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GREATEST COMMON SUBGRAPHS OF GRAPHS

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Summary. A graph G without isolated vertices is a greatest common subgraph of a set $\mathcal{G}$ of graphs, all having the same size, if G is a graph of maximum size that is isomorphic to some subgraph of every graph in $\mathcal{G}$. A number of results concerning greatest common subgraphs are presented. In particular, it is shown that for integers $m \geqq 3$ and $n \geqq 1$, there exists a set of m graphs of equal size having exactly n greatest common subgraphs. Furthermore, it is shown that for any graph G without isolated vertices, there exist graphs G_1 and G_2 of equal size having G as their unique common subgraph. A further investigation of this result gives rise to a parameter, called the greatest common subgraph index of a graph.

1. INTRODUCTION

In [2] the authors introduced the concept of a greatest common subgraph of two graphs G_1 and G_2 of the same size (having the same number of edges) for the purpose of studying a distance between G_1 and G_2. This concept can be generalized as follows:

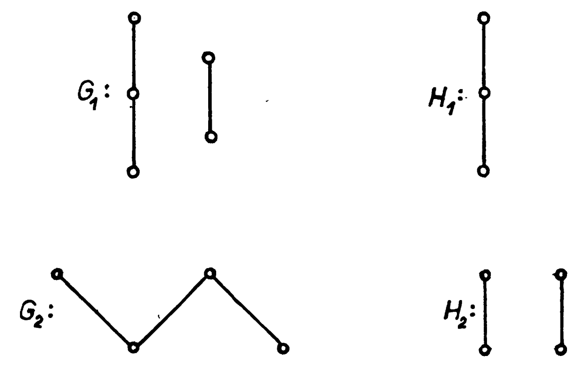

Figure 1

Given a set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $n \geqq 2$, of graphs, all of the same size, a *greatest common subgraph* of $\mathcal{G}$ is a graph of maximum size and without isolated vertices that is isomorphic to some subgraph of every graph in $\mathcal{G}$. The set of all greatest

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common subgraphs of $\mathscr{G}$ is denoted by

$$\text{gcs}\,\mathscr{G} = \text{gcs}(G_1, G_2, \ldots, G_n)\,.$$

If $\mathscr{G} = \{G_1, G_2\}$, where G_1 and G_2 are shown in Figure 1, then gcs $\mathscr{G} = \{H_1, H_2\}$, where H_1 and H_2 are also shown in Figure 1. (All definitions and terminology not presented here may be found in [1].)

2. GREATEST COMMON SUBGRAPHS OF GRAPHS

We first show that the number of greatest common subgraphs of the two graphs can be arbitrarily large.

Proposition 1. *For every positive integer n, there exist graphs G_n and H_n such that* $|\text{gcs}(G_n, H_n)| = n$.

Proof. First we note that if we define $G_1 = P_3$ and $H_1 = 2K_2$, then gcs $(G_1, H_1) = \{K_2\}$. For $n \geqq 2$, define $G_n' = S(K(1, n))$, the subdivision of the star $K(1, n)$, i.e., each edge uv of $K(1, n)$ is replaced by a new vertex w and two edges uw and wv. The graph G_n is then obtained from G_n' by identifying two endvertices of G_n'. Define $H_n = K(1, n) \cup nK_2$. Observe that each of G_n and H_n has size $2n$. The graphs G_4 and H_4 are shown in Figure 2.

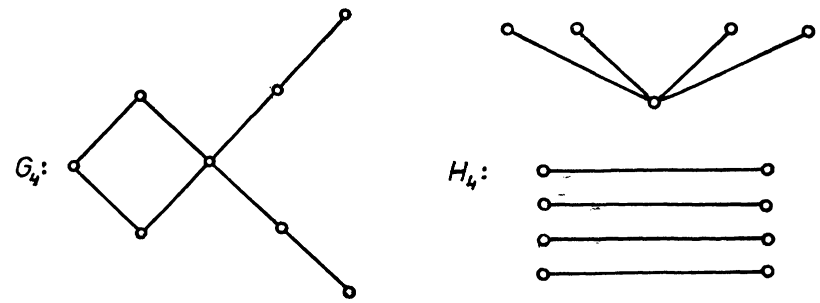

Figure 2

Observe that every subgraph (without isolated vertices) of H_n is of the type $K(1, r)$, sK_2 or $K(1, r) \cup sK_2$, where $1 \leqq r \leqq n$ and $1 \leqq s \leqq n$. Since each of G_n and H_n contains $K(1, n)$ as a subgraph, every greatest common subgraph of G_n and H_n has size at least n. Further, the edge independence number $\beta_1(G_n)$ of G_n is n; while $\beta_1(H_n) = n + 1$ so that nK_2 is also a common subgraph of G_n and H_n. Let $K(1, r) \cup sK_2$ be a common subgraph of G_n and H_n $(r, s \geqq 1)$ of maximum size. If $r = 1$, then $s = n - 1$. For any subgraph $K(1, r)$, $r \geqq 2$, of G_n, there are at most $n - r$ independent edges of G_n that neither are adjacent to nor are themselves the edges of $K(1, r)$. Hence $r + s \leqq n$, which implies that every greatest common subgraph of G_n and H_n has size n. It now follows that

$$\text{gcs}(G_n, H_n) = \{K(1, n)\} \cup \{nK_2\} \cup \{K(1, r) \cup (n - r) K_2 \quad r = 2, 3, \ldots, n - 1\} ;$$

consequently, $|\text{gcs}(G_n, H_n)| = n$. □

A *branch* of a graph G at a vertex v is a maximal connected subgraph of G containing v as a non-cut-vertex. Thus, if v is not a cut-vertex, then there is only one branch at v, namely the component of G containing v; otherwise, the number of branches at v equals the number of blocks to which v belongs.

We are now prepared to present a much stronger result than Proposition 1 in the case where $n = 1$.

Proposition 2. *For every graph G without isolated vertices, there exist graphs G_1 and G_2 of equal size such that* $\text{gcs}(G_1, G_2) = \{G\}$.

Proof. Let G be a graph without isolated vertices having size $q(\geqq 1)$, and let v be a vertex of maximum degree in G. We consider two cases.

Case 1. *Suppose that no branch of G at v is isomorphic to P_3.* In this case we construct a graph G_1 by adding a new vertex u to G and joining it to v. Define $G_2 = G \cup K_2$, where $E(G_2) - E(G) = \{e\}$. Clearly $G_1 \not\cong G_2$. Each of G_1 and G_2 has size $q + 1$, and since G has size q and is a common subgraph of G_1 and G_2, it follows that $G \in \text{gcs}(G_1, G_2)$.

We now show that $\text{gcs}(G_1, G_2) = \{G\}$. Assume, to the contrary, that $G' \in \text{gcs}(G_1, G_2)$ and $G' \not\cong G$. Then G' has size q. Since G' is a subgraph of G_2, the graph G' is obtained by deleting an edge f from G_2 (and any resulting isolated vertices), where $f \neq e$. The edge f cannot belong to a component isomorphic to K_2; for otherwise $G \simeq G'$. Hence f must belong to a component with two or more edges, which implies that G' has more components isomorphic to K_2 than does G. Since G' is a subgraph of G_1, the graph G' is obtained by deleting an edge f' from G_1 (and any resulting isolated vertices), where $f' \neq uv$. Since $\Delta(G') \leqq \Delta(G_2) < \Delta(G_1)$, it follows that f' is incident with v. However, G contains no branches at v isomorphic to P_3; therefore, G' and G have the same number of components isomorphic to K_2, and this produces a contradiction.

Case 2. *Suppose that G contains branches at v that are isomorphic to P_3.* Let B be a branch at v isomorphic to P_3, where u is the vertex of B adjacent to v and w is the remaining vertex of B. Define $G_1 = G + vw$ and let $G_2 = G \cup K_2$, where $E(G_2) - E(G) = \{e\}$. Then $G_1 \not\cong G_2$, and each of G_1 and G_2 has size $q + 1$. Since G is a common subgraph of G_1 and G_2, we conclude that $G \in \text{gcs}(G_1, G_2)$.

Next we show that $\text{gcs}(G_1, G_2) = \{G\}$. Assume, to the contrary, that $G' \in \text{gcs}(G_1, G_2)$, where $G' \not\cong G$. Then G' has size q. Suppose that G has k components isomorphic to K_2 and t subgraphs isomorphic to K_3. Since G' is a subgraph of G_2, the graph G' is obtained by deleting an edge f from G_2 (and any resulting isolated vertices), where $f \neq e$. Since f cannot belong to a component isomorphic to K_2, it implies that G' has at least $k + 1$ components isomorphic to K_2. Further, since deleting an edge from a graph does not increase the number of subgraphs isomorphic to K_3, it follows that G' has at most t subgraphs isomorphic to K_3. Now, since G' is a subgraph of G_1, the graph G' is obtained by deleting an edge f' from G_1 (and any isolated vertices), where $f' \neq vw$. Since $\Delta(G') \leqq \Delta(G_2) < \Delta(G_1)$, we see that f' must be incident with v. Moreover, since G_1 has k components isomorphic to K_2 and G' has at least $k + 1$ components isomorphic to K_2, it follows that f' must belong to a branch isomorphic to P_3. However, this implies that the number of subgraphs of G' isomorphic to K_3 must equal that in G_1, which is $t + 1$. This produces the desired contradiction.

We now show that the above result has no analogue where two graphs are prescribed.

Proposition 3. *Let $H_1 \simeq K(1, 6)$ and $H_2 \simeq K_4$. Then for every two graphs G_1 and G_2 of equal size,* $\operatorname{gcs}(G_1, G_2) \neq \{H_1, H_2\}$.

Proof. Suppose, to the contrary, that there exist graphs G_1 and G_2 of equal size such that $\operatorname{gcs}(G_1, G_2) = \{H_1, H_2\}$. Observe that not both G_1 and G_2 have a component isomorphic to K_4; for otherwise, each has a component containing a subgraph isomorphic to $K(1, 6)$, which implies that $K_4 \cup K(1, 6)$ is a common subgraph of G_1 and G_2. However, since $K_4 \cup K(1, 6)$ has size 12, $H_i \notin \operatorname{gcs}(G_1, G_2)$ for $i = 1, 2$, which produces a contradiction. On the other hand, if neither G_1 nor G_2 has a component isomorphic to K_4, then both must contain a subgraph isomorphic to the graph G of Figure 3. Since G has size 7, however, we again have a contradiction.

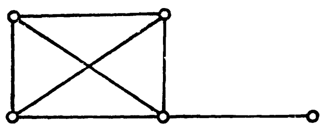

Figure 3

Therefore, we may now assume that exactly one of G_1 and G_2, say G_1, has a component isomorphic to K_4. In G_1, then, there is another component containing a subgraph isomorphic to $K(1, 6)$. In G_2, let F be a subgraph isomorphic to K_4, and let v be a vertex of G_2 having degree at least 6. If $v \in V(F)$, then each of G_1 and G_2 has a subgraph isomorphic to $K_3 \cup K(1, 3)$, which has size 6, so that $\operatorname{gcs}(G_1, G_2) \neq \{H_1, H_2\}$. If $v \notin V(F)$, then there are at least two vertices in $V(G_2) - V(F)$ that are adjacent to v so that G_1 and G_2 have a subgraph isomorphic to $K_4 \cup P_3$, which has size 8, and $H_i \notin \operatorname{gcs}(G_1, G_2)$ for $i = 1, 2$. □

We present yet another extension of Proposition 1.

Proposition 4. *For every pair m, n of integers with $m \geqq 2$ and $n \geqq 1$, there exist pairwise nonisomorphic graphs $G_1, G_2, \ldots, G_m$ of equal size such that*

$$|\mathrm{gcs}\,(G_1, G_2, \ldots, G_m)| = n\,.$$

Proof. The result is true for $m = 2$ by Proposition 1. Otherwise, we proceed by cases.

Case 1. *Assume that* $n = 1$. Define

$$G_i = K(1, m + 2 - i) \cup iK_2$$

for $i = 1, 2, \ldots, m$. Then G_i has maximum degree $\Delta(G_i) = m + 2 - i$ so that $\Delta(G) \leqq 2$ whenever $G \in \mathrm{gcs}\,\mathscr{G}$, where

$$\mathscr{G} = \{G_1, G_2, \ldots, G_m\}\,.$$

Moreover, the edge independence number of G_i is $\beta_1(G_i) = i + 1$ for $i = 1, 2, \ldots, m$. Therefore, $\beta_1(G) \leqq 2$ for $G \in \mathrm{gcs}\,\mathscr{G}$, and so $G = K(1, 2) \cup K_2$ is the unique member of gcs $\mathscr{G}$.

Case 2. *Assume that* $n = 2$. At this point, it is convenient to introduce a class of graphs. For nonnegative integers i and j, not both zero, we denote by $S_i(1, i + j)$ that graph obtained by subdividing i edges in the graph $K(1, i + j)$.

For $i = 1, 2, \ldots, m$, define

$$G_i = S_1(1, m + 2 - i) \cup iK_2\,,$$

and let $\mathscr{G} = \{G_i\}$. If $G \in \mathrm{gcs}\,\mathscr{G}$, then $\Delta(G) \leqq 2$ and $\beta_1(G) \leqq 3$. Since $P_4 \cup K_2 \subset G_i$ for all i, the size $q(G)$ of G satisfies $q(G) \geqq 4$. Now $\Delta(G) = 2$; for otherwise $G = tK_2$ for some $t \geqq 4$, which contradicts the fact that $\beta_1(G) \leqq 3$. Since the length of a longest path in each G_i is 3, either $G = P_4 \cup K_2$ or $G = P_3 \cup 2K_2$ so that

$$|\mathrm{gcs}\,\mathscr{G}| = 2\,.$$

Case 3. *Assume that* $3 \leqq n \leqq m - 1$. Here we define

$$G_i = S_{n-1}(1, m + n - i) \cup iK_2$$

for $i = 1, 2, \ldots, m$, and let $\mathscr{G} = \{G_i\}$. If $G \in \mathrm{gcs}\,\mathscr{G}$, then $\beta_1(G) \leqq n + 1$ and $\Delta(G) \leqq n$. Since $S_{n-1}(1, n) \cup K_2 \subset G_i$ for all i, it follows that $q(G) \geqq 2n$ for any such graph G. If $\Delta(G) < n$, then the structure of the graphs G_i implies that $\beta_1(G) > n + 1$, which produces a contradiction. Therefore, $\Delta(G) = n$ whenever $G \in \mathrm{gcs}\,\mathscr{G}$. If $q(G) > 2n$, then since $\Delta(G) = n$, it follows that $\beta_1(G) > n + 1$ which is impossible. These observations imply that

$$\mathrm{gcs}\,\mathscr{G} = \{S_{n-i}(1, n) \cup iK_2 \mid i = 1, 2, \ldots, n\}\,.$$

Case 4. *Assume that* $3 \leqq m \leqq n$. For $i = 1, 2, \ldots, n$, define

$$G_i = S_{n-i+1}(1, n) \cup (i-1) K_2 .$$

Consider first gcs (G_1, G_n). Since $S_1(1, n)$ is a subgraph of both G_1 and G_n, it follows that if $G \in$ gcs (G_1, G_n), then $q(G) \geqq n + 1$. We cannot, however, have $q(G) \geqq n + 2$, for this would imply that G has a path of length 3, which is not present in G_n. Therefore, if $G \in$ gcs (G_1, G_n), then $q(G) = n + 1$. If we define

$$H_i = S_1(1, n - i + 1) \cup (i - 1) K_2$$

for $i = 1, 2, \ldots, n$, then it is easy to see that

$$\text{gcs}\,(G_1, G_n) = \{H_i \mid i = 1, 2, \ldots, n\} .$$

However, since $H_j \subset G_i$ for all $i, j \in \{1, 2, \ldots, n\}$, it follows that

$$\text{gcs}\,(G_1, G_2, \ldots, G_{m-1}, G_n) = \{H_i \mid i = 1, 2, \ldots, n\} ,$$

thereby completing the proof. □

3. THE GCS INDEX OF A GRAPH

In Proposition 2 we showed that for every graph G without isolated vertices, there exist graphs G_1 and G_2 of equal size such that gcs $(G_1, G_2) = \{G\}$. By a similar argument, the following result, whose proof we omit, can be verified.

Proposition 5. *For every graph G without isolated vertices, there exist pairwise nonisomorphic graphs G_1, G_2 and G_3 of equal size such that*

$$\text{gcs}\,(G_1, G_2, G_3) = \{G\} .$$

Propositions 2 and 5 suggest the question that for a given graph G without isolated vertices and a given integer $n \geqq 2$ as to whether there exists a set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of n pairwise nonisomorphic graphs of equal size such that gcs $\mathcal{G} = \{G\}$. Certainly if n is large, then the graphs in $\mathcal{G}$ must have large size. By introducing a new graphical parameter, we shall see that the answer to this question depends on the given graph G.

For a graph G without isolated vertices, the *greatest common subgraph index* or *gcs index* of G, denoted $i(G)$, is the least positive integer q_0 such that for any integer $q > q_0$ and any set

$$\mathcal{G} = \{G_1, G_2, \ldots, G_n\} , \quad n \geqq 2 ,$$

of graphs of size q for which $G \in$ gcs $\mathcal{G}$, it follows that $|\text{gcs}\,\mathcal{G}| > 1$, i.e., gcs $\mathcal{G}$ contains an element different from G. If no such q_0 exists, then we write $i(G) = \infty$; it is for such graphs G that Propositions 2 and 5 can be extended. We illustrate this idea now.

Proposition 6. *For integers $r \geqq 1$ and $n \geqq 4$,*
(a) $i(K(1, r)) = \infty$,
(b) $i(rK_2) = \infty$, *and*
(c) $i(K_n) = \infty$.

Proof. (a) Suppose, to the contrary, that $i(K(1, r))$ is defined, say $i(K(1, r)) = q_0$ for some positive integer q_0. Let q be an integer such that $q > \max\{q_0, r\}$. Let

$$G_1 = K(1, q) \quad \text{and} \quad G_2 = K(1, r) \cup (q - r) K_2 .$$

Then

$$\text{gcs}(G_1, G_2) = \{K(1, r)\} ,$$

a contradiction.

(b) Suppose that $i(rK_2) = q_0$ for some positive integer q_0, and let q be an integer such that $q > \max\{q_0, r\}$. Let

$$G_1 = qK_2 \quad \text{and} \quad G_2 = K(1, q - r + 1) \cup (r - 1) K_2 .$$

Then $\text{gcs}(G_1, G_2) = \{rK_2\}$, which contradicts the fact that $|\text{gcs}(G_1, G_2)| > 1$.

(c) Suppose that $i(K_n) = q_0$ for some positive integer q_0, and let q be an integer such that

$$q > \max\{q_0, q_n\} .$$

where $q_n = \binom{n}{2}$. Define

$$G_1 = K_1 + (K_{n-1} \cup \bar{K}_{q-n}) \quad \text{and} \quad G_2 = K_n \cup (q - q_n) K_2 .$$

Then $\text{gcs}(G_1, G_2) = \{K_n\}$, which is impossible. □

That the condition $n \geqq 4$ is required in Proposition 6(c) is now verified.

Proposition 7. *The* gcs *index of K_3 is* 6.

Proof. For $q > 6$, let

$$\mathcal{G} = \{G_1, G_2, \ldots, G_n\} , \quad n \geqq 2 ,$$

be any set of graphs of size q for which $K_3 \in \text{gcs}\,\mathcal{G}$. We show that $K_2 \cup P_3 \in \text{gcs}\,\mathcal{G}$ so that $|\text{gcs}\,\mathcal{G}| > 1$.

For each i $(1 \leqq i \leqq n)$ such that G_i has at least two components, it is obvious that $K_2 \cup P_3 \subset G_i$. Suppose then that G_j $(1 \leqq j \leqq n)$ is connected. Let v_1, v_2 and v_3 be the vertices of a triangle in G_j. If $\deg_{G_j} v_i \geqq 4$ for some i $(1 \leqq i \leqq 3)$, then $K_2 \cup \cup P_3 \subset G_j$. On the other hand, if $\deg_{G_j} v_i \leqq 3$ for all i, then since $q > 6$, G_j must contain an edge incident with none of the vertices v_i so that $K_2 \cup P_3 \subset G_j$. Hence $K_2 \cup P_3 \in \text{gcs}\,\mathcal{G}$, as claimed.

Therefore, $i(K_3) \leqq 6$. Suppose, to the contrary, that $i(K_3) = q_0 < 6$. Necessarily, $q_0 > 3$ since $K_3 \in \text{gsc}\,\mathcal{G}$. Now $q_0 \neq 4$ since each of $G_1 = K_3 \cup 2K_2$ and $G_2 =$

$= K_4 - e$ has size 5 and $\text{gcs}(G_1, G_2) = \{K_3\}$. Further, $q_0 \neq 5$, since $H_1 = K_3 \cup 3K_2$ and $H_2 = K_4$ have six edges and $\text{gcs}(H_1, H_2) = \{K_3\}$. Consequently, $i(K_3) = 6$. □

We conclude by determining the gcs index of every path.

Proposition 8. *The* gcs *index of a path is given by*

$$i(P_n) = \begin{cases} \infty & \text{if} \quad n \neq 4 . \\ 6 & \text{if} \quad n = 4 \end{cases}$$

Proof. By Proposition 6(a), $i(P_n) = \infty$ for $n = 2, 3$. Suppose, then, that $n \geqq 5$ and assume, to the contrary, that $i(P_n) = q_0$ for some positive integer q_0. Let q be any integer such that $q > \max\{q_0, n - 1\}$. Let G_1 be that graph obtained by subdividing an edge of $K(1, q - n + 3)$ a total of $n - 3$ times, and let

$$G_2 = P_n \cup (q - n + 1) K_2 .$$

Then $\text{gcs}(G_1, G_2) = \{P_n\}$, which is impossible.

The proof that $i(P_4) = 6$ is very similar to the proof of Proposition 7 and is therefore omitted. □

References

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Souhrn

NEJVĚTŠÍ SPOLEČNÉ PODGRAFY GRAFŮ

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Graf G bez izolovaných vrcholů je největším společným podgrafem množiny $\mathcal{G}$ grafů, které mají všechny stejnou velikost, jestliže G je graf maximální velikosti, který je izomorfní s nějakým podgrafem každého grafu z $\mathcal{G}$. Je podána řada výsledků týkajících se největších společných podgrafů. Zejména je ukázáno, že pro každý graf G bez izolovaných vrcholů existují takové grafy G_1, G_2 stejné velikosti, že G je jejich jediný největší společný podgraf. Další vyšetřování tohoto výsledku vede k zavedení parametru, který se nazývá index největšího společného podgrafu grafu.

Резюме

НАЙБОЛЬШИЕ ОБЩИЕ ПОДГРАФЫ ГРАФОВ

GARY CHARTRAND, FARROKH SABA, HUNG-BÍN ZOU

Граф G без изолированных вершин называется найбольшим общим подграфом множества $\mathcal{G}$ графов одинаковой величины, если G есть граф максимальной величины, которой изоморфен некоторому подграфу каждого графа из $\mathcal{G}$. В статье доказан целый рад результатов о наибольших общих подграфах. В частности здесь показано, что для каждого графа G без изолированных вершин существуют такие графы G_1, G_2 одинаковой величины, что G является их единственным наибольшим общим подграфом. Дальнейшее исследование этого результата приводит к определению параметра, которой называется индексом наибольшего общего подграфа.

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