right), & 2 \leqq k \leqq n - 1, \\ n/2 \quad for \quad k = n. \end{cases}$$

The proof of Theorem 5.1 is based on a result formulated by B. Zelinka [21] but as the proof contained a mistake we had to do it new [14].

Theorem 5.2. [14]. *The maximal number of edges of a connected graph of* n *vertices any spanning tree of which has not more than* $n - 3$ *end edges, is equal to* $(n^2 - 5n + 10)/2$ *for* $n \geqq 5$, *and all extremal graphs are given in Fig.* 3.

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Souhrn

NOVÉ VÝSLEDKY NOVOSIBIRSKÝCH MATEMATIKŮ V TEORII GRAFŮ

L. S. Mělnikov

Práce podává přehled nových výsledků skupiny novosibirských matematiků (Aksjonov, Borodin, Kostočka, Mělnikov, Ponomarev, Taškinov) v teorii grafů. Jsou pojednána tato témata: barvení, intervalové reprezentace, topologická vnoření, Hadwigerovo číslo, Bergeova hypotéza o regulárních podgrafech regulárních grafů a jeden problém o kostrách.

Резюме

НОВЫЕ РЕЗУЛЬТАТЫ НОВОСИБИРСКИХ МАТЕМАТИКОВ В ТЕОРИИ ГРАФОВ

L. S. Mělnikov

В работе дается обзор новых результатов группы новосибирских математиков (Аксенов, Бородин, Косточка, Мельников, Пономарев, Ташкинов) в теории графов. Рассмотрены следующие темы: раскраски, интервальные представления, топологические вложения, число Хадвигера, гипотеза Бержа о регулярных подграфах регулярных графов и одна проблема связанная с каркасами графа.

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