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ANTI-NETS II

(COLLINEATIONS OF ANTI-NETS)

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Summary. This paper is meant as an immediate continuation of the paper "Anti-nets I". Its aim is to investigate the so-called collineations of an anti-net. After describing the groups of special collineations (the group of translations, the group of homotheties with given centre, the group of perspective affinities with given axis), the group of all possible collineations of a given anti-net is characterized, making use of the automorphisms of the partition of the group of translations of that anti-net. The conditions of existence of the groups of collineations are expressed in terms of closure conditions and their algebraic equivalents, discussed in "Anti-nets I".

Keywords: Anti-net, automorphism and collineation of anti-net, translation group, perspective affinity, partition of group, equivalence and automorphism of partition.

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INTRODUCTION

The aim of this part of our treatise about anti-nets is the determination of the groups of some special automorphisms of an anti-net. By an automorphism of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in I})$ we mean any mapping $\pi: \mathbf{P} \to \mathbf{P}$ having the following properties:

(i) π is a bijective mapping;

(ii) for any line $\mathbf{g} \in \mathscr{L}$ there exists exactly one line $\mathbf{g}' \in \mathscr{L}$ and exactly one line $\mathbf{g}'' \in \mathscr{L}$ so that $\{X^\pi \mid X \in \mathbf{g}\} \subseteq \mathbf{g}'$ as well as $\{X^{\pi^{-1}} \mid X \in \mathbf{g}\} \subseteq \mathbf{g}''$.

Moreover, it can be easily shown that any automorphism π of the anti-net $\mathscr{A}$ and any line $\mathbf{g} \in \mathscr{L}$ satisfy $\mathbf{g}^\pi = \{X^\pi \mid X \in \mathbf{g}\} = \mathbf{g}'$ as well as $\mathbf{g}^{\pi^{-1}} = \{X^{\pi^{-1}} \mid X \in \mathbf{g}\} = \mathbf{g}''$.

The assertions a)–c) follow from the just introduced property of an automorphism of $\mathscr{A}$.

a) If $\mathbf{g} = AB$ (A, B are two distinct points of $\mathbf{P}$) then $\mathbf{g}^\pi = \mathbf{A}^\pi\mathbf{B}^\pi$.

b) If $Q = \mathbf{a} \cap \mathbf{b}$ ($\mathbf{a}, \mathbf{b}$ are two distinct lines of $\mathscr{L}$) then $Q^\pi = \mathbf{a}^\pi \cap \mathbf{b}^\pi$.

c) For any principal line $\mathbf{h}_\iota$, $\iota \in I$ the line $\mathbf{h}_\iota^\pi$ is also a principal line. For any ordinary line $\mathbf{g}$ the line $\mathbf{g}^\pi$ is also an ordinary line.

An automorphism γ of $\mathscr{A}$ will be called a collineation of $\mathscr{A}$ if for any lines $\mathbf{a}, \mathbf{b} \in \mathscr{L}$ the condition $\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a}^\gamma \parallel \mathbf{b}^\gamma$ is fulfilled.

The set $\mathbf{\Gamma}$ of all collineations of the anti-net $\mathscr{A}$ together with the operation "the composition of mappings" is a group $(\mathbf{\Gamma},\ )$. The neutral element of this group is the identical automorphism, i.e. the mapping ε such that $\mathsf{B}^\varepsilon = \mathsf{B}$ and $\mathbf{g}^\varepsilon = \mathbf{g}$ holds for any $\mathsf{B} \in \mathbf{P}$ and any $\mathbf{g} \in \mathscr{L}$.

Let $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota \in I \cup \{\infty\}})$ be the extension of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in I})$. Similarly to the case of $\mathscr{A}$, by an automorphism of $\mathscr{A}'$ we will mean any bijection $\pi: \mathbf{P} \cup \mathbf{h}_\infty \to \mathbf{P} \cup \mathbf{h}_\infty$ with the following property: for any $\mathbf{g} \in \mathscr{L} \cup \{\mathbf{h}_\infty\}$ there exists a uniquely determined line $\mathbf{g}' \in \mathscr{L} \cup \{\mathbf{h}_\infty\}$ and a uniquely determined $\mathbf{g}'' \in \mathscr{L} \cup \{\mathbf{h}_\infty\}$ such that $\{\mathsf{X}^\pi \mid \mathsf{X} \in \mathbf{g}\} \subseteq \mathbf{g}'$ as well as $\{\mathsf{X}^{\pi^{-1}} \mid \mathsf{X} \in \mathbf{g}\} \subseteq \mathbf{g}''$ is true. The automorphisms of the extension $\mathscr{A}'$ have properties analogous to those of the automorphisms of $\mathscr{A}$ introduced above.

Any collineation γ of $\mathscr{A}$ induces in a very natural way an automorphism (denoted also by γ) of $\mathscr{A}'$. Such an automorphism of $\mathscr{A}'$ reproduces the principal line $\mathbf{h}_\infty$. Conversely, let an automorphism γ of $\mathscr{A}'$ with the property $\mathbf{h}_\infty^\gamma = \mathbf{h}_\infty$ be given. The restriction of γ to $\mathscr{A}$ (more exactly, to $\mathbf{P}$) is an automorphism of the anti-net $\mathscr{A}$ denoted also by γ. Such an automorphism γ of $\mathscr{A}$ preserves the parallelity relation. Hence any collineation of $\mathscr{A}$ may be investigated as an automorphism of the extension $\mathscr{A}'$ reproducing the improper principal line $\mathbf{h}_\infty$. Such an automorphism of $\mathscr{A}'$ will be also called the collineation of the extension $\mathscr{A}'$. The group of all collineations of the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ will be also denoted by $(\mathbf{\Gamma},\ )$.

It follows from the property c) of the automorphism of $\mathscr{A}$ that the common improper point $\mathsf{L} \in \mathbf{h}_\infty$ of all principal lines $\mathbf{h}_\iota$, $\iota \in I$ will be fixed under all collineations of $\mathscr{A}'$.

As every anti-net $\mathscr{A}$ is a parallel structure in the sense of [1] we will use the same terminology as in [1] when studying collineations of anti-nets. We will give a short survey of them together with some principal properties of collineations of an anti-net.

Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ be given. The collineation $\pi \in \mathbf{\Gamma}$ of $\mathscr{A}'$ fixing every point of some principal line $\mathbf{h}_\iota$ will be called an *axial collineation.* The line $\mathbf{h}_\iota$ is an *axis* of π. Obviously the set $\mathbf{\Pi}_{\mathbf{h}_\iota}$ of all axial collineations with the same axis $\mathbf{h}_\iota$ together with the operation "the composition of mappings" is a subgroup of the group $(\mathbf{\Gamma},\ )$. According to Theorem 4.1 in [1], a non-identical axial collineation π with the axis $\mathbf{h}_\iota$ fixes at most one point not lying on $\mathbf{h}_\iota$.

A collineation $\zeta \in \mathbf{\Gamma}$ of the extension $\mathscr{A}'$ which fixes every line going through a fixed point $\mathsf{Z} \in \mathbf{P} \cup \mathbf{h}_\infty$ will be called a *central collineation with the centre* Z. Evidently the set of all central collinestions of $\mathscr{A}'$ with the same centre Z together with the operation "the composition of mappings" is a subgroup of the group $(\mathbf{\Gamma},\ )$ of all collineations.

If the axis of an axial collineation π of $\mathscr{A}'$ is the improper principal line $\mathbf{h}_\infty$ then for any line $\mathbf{g} \in \mathscr{L}$, $\mathbf{g}^\pi \parallel \mathbf{g}$ is true. According to [1], p. 97 the set $\mathbf{\Pi}$ of all axial collineations with the axis $\mathbf{h}_\infty$ (together with the operation "the composition of mappings") is a normal subgroup of the group $(\boldsymbol{\Gamma}, \ )$.

Let a non-identical axial collineation π with the axis $\mathbf{h}_\infty$ of $\mathscr{A}'$ be given. Then the following two cases are possible: either no proper point is fixed under π or π fixes a unique proper point Z of $\mathscr{A}'$. In the first case π will be called a *translation*, in the second π will be called a *homothety with the centre* Z. The set of all translations will be denoted by $\mathbf{T}$, the set of all homotheties with the same centre Z will be denoted by $\boldsymbol{\Pi}_{\mathrm{Z}}$.

In virtue of Theorem 4.2 in [1], any homothety of $\mathscr{A}'$ with the centre Z is a central collineation of $\mathscr{A}'$ with the centre Z. On the other hand, a translation τ of $\mathscr{A}'$ need not be a central collineation of $\mathscr{A}'$. If a translation τ is a central collineation of $\mathscr{A}'$ then it has an improper centre Z. In this case the translation τ fixes, in addition to the improper line $\mathbf{h}_\infty$, just the proper lines going through Z. Hence a non-identical translation τ of $\mathscr{A}'$ is a central collineation iff

1. there exists a proper line $\mathbf{g} \in \mathscr{L}$ such that $\mathbf{g}^\tau = \mathbf{g}$;
2. for any other proper line $\mathbf{g}' \in \mathscr{L}$, $\mathbf{g}'^\tau = \mathbf{g}' \Leftrightarrow \mathbf{g}' \parallel \mathbf{g}$ is true.

A non-identical central translation τ of $\mathscr{A}'$, i.e. a non-identical translation τ which is a central collineation of $\mathscr{A}'$, may be characterized by: $\mathsf{AA}^\tau \parallel \mathsf{BB}^\tau$ for all $\mathsf{A}, \mathsf{B} \in \mathbf{P}$.

The set $\boldsymbol{\Pi}_{\mathrm{Z}}$ of all homotheties of $\mathscr{A}'$ with the same centre Z together with the operation "the composition of mappings" is a subgroup of $(\boldsymbol{\Pi}, \ )$. On the other hand, it is not known whether the set of all translations of $\mathscr{A}'$ is a subgroup of $(\boldsymbol{\Pi}, \ )$. J. André introduced some necessary and sufficient conditions for the set of all translations of an affine-parallel structure to be a group. According to [1] p. 97, the following implication holds: If the set of all translations of $\mathscr{A}'$ is a group then it is also a normal subgroup of the group $(\boldsymbol{\Pi}, \ )$. The following result can be proved without troubles: If the set of all translatinos of $\mathscr{A}'$ is a group $(\mathbf{T}, \ )$ then $(\mathbf{T}, \ )$ is a normal subgroup of $(\boldsymbol{\Gamma}, \ )$.

In virtue of Theorem 4.4 in [1] any central translation τ of $\mathscr{A}'$ is uniquely determined by its operating on an arbitrarily chosen proper point of $\mathscr{A}'$. It may be shown without difficulties that also any homothety of $\mathscr{A}'$ with a given centre Z is uniquely determined by its operating on an arbitrarily chosen proper point of $\mathscr{A}'$ different from Z.

In the extension $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota \in I \cup \{\infty\}})$ of an anti-net $\mathscr{A}$ the statement that any axial collineation is also a central one and conversely (which holds e.g. in the case of projective planes) is in general not true (cf. [1] p. 95–96). If the extension $\mathscr{A}'$ has a degree $\geqq 4$ and if it possesses the properties $(\boldsymbol{rs})$, $(\boldsymbol{D}_\infty)$ and $(\boldsymbol{dg})$ then any translation $\tau_{(\varkappa,a)}$, $\varkappa \in J$, $a \in S \setminus \{0\}$ defined by (3) in Sec. 1 is an axial collineation with the axis $\mathbf{h}_\infty$; but $\tau_{(\varkappa,a)}$ is not generally a central collineation. The automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of the partition of the group $(\mathbf{P}, ++)$ where $\varphi_1 = \mathrm{id}_J$, $\varphi_2 \in \mathbf{Z}(\Sigma^*)$ (see Sections 3 and 4) is an example of a central collineation of $\mathscr{A}'$ which is not an axial collineation of $\mathscr{A}'$ ($\mathscr{A}'$ has a degree $\geqq 4$ and possesses the properties $(\boldsymbol{rs})$, $(\boldsymbol{D}_\infty)$, $(\boldsymbol{D}_\beta)$, $(\boldsymbol{dg})$ and $(\boldsymbol{R})$). The centre of this collineation is the improper point

$\mathsf{L} = (\infty)$ but if $\varphi_2 \neq \mathrm{id}_S$ then there exists no principal line $\mathbf{h}_\iota$, $\iota \in I \cup \{\infty\}$ whose points are fixed under the automorphism $\mathbf{f}_0$, i.e. if $\varphi_2 \neq \mathrm{id}_S$ the collineation $\mathbf{f}_0$ has no axis.

An axial collineation π of $\mathscr{A}'$ with the axis $\mathbf{h}_\iota$ $(\iota \in I \cup \{\infty\})$ which at the same time is a central collineation of $\mathscr{A}'$ with the centre $\mathsf{Z} \in \mathbf{P} \cup \mathbf{h}_\infty$ will be called a *perspective collineation* of the extension $\mathscr{A}'$. The set of all perspective collineations of $\mathscr{A}'$ with the same axis $\mathbf{h}_\iota$ and with the same centre Z (together with the operation "the composition of mappings", of course) is a group. Using this property of the perspective collineations of $\mathscr{A}'$ and Theorem 4.1 in [1] we can easily prove: Any non-identical perspective collineation π with the centre $\mathsf{Z} \in \mathbf{P} \cup \mathbf{h}_\infty$ and with the axis $\mathbf{h}_\iota$ $(\iota \in I \cup \cup \{\infty\})$ is uniquely determined by an arbitrary point $\mathsf{B} \in \mathbf{P}$, $\mathsf{B} \neq \mathsf{Z}$, $\mathsf{B} \notin \mathbf{h}_\iota$ and its image B^π, $\mathsf{B}^\pi \neq \mathsf{Z}$, $\mathsf{B}^\pi \notin \mathbf{h}_\iota$ with "ZBB^π".

As has been shown above there exists no non-identical perspective collineation of $\mathscr{A}'$ with the proper axis $\mathbf{h}_\iota$ $(\iota \in I)$ and proper centre $\mathsf{Z} \in \mathbf{P}$. Hence the following types of perspective collineations of $\mathscr{A}'$ are possible:

a) perspective collineations with the improper axis $\mathbf{h}_\infty$ and with an improper centre $\mathsf{Z} \in \mathbf{h}_\infty$; these collineations are central translations of $\mathscr{A}'$ studied in Sec. 1;

b) perspective collineations with the improper axis $\mathbf{h}_\infty$ and with a proper centre $\mathsf{Z} \in \mathbf{P}$; these ones are homotheties of $\mathscr{A}'$ with the centre Z described in Sec. 6;

c) perspective collineations with the proper axis $\mathbf{h}_\iota$ and with an improper centre $\mathsf{Z} \in \mathbf{h}_\infty$; they will be termed the *perspective affinities* of $\mathscr{A}'$. They are investigated in Sec. 7.

In section 1 of this work we determine the translation group $(\mathbf{T}',)$ of the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$. This group operates strictly simply transitively (=effectively and freely) on the set $\mathbf{P}$ of all proper points of $\mathscr{A}'$ (see Theorems 1.1 and 1.2).

With help of the translation group $(\mathrm{T}',)$ of $\mathscr{A}'$ and the relation (4) we define in Sec. 2 a binary operation "$++$" on the set $\mathbf{P}$ of all proper points of $\mathscr{A}'$. Then the structure $(\mathbf{P}, ++)$ is a group whose neutral element is the point $(\beta, 0)$ (see Theorem 2.1). For any line $\mathbf{g} \in \mathscr{L}$ going through the "zero point" $(\beta, 0)$, the set of all points of $\mathbf{g}$ together with the operation "$++$" is a proper subgroup of $(\mathbf{P}, ++)$. The system of these subgroups forms a partition $\mathscr{P}$ of $(\mathbf{P}, ++)$. The left cosets of the decompositions of the group $(\mathbf{P}, ++)$ with respect to all components of the partition $\mathscr{P}$ are precisely all proper lines (without improper points) of the structure $\mathscr{A}'$ (see Theorem 2.3).

In Section 3 some groups of automorphisms of groups $(S, +)$ and $(J', +_\circ)$ of $\mathscr{A}'$ are studied (see Lemmas 3.1–3.4).

Section 4 deals with equivalences and automorphisms of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$. Theorem 4.1 describes all automorphisms $\mathbf{f}_0$ of $\mathscr{P}$ sending the components $(\mathbf{P}_\mathbf{L}, ++)$ and $(\mathbf{P}_{(0)}, ++)$ onto themselves.

By using the automorphisms $\mathbf{f}_0$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ and the translations of the translation group $(\mathrm{T}',)$ of $\mathscr{A}'$, the group of all collineations of $\mathscr{A}'$ fixing the improper points $\mathsf{L} = (\infty)$ and $(0) = \mathbf{o} \cap \mathbf{h}_\infty$ is described in Section 5 (see Theorem 5.1).

As a special case of these collineations, the transformation formulas for the homothety with the centre $(\beta, 0) = \mathsf{Z}$ are determined in Section 6. Further, the group of all homotheties of $\mathscr{A}'$ with an arbitrary given proper centre Z is described (see Theorem 6.1 and Remark 6.1).

Section 7 is devoted to the characterization of all perspective affinities of $\mathscr{A}'$ with a given (proper) axis $\mathbf{h}_\iota$ ($\iota \in \boldsymbol{I}$) and with a given improper centre $\mathsf{Z} \in \mathbf{h}_\infty$. First, all perspective affinities with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{Z} = (0) \in \mathbf{h}_\infty$ (as the special case of automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$) are determined (see Lemma 7.1). Further, in Lemma 7.2 the group of all perspective affinities with the axis $\mathbf{h}_\infty$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ is described. The perspective affinities with the centres (0) and L form the group of all perspective affinities with the axis $\mathbf{h}_\infty$ and the centre $\mathsf{Z} \in \mathbf{h}_\infty$ (cf. Theorems 7.1 and 7.2). Using the translations of the translation group (T′,) of the extension $\mathscr{A}'$of the given anti--net $\mathscr{A}'$, the group of all perspective affinities with an arbitrary axis $\mathbf{h}_\iota$, $\iota \in \boldsymbol{I}$ is described.

Section 8 is a continuation of the investigation of the equivalence of automorphisms of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ started in Section 4. Let $\boldsymbol{\alpha}_{(\gamma,a)}$ be a perspective affinity of $\mathbf{A}_\mathsf{L}$, $a \in S$ where $\boldsymbol{\alpha}_{(\gamma,a)}$ does not belong to the group $\mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ of all automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ determined in Sec. 4 (see Lemma 8.1). Then $\boldsymbol{\alpha}_{(\gamma,a)}$ is also an automorphism of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$. Moreover, the composition of the automorphism $\mathbf{f}_0 \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ and the perspective affinity $\boldsymbol{\alpha}_{(\gamma,a)} \in \mathbf{A}_\mathsf{L}$ (in this order) is an automorphism of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$ (see Lemma 8.2). By interchanging the order of the automorphisms $\mathbf{f}_0$ and $\boldsymbol{\alpha}_{(\gamma,a)}$ in the composition we get new automorphisms of the partition $\mathscr{P}$ of $\mathbf{P}(, ++)$ (see Lemma 8.3). All such compound automorphisms $\mathbf{f} = \mathbf{f}_0\boldsymbol{\alpha}_{(\gamma,a)}$ form the group $\mathbf{Aut}(\mathscr{P}, \mathbf{P})$ (see Lemma 8.4) isomorphic with the group of all automorphisms of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ (see Theorem 8.1). Any collineation $\gamma \in \boldsymbol{\Gamma}$ of $\mathscr{A}'$ may be expressed in terms of the automorphism $\mathbf{f} = \mathbf{f}_0\boldsymbol{\alpha}_{(\gamma,a)} \in \mathbf{Aut}(\mathscr{P}, \mathbf{P})$ and the translation $\boldsymbol{\tau} \in \mathbf{T}'$ in the form $\gamma = \mathbf{f}_0\boldsymbol{\alpha}_{(\gamma,a)}\boldsymbol{\tau}$ (see Theorem 8.2).

1. TRANSLATIONS OF AN ANTI-NET

According to the remarks in Introduction we can investigate any translation $\boldsymbol{\tau}$ of an anti-net $\mathscr{A} = (\mathbf{P}, \mathscr{L}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I}})$ as an axial collineation of the extension $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota \in \boldsymbol{I} \cup \{\infty\}})$ with the axis $\mathbf{h}_\infty$ in which either all proper points of $\mathbf{P}$ are fixed or none of them is.

In what follows we will always suppose that the extension $\mathscr{A}'$ has the property ($\boldsymbol{rs}$), i.e. the triple $(\mathbf{o}; \beta, \gamma)$ is its frame and $\mathfrak{A} = (S, 0, (\sigma_\iota)_{\iota \in \boldsymbol{J}}, (+_\iota)_{\iota \in \boldsymbol{J}})$ is the coordinate algebra with respect to the frame $(\mathbf{o}; \beta, \gamma)$. The index set $\boldsymbol{J}$ of the coordinate algebra $\mathfrak{A}$ will be supposed to be $\boldsymbol{J} = \boldsymbol{I} \setminus \{\beta\}$.

For any element $a \in S$ let us define the mapping $\tau_{(\beta,a)}$ by the rules

$$(1) \qquad \begin{array}{lll} \{\xi, y\} \mapsto (\xi, y +_\gamma a) & \forall \xi \in \boldsymbol{I}, & \forall y \in S\,; \\ (k) \mapsto (k) & \forall k \in S\,; & \\ \mathsf{L} \mapsto \mathsf{L}\,; & & \\ \{k, q\} \mapsto \{k, q +_\gamma a\} & \forall k, q \in S\,; & \\ \mathbf{h}_\iota \mapsto \mathbf{h}_\iota & \forall \iota \in \boldsymbol{I} \cup \{\infty\}\,. & \end{array}$$

Lemma 1.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 3$ having the properties $(\mathbf{rs})$ and $(\mathbf{D}_\infty)$. Then any mapping $\tau_{(\beta,a)}$, $a \in S$ defined by (1) is a central translation with the centre L of $\mathscr{A}'$. Conversely, any central translation of the structure $\mathscr{A}'$ with the centre L satisfies the relation (1) for a suitable $a \in S$.*

Proof. a) We will prove that any mapping $\tau_{(\beta,a)}$ defined by (1) is central translation of $\mathscr{A}'$ with the centre L.

If $a = 0$ then $\tau_{(\beta,0)} = \varepsilon$. If $a \in S \setminus \{0\}$ then (1) immediately implies that $\tau_{(\beta,a)}$ fixes any point of the improper line $\mathbf{h}_\infty$ but not proper point. As $\mathscr{A}'$ has the property $(\mathbf{D}_\infty)$, $(S, +)$ is a group and hence $\tau_{(\beta,a)}$ induces the permutation of the sets $\mathbf{P} \cup \mathbf{h}_\infty$ and $\mathscr{L} \cup \{\mathbf{h}_\infty\}$. The mapping $\tau_{(\beta,a)}$ preserves the incidence relation between the points and lines of $\mathscr{A}'$ — to verify this fact it is sufficient to use the associativity of the addition in the group $(S, +)$. Any principal line $\mathbf{h}_\iota$, $\iota \in \boldsymbol{I}$ is mapped (globally) onto itself under the mapping $\tau_{(\beta,a)}$. Summarizing these facts we get that for any $a \in S$, $\tau_{(\beta,a)}$ is a central translation of $\mathscr{A}'$ with the centre L.

b) Let τ be a non-identical central translation of $\mathscr{A}'$ with the centre L. Then the second, third, and fifth relation of (1) must be true for τ. The translation τ is uniquely determined by its operating on some fixed proper point of $\mathscr{A}'$. Let us suppose that this point is $(\beta, 0)$ with $(\beta, 0)^\tau = (\beta, a)$, $a \in S \setminus \{0\}$. We may easily verify by direct calculation that τ fulfils the first and the fourth relation of (1), too. It suffices to use the assumption that $\mathscr{A}'$ has the property $(\mathbf{D}_\infty)$. Therefore $\tau = \tau_{(\beta,a)}$ for some $a \in S \setminus \{0\}$. ∎

We will denote by $\mathbf{T}_\mathsf{L}$ the set of all central translations $\tau_{(\beta,a)}$ with the same centre L determined by (1). The composition of two translations $\tau_{(\beta,a)}$, $\tau_{(\beta,b)}$ of $\mathbf{T}_\mathsf{L}$ is a translation $\tau_{(\beta,a)}\tau_{(\beta,b)} = \tau_{(\beta,a+b)} \in \mathbf{T}_\mathsf{L}$. The operation "composition of central collineations" has the neutral element $\varepsilon = \tau_{(\beta,0)}$ — the identical collineation. The inverse mapping to the translation $\tau_{(\beta,a)} \in \mathbf{T}_\mathsf{L}$ is again a translation, viz. $\tau^{-1}_{(\beta,a)} = \tau_{(\beta,-a)} \in \mathbf{T}_\mathsf{L}$. As $(S, +)$ is a group ($\mathscr{A}'$ has the property $(\mathbf{D}_\infty)$), the composition of translations of $\mathbf{T}_\mathsf{L}$ is associative. Thus we have obtained

Lemma 1.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 3$ have the properties $(\mathbf{rs})$ and $(\mathbf{D}_\infty)$. Then the set $\mathbf{T}_\mathsf{L}$ of all central translations $\tau_{(\beta,a)}$, $a \in S$ determined by (1) together with the operation "the composition of mappings" is a group $(\mathbf{T}_\mathsf{L}, \ )$.*

Now let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$ and $(\boldsymbol{dg})$. According to the corollary of Theorem 4.1 in [4] the formulas

$$(2)\qquad \begin{array}{lll} (\xi, y) \mapsto (\varkappa +_\circ \xi, y) & \forall \xi \in \boldsymbol{I}, & \forall y \in S\,; \\ (k) \mapsto (k) & \forall k \in S\,; & \\ \mathsf{L} \mapsto \mathsf{L}\,; & & \\ \{k, q\} \mapsto \{k, (k)\,\sigma_\varkappa + q\} & \forall k, q \in S\,; & \\ \mathbf{h}_\iota \mapsto \mathbf{h}_{\varkappa +_\circ \iota} & \forall \iota \in \boldsymbol{I}\,; & \\ \mathbf{h}_\infty \mapsto \mathbf{h}_\infty & & \end{array}$$

correctly define a mapping $\tau_{(\varkappa,0)}$, $\varkappa \in \boldsymbol{I}$.

Lemma 1.3. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$ and $(\boldsymbol{dg})$. Then any mapping $\tau_{(\varkappa,0)}$, $\varkappa \in \boldsymbol{I}$ defined by (2) is a central translation of $\mathscr{A}'$ with the centre $\mathsf{Z} = (0)$. Conversely, any central translation of $\mathscr{A}'$ with the centre (0) satisfies the relations (2) for some $\varkappa \in \boldsymbol{I}$.*

Proof. a) We will prove that any mapping $\tau_{(\varkappa,0)}$ defined by (2) is a central translation of $\mathscr{A}'$ with the centre (0). If $\varkappa = \beta$ then $\tau_{(\beta,0)} = \boldsymbol{\varepsilon}$. If $\varkappa \in \boldsymbol{J} = \boldsymbol{I} \setminus \{\beta\}$ then the relations (2) imply that the mapping $\tau_{(\varkappa,0)}$ fixes any point of the improper principal line $\mathbf{h}_\infty$ but no proper point. From the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$ and $(\boldsymbol{dg})$ of the structure $\mathscr{A}'$ and from the corollary of Theorem 4.1 in [4] it follows that $\tau_{(\varkappa,0)}$ is a bijective mapping of the sets $\mathbf{P} \cup \mathbf{h}_\infty$ and $\mathscr{L} \cup \{\mathbf{h}_\infty\}$, respectively. The permutation $\tau_{(\varkappa,0)}$ preserves the incidence relation between the points and lines of $\mathscr{A}'$. If e.g. a point (ξ, y) lies on a line $\{k, q\}$ and therefore $(k)\,\sigma_\xi + y = q$ then its image $(\varkappa +_\circ \xi, y)$ lies on the line $\{k, (k)\,\sigma_\varkappa + q\}$. In fact, $(k)\,\sigma_{\varkappa +_\circ \xi} + y = (k)(\sigma_\varkappa \dotplus \sigma_\xi) + y = ((k)\,\sigma_\varkappa + (k)\,\sigma_\xi) + y = (k)\,\sigma_\varkappa + ((k)\,\sigma_\xi + y) = (k)\,\sigma_\varkappa + q$. (We have used the relations (9), (7) and Theorem 4.1 in [4] and finally, the associativity of the addition in the group $(S, +)$). The other cases of the incidence of points and lines are obvious. It follows from the axiom $(\boldsymbol{\alpha 1})$ in Sec. 2 of [4] that the mapping $\tau_{(\varkappa,0)}$ reproduces every line $\{0, q\}$. Summarizing these results we obtain that for any index $\varkappa \in \boldsymbol{I}$ the mapping $\tau_{(\varkappa,0)}$ is a central translation of $\mathscr{A}'$ with the centre $(0) \in \mathbf{h}_\infty$.

b) Let τ be a non-identical central translation of $\mathscr{A}'$ with the centre $(0) \in \mathbf{h}_\infty$. Then τ satisfies the second, third, and sixth relations of (2). The translation τ is uniquely determined by the image of a given proper point of $\mathscr{A}'$. Let e.g. $(\beta, 0)^\tau = (\varkappa, 0)$, $\varkappa \in \boldsymbol{I} \setminus \{\beta\}$. By a simple calculation we can verify that τ fulfils the remaining relations of (2) as well, it suffices to use the properties $(\mathbf{D}_\infty)$ and $(\boldsymbol{dg})$ of the structure $\mathscr{A}'$. Thus $\tau = \tau_{(\varkappa,0)}$ for some index $\varkappa \in \boldsymbol{I} \setminus \{\beta\}$. ∎

Let us denote by $\mathbf{T}_{(0)}$ the set of all central translations $\tau_{(\varkappa,0)}$ with centre $(0) \in \mathbf{h}_\infty$ determined by the relations (2). The composition of two translations $\tau_{(\varkappa,0)}, \tau_{(\lambda,0)} \in \mathbf{T}_{(0)}$ gives the translation $\tau_{(\varkappa,0)}\tau_{(\lambda,0)} = \tau_{(\lambda +_\circ \varkappa,0)} \in \mathbf{T}_{(0)}$. The neutral element of this composition is $\tau_{(\beta,0)} = \boldsymbol{\varepsilon}$, the inverse translation to $\tau_{(\varkappa,0)} \in \mathbf{T}_{(0)}$ is the translation $\tau^{-1}_{(\varkappa,0)} =$

$= \tau_{(-_0\varkappa,0)} \in \mathbf{T}_{(0)}$ where $-_0\varkappa$ is the opposite element to the element $\varkappa$ in the group $(J', +_0)$, $(J' = J \cup \{\beta\} = I)$.

Lemma 1.4. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties* (**rs**). ($\mathbf{D}_\infty$) *and* (**dg**) *be given. Then the set $\mathbf{T}_{(0)}$ of all central translations $\tau_{(\varkappa,0)}$, $\varkappa \in I$ determined by* (2) *together with the operation "composition of mapping" is a group* $(\mathbf{T}_{(0)},\ )$.

The result of the composition of two central translations τ, τ' of the structure $\mathscr{A}'$ with different centres is, according to Theorem 4.4 of [1] and its corollary, again a translation. However, we do not know if it is necessarily a central translation. Using Theorem 4.5 of [1] we get that $\tau\tau' \doteq \tau'\tau$.

Let $\tau_{(\beta,a)} \in \mathbf{T_L}$ and $\tau_{(\varkappa,0)} \in \mathbf{T}_{(0)}$ be two central translations of the structure $\mathscr{A}'$ with the centres L and (0), respectively. Let us put $\tau_{(\varkappa,a)} = \tau_{(\beta,a)}\tau_{(\varkappa,0)} = \tau_{(\varkappa,0)}\tau_{(\beta,a)}$. For the mapping $\tau_{(\varkappa,a)}$ we have

$$(3)\qquad \begin{array}{lll} (\xi, y) & \mapsto (\varkappa +_0 \xi,\ y + a) & \forall \xi \in I,\ \ \forall y \in S\,; \\ (k) & \mapsto (k) & \forall k \in S\,; \\ \mathsf{L} & \mapsto \mathsf{L}\,; & \\ \{k, q\} & \mapsto \{k, (k)\,\sigma_\varkappa + q + a\} & \forall k, q \in S\,; \\ \mathbf{h}_\iota & \mapsto \mathbf{h}_{\varkappa +_0 \iota} & \forall \iota \in I\,; \\ \mathbf{h}_\infty & \mapsto \mathbf{h}_\infty\,. & \end{array}$$

We can verify without troubles that the mapping $\tau_{(\varkappa,a)}$, $\varkappa \in I$, $a \in S$ is indeed a translation of the structure $\mathscr{A}'$, i.e. that it induces permutations of the sets $\mathbf{P} \cup \mathbf{h}_\infty$ and $\mathscr{L} \cup \cup \{\mathbf{h}_\infty\}$ preserving the incidence relation between the points and lines and fixing each point of the improper principal line $\mathbf{h}_\infty$ but no proper point (in the case $(\varkappa, \alpha) \neq \neq (\beta, 0)$).

Using (3) we have

$$\begin{aligned} \{-(a)\,\sigma_\varkappa^{-1}, q\}^{\tau(\varkappa,a)} &= \{-(a)\,\sigma_\varkappa^{-1}, (-(a)\,\sigma_\varkappa^{-1})\,\sigma_\varkappa + q + a\} = \\ &= \{-(a)\,\sigma_\varkappa^{-1}, -(a)\,(\sigma_\varkappa^{-1}\sigma_\varkappa) + q + a\} = \\ &= \{-(a)\,\sigma_\varkappa^{-1}, -a + a + q\} = \{-(a)\,\sigma_\varkappa^{-1}, q\}\,. \end{aligned}$$

In virtue of Theorems 4.2 and 5.2 of [4] we may conclude: If the structure $\mathscr{A}'$ has, in addition to the properties (**rs**), ($\mathbf{D}_\infty$), (**dg**), also the properties ($\mathbf{D}_\beta$) and (**R**) then any line $\{-(a)\,\sigma_\varkappa^{-1}, q\}$ is invariant under the translation $\tau_{(\varkappa,a)}$, $\varkappa \in J$, $a \in S \setminus \{0\}$. Hence we obtain

Theorem 1.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties* (**rs**), ($\mathbf{D}_\infty$) *and* (**dg**). *Then any mapping $\tau_{(\varkappa,a)}$, $\varkappa \in I$, $a \in S$ determined by* (3) *is a translation of the structure $\mathscr{A}'$.*

If $\varkappa = \beta$ then $\tau_{(\beta,a)}$ is a central translation with the centre $\mathsf{L} \in \mathbf{h}_\infty$. If $a = 0$ then $\tau_{(\varkappa,0)}$ is a central translation with the centre $(0) \in \mathbf{h}_\infty$. If $\varkappa \neq \beta$, $a \neq 0$ and moreover

$\mathscr{A}'$ has the properties $(\mathbf{D}_\beta)$ and $(\mathbf{R})$ then $\tau_{(\varkappa,a)}$ is a central translation with the centre $Z = (-(a)\,\sigma_\varkappa^{-1}) \in \mathbf{h}_\infty$.

Let us denote by $\mathbf{T}'$ the set of all translations $\tau_{(\varkappa,a)}$, $\varkappa \in I$, $a \in S$ of the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ determined by the relations (3). Composing two translations $\tau_{(\varkappa,a)}, \tau_{(\lambda,b)} \in \mathbf{T}'$ we again get a translation $\tau_{(\varkappa,a)}\tau_{(\lambda,b)} = \tau_{(\lambda+_\circ\varkappa,a+b)} \in \mathbf{T}'$. The neutral element of this composition is $\tau_{(\beta,0)} = \varepsilon \in \mathbf{T}'$. The inverse translation to $\tau_{(\varkappa,a)} \in \mathbf{T}'$ is the translation $\tau_{(\varkappa,a)}^{-1} = \tau_{(-_\circ\varkappa,-a)} \in \mathbf{T}'$. For any two proper points $(\varkappa, a), (\lambda, b) \in \mathbf{P}$ of the structure $\mathscr{A}'$ there exists exactly one translation $\tau_{(\lambda-_\circ\varkappa,-a+b)} \in \mathbf{T}'$ carrying the point $(\varkappa, a)$ into the point (λ, b). Hence we arrive at

Theorem 1.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$ and $(\mathbf{dg})$ be given. Then the set $\mathbf{T}' = \{\tau_{(\varkappa,a)} \mid \varkappa \in I,\ a \in S\}$ of all translations of the structure $\mathscr{A}'$ determined by (3) together with the operation "the composition of mappings" is a group $(\mathbf{T}',\ )$. The group $(\mathbf{T}',\ )$ operates freely and effectively on the set $\mathbf{P}$ of all proper points of the structure $\mathscr{A}'$.*

Remark 1.1. The group $(\mathbf{T}',\ )$ from Theorem 1.2 will be called the *translation group* of the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$. The extension $\mathscr{A}'$ from Theorem 1.2 is an extended translation structure with the translation group $(\mathbf{T}',\ )$ in the sense of [2]. The problem is whether $\mathbf{T}'$ is the set of all possible translations of the structure $\mathscr{A}'$. Certainly, we always have $\mathbf{T}' \subseteq \mathbf{T}$. If all possible translations of $\mathscr{A}'$ form the group $(\mathbf{T},\ )$ then $\mathbf{T} = \mathbf{T}'$.

Using the results of [2] on translation structures and Theorem 4.2 of [4] we get the following theorem

Theorem 1.3. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then* a) *the translation group $(\mathbf{T}',\ )$ of the structure $\mathscr{A}'$ is abelian and $\mathscr{A}'$ is an extended central translation structure;*
b) *the translation group $(\mathbf{T}',\ )$ is the group of all translations of $\mathscr{A}'$, i.e. $\mathbf{T}' = \mathbf{T}$.*

2. THE PARTITION OF THE GROUP OF PROPER POINTS OF $\mathscr{A}'$

Let the extension $\mathscr{A}'$ of the anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it possess the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$ and $(\mathbf{dg})$. According to Theorem 1.2 we may correctly define the binary operation "$++$" (the so-called addition) on the set $\mathbf{P}$ of all proper points of $\mathscr{A}'$ as follows:

Let $(\varkappa, a)$, (λ, b) be two points of $\mathbf{P}$, then we put

$$(4) \qquad (\varkappa, a) ++ (\lambda, b) = (\varkappa, a)^{\tau(\lambda,b)} = (\lambda +_\circ \varkappa,\ a + b)\,.$$

As for any point $(\varkappa, a) \in \mathbf{P}$ the identity $(\varkappa, a) = (\beta, 0)^{\tau(\varkappa,a)}$ is true, we have $(\varkappa. a) ++ (\lambda, b) = (\varkappa, a)^{\tau(\lambda,b)} = (\beta, 0)^{\tau(\varkappa,a)\,\tau(\lambda,b)}$. Consequently, the mapping $\mathbf{T}' \to \mathbf{P}$, $\tau_{(\varkappa,a)} \mapsto (\varkappa, a)$ is an isomorphism of the translation group $(\mathbf{T}',\ )$ onto $(\mathbf{P}, ++)$. Hence $(\mathbf{P}, ++)$ is a group. Thus we have

Theorem 2.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties* ($\mathbf{rs}$), ($\mathbf{D}_\infty$) *and* ($\mathbf{dg}$). *Then the set of all proper points of the structure $\mathscr{A}'$ together with the operation "$++$" defined by* (4) *is a group* ($\mathbf{P}$, $++$) *with the neutral-zero element* $(\beta, 0)$.

If besides the properties ($\mathbf{rs}$), ($\mathbf{D}_\infty$) and ($\mathbf{dg}$) the structure $\mathscr{A}'$ has also the properties ($\mathbf{D}_\beta$) and ($\mathbf{R}$) then according to Theorem 4.2 in [4], ($\mathbf{P}$, $++$) is an abelian group. Defining the product of a "scalar" $\varkappa \in \boldsymbol{J}'(\boldsymbol{J}' = \boldsymbol{J} \cup \{\beta\} = \boldsymbol{I})$ and an element $(\xi, y) \in \mathbf{P}$ by the formula $(\xi, y) \,.\, \varkappa = (\xi \circ \varkappa, (y)\, \sigma_\varkappa)$ we establish that ($\mathbf{P}$, $++$) has, moreover, the structure of a vector space over the skewfield $(\boldsymbol{J}', +_\circ, \circ, \beta, \gamma)$. The axioms of the vector space may be verified without troubles. Moreover, if m denotes the dimension of the vector space $(S, +)$ over the skewfield $(\boldsymbol{J}', +_\circ, \circ, \beta, \gamma)$ from Theorem 5.4 in [4] then the dimension of the vector space ($\mathbf{P}$, $++$) is $m + 1$. Thus we have proved

Theorem 2.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties* ($\mathbf{rs}$), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$), ($\mathbf{dg}$) *and* ($\mathbf{R}$). *Let m denote the dimension of the vector space $(S, +)$ over $(\boldsymbol{J}', +_\circ, \circ, \beta, \gamma)$. Then the group* ($\mathbf{P}$, $++$) *of all proper points of $\mathscr{A}'$ is a vector space over the skewfied $(\boldsymbol{J}', +_\circ, \circ, \beta, \gamma)$.*

Let the extension $\mathscr{A}'$ have a degree $\geqq 4$ and let it posses the properties ($\boldsymbol{rs}$), ($\mathbf{D}_\infty$) and ($\mathbf{dg}$). We will determine the *partition $\mathscr{P}$ of the group* ($\mathbf{P}$, $++$) of all proper points of $\mathscr{A}'$. By a partition of a non-trivial group ($\mathbf{G}$, $+$) with the neutral element $\mathbf{o}$ we mean a system $\mathscr{G}$ of proper non-trivial subgroups of the group ($\mathbf{G}$, $+$) such that any element of ($\mathbf{G}$, $+$) different from the neutral element $\mathbf{o}$ belongs to exactly one subgroup of $\mathscr{G}$. The subgroups of ($\mathbf{G}$, $+$) are the so-called *components of the partition $\mathscr{G}$.*

Let us denote by $\mathbf{P}_\mathbf{L}$ the set $\{(\beta, x) \mid x \in S\}$ of all proper points of the principal line $\mathbf{h}_\beta \in \mathscr{L}$. Further, let

$$\mathbf{P}_{(k)} = \{(\xi, y) \mid \xi \in \boldsymbol{I},\; y \in S,\; (k)\, \sigma_\xi + y = 0\}$$

be the set of all proper points of the line $\mathbf{g} = \{k, 0\} \in \mathscr{L}$. It may be easily shown that ($\mathbf{P}_\mathbf{L}$, $++$) as well as ($\mathbf{P}_{(k)}$, $++$), $k \in S$ are proper subgroups of ($\mathbf{P}$, $++$). The point – the element $(\xi, y) \in \mathbf{P}$ – belongs to the subgroup ($\mathbf{P}_\mathbf{L}$, $++$) iff $\xi = \beta$. The element $(\xi, y) \in \mathbf{P}$ is contained in a subgroup ($\mathbf{P}_{(k)}$, $++$), $k \in S$ iff $y = -(k)\, \sigma_\xi$. In the sequel, let us put

$$\mathscr{P} = \{(\mathbf{P}_\mathbf{L}, ++)\} \cup \{(\mathbf{P}_{(k)}, ++) \mid k \in S\}\,. \tag{5}$$

Theorem 2.3. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties* ($\mathbf{rs}$), ($\mathbf{D}_\infty$) *and* ($\mathbf{dg}$). *Then the set $\mathscr{P}$ defined by* (5) *is a partition of the group* ($\mathbf{P}$, $++$) *of all proper points of the structure $\mathscr{A}'$. The left cosets of the decomposition of* ($\mathbf{P}$, $++$) *with respect to the components of the partition $\mathscr{P}$ are exactly all the proper lines (without improper points) of $\mathscr{A}'$.*

Proof. The theorem follows from Theorem 2.1, from the considerations before Theorem 2.3 and from Theorem 1.3 in [2]. Let $\mathbf{U} = (\mathbf{P}_{(k)}, ++)$, $k \in S$ be a component of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ defined by (5) and let $(\xi_0, y_0) \in \mathbf{P}$ be a proper point of $\mathscr{A}'$. Then the left coset of the decomposition of $(\mathbf{P}, ++)$ with respect to $\mathbf{U}$

$$\mathbf{U} ++ (\xi_0, y_0) = \{(\xi, -(k)\,\sigma_\xi) \mid \xi \in \boldsymbol{I}\} ++ (\xi_0, y_0) =$$
$$= \{(\xi_0 + \xi, -(k)\,\sigma_\xi + y_0) \mid \xi \in \boldsymbol{I}\}$$

is an ordinary line $\{k, q\} \in \mathscr{L}$, $q = (k)\,\sigma_{\xi_0} + y_0$ (without the improper point (k)). ∎

Remark 2.1. Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ defined by (5) consists of the vector space $(S, +)$ over the skewfield $(\boldsymbol{J}', +_\circ, \circ, \beta, \gamma)$ and of all one-dimensional subspaces of the vector space $(\mathbf{P}, ++)$ over $(\boldsymbol{J}', +_\circ, \circ, \beta, \gamma)$ which are not subspaces of $(S, +)$.

3. AUTOMORPHISMS OF THE GROUPS $(S, +)$ AND $(J', +_\circ)$

Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ a degree $\geqq 3$ and let it possess the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$ and $(\mathbf{R})$. Let $\Sigma^* = \{\sigma_\iota \mid \iota \in \boldsymbol{J}\}$. According to Theorem 5.2 in [4], $(\Sigma^*, \ )$ is a group with the neutral element σ_γ. It follows from Theorem 3.2 in [4] that any permutation σ_ι, $\iota \in \boldsymbol{J}$ is an automorphism of the group $(S, +)$. Hence $(\Sigma^*, \ )$ is the group of automorphisms of $(S, +)$.

We will denote by $\mathbf{Aut}(S, +)$ the group of all automorphisms of the group $(S, +)$. Evidently $\Sigma^* \subseteq \mathbf{Aut}(S, +)$. Let us denote by $\mathbf{Z}(\Sigma^*)$ the centralizer $\{\varphi \in \mathbf{Aut}(S, +) \mid \varphi\sigma_\iota = \sigma_\iota\varphi,\ \forall\sigma_\iota \in \Sigma^*\}$ of the set Σ^* in the group $\mathbf{Aut}(S, +)$, i.e. the set of all automorphisms $\varphi \in \mathbf{Aut}(S, +)$ commuting with any automorphism $\sigma_\iota \in \Sigma^*$. The next lemma is well known from the group theory (see e.g. [3]).

Lemma 3.1. *The centralizer* $\mathbf{Z}(\Sigma^*)$ *of the set* Σ^* *in the group* $\mathbf{Aut}(S, +)$ *is a subgroup of the group* $\mathbf{Aut}(S, +)$.

Let $\mathrm{N}(\Sigma^*)$ denote the normalizer $\{\psi \in \mathbf{Aut}(S, +) \mid \psi^{-1}\sigma_\iota\psi \in \Sigma^*,\ \forall\sigma_\iota \in \Sigma^*\}$ of the set Σ^* in the group $\mathbf{Aut}(S, +)$, i.e. the set of all automorphisms $\psi \in \mathbf{Aut}(S, +)$ for which $\psi^{-1}\sigma_\iota\psi \in \Sigma^*$, $\forall\sigma_\iota \in \Sigma^*$ is true. The next result is again well known from the group theory (see e.g. [3]).

Lemma 3.2. *The normalizer* $\mathrm{N}(\Sigma^*)$ *of the set* Σ^* *in the group* $\mathbf{Aut}(S, +)$ *is a subgroup of the group* $\mathbf{Aut}(S, +)$. *Moreover, the groups* $(\Sigma^*, \ )$, $(\mathbf{Z}(\Sigma^*), \ )$ *and* $(\mathbf{Z}(\Sigma^*) \,.\, \Sigma^*, \ )$ *are subgroups of the group* $(\mathrm{N}(\Sigma^*), \ )$ *where* $\mathbf{Z}(\Sigma^*) \,.\, \Sigma^* = \{\varphi\sigma_\iota \mid \varphi \in \mathbf{Z}(\Sigma^*), \sigma_\iota \in \Sigma^*\}$.

Now let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it possess the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$, $(\mathbf{R})$. By virtue of the results from Sections 4 and 5 in [4], $(J', +_\circ)$ is an abelian group (cf. Theorem 4.2 in [4]) and $(J, \circ)$ is a group (cf. Theorem 5.2 in [4]). Let us define the mapping R_ι for each index ι, $\iota \in J$ by

$$(6) \qquad J' \to J', \quad \xi \mapsto (\xi)\,R_\iota = \xi \circ \iota\,.$$

Lemma 3.3. *Each mapping* R_ι, $\iota \in J$ *is an automorphism of the group* $(J', +_\circ)$.

Proof. According to part a) of Lemma 5.1 in [4] and the relation (13) in [4], $\beta \circ \iota = \beta$ holds for any index $\iota \in J$. Since $(J, \circ)$ is a group, each mapping R_ι, $\iota \in J$, is a permutation of J', $J' = J \cup \{\beta\} = I$. Finally, part b) of Lemma 5.4 in [4] implies that for any two indices $\xi, \eta \in J'$ we have

$$(\xi +_\circ \eta)\,R_\iota = (\xi +_\circ \eta) \circ \iota = \xi \circ \iota +_\circ \eta \circ \iota = (\xi)\,R_\iota +_\circ (\eta)\,R_\iota\,. \quad \blacksquare$$

Analogously we can define the mapping L_ι by

$$(6') \qquad J' \to J', \quad \xi \mapsto (\xi)\,L_\iota = \iota \circ \xi$$

and prove

Lemma 3.4 *Each mapping* L_ι, $\iota \in J$ *is an automorphism of the group* $(J', +_\circ)$.

4. THE EQUIVALENCE AND THE AUTOMORPHISM OF THE PARTITION $\mathscr{P}$ OF THE GROUP $(\mathbf{P}, ++)$

In [5] W. Seier defined an *equivalence of an partition* $\mathscr{G}$ of the group $(\mathbf{G}, +)$ as a permutation $\mathbf{f}$ of the set $\mathbf{G}$ fulfilling the following conditions:

(a) If $\mathbf{U}$ is a component of the partition $\mathscr{G}$, then $(\mathbf{U})\,\mathbf{f}$ is a component, too;

(b) $(\mathbf{U} + x)\,\mathbf{f} = (\mathbf{U})\,\mathbf{f} + (x)\,\mathbf{f}$ is true for each element $x \in \mathbf{G}$ and for each component $\mathbf{U} \in \mathscr{G}$.

He regarded the automorphism $\mathbf{f}$ of the group $(\mathbf{G}, +)$ as an *automorphism of the partition* $\mathscr{G}$ of $(\mathbf{G}, +)$, if this automorphism was simultaneously an equivalence of the partition $\mathscr{G}$. He determined the group of all collineations of a translation structure associated to the group partition $\mathscr{G}$.

Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\boldsymbol{dg})$, $(\mathbf{R})$. We shall determine some automorphisms of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ of all proper points of the structure $\mathscr{A}'$ ($\mathscr{P}$ is determined in (5)). As $\mathbf{P} = \{(\xi, y) \mid \xi \in I,\ y \in S\}$, any ordered pair $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of automorphisms φ_1 of the group $(J', +_\circ)$ and φ_2 of the group $(S, +)$ is an automorphism of the group $(\mathbf{P}, ++)$, which means: if $\mathbf{f}_0 = (\varphi_1, \varphi_2)$, $\varphi_1 \in \mathbf{Aut}(J', +_\circ)$, $\varphi_2 \in \mathbf{Aut}(S, +)$ then $\mathbf{f}_0 \in \mathbf{Aut}(\mathbf{P}, ++)$.

We shall establish the condition under which the automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of $(\mathbf{P}, ++)$ is an equivalence of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$.

Let $\mathbf{U} = (\mathbf{P}_{(k)}, ++)$, $k \in S$, be an arbitrary component of the partition $\mathscr{P}$ and let $(\xi, y) \in \mathbf{P}$ be some of its points different from $(\beta, 0)$ (i.e. $y = -(k)\,\sigma_\varepsilon$, $\xi \neq \beta$).

Applying the automorphism $\mathbf{f}_0$ of $(\mathbf{P}, ++)$ we get $(\xi, y)\,\mathbf{f}_0 = (\xi, -(k)\,\sigma_\xi)\,\mathbf{f}_0 = ((\xi)\,\varphi_1, -(k)\,\sigma_\xi\varphi_2)$ as well as $(\boldsymbol{U})\,\mathbf{f}_0 = (\mathbf{P}_{(k')}, ++)$, $k' \in S$. The automorphism $\mathbf{f}_0$ becomes an equivalence of the partition $\mathscr{P}$ iff it possesses the properties (a), (b). According to (a) we obtain

$(\xi, y)\,\mathbf{f}_0 \in (\boldsymbol{U})\,\mathbf{f}_0$, i.e. $(k')\,\sigma_{(\xi)\varphi_1} = (k)\,\sigma_\xi\varphi_2$ for any element $k \in S$. With respect to (b), Construction 2.1 in [4] and Definition 2.3 in [4], for any $k \in S$ we necessarily have

$$(k')\,\sigma_{(\gamma)\varphi_1} = (k)\,\varphi_2\,, \quad \text{i.e.} \quad k' = (k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\,.$$

This equality and the previous one give

$$(k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\xi)\varphi_1} = (k)\,\sigma\varphi_2\,.$$

The last equality holds for any $k \in S$, hence

$$\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\xi)\varphi_1} = \sigma_\xi\varphi_2$$

and therefore

$$\sigma_{(\xi)\varphi_1} = \sigma_{(\gamma)\varphi_1}\varphi_2^{-1}\sigma_\xi\varphi_2\,. \tag{7}$$

Using the results of the previous section we get that the automorphism φ_2 must belong to the normalizer $\mathrm{N}(\Sigma^*)$ of the set Σ^* in the group $\mathbf{Aut}(S, +)$.

Consequently, the automorphism φ_1 of the group $(\boldsymbol{J}', +_\circ)$ is uniquely determined by (7) and by the element $(\gamma)\,\varphi_1 = \lambda$, $\lambda \in \boldsymbol{J}$. In fact, the automorphism φ_2 of the group $(S, +)$, $\varphi_2 \in \mathrm{N}(\Sigma^*)$, uniquely determines the permutation φ_1, $\varphi_1\colon \boldsymbol{J}' \to \boldsymbol{J}'$, $\xi \mapsto (\xi)\,\varphi_1$, fixing the element β and sending γ to λ. Moreover, (7) is fulfilled. The permutation φ_1 is an automorphism of the group $(\boldsymbol{J}', +_\circ)$ since for any $k \in S$ we have

$$(k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\xi+_\circ\eta)\varphi_1} = (k)\,\sigma_{\xi+_\circ\eta}\varphi_2 = (k)\,\sigma_\xi\varphi_2 + (k)\,\sigma_\eta\varphi_2 =$$
$$= (k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\xi)\varphi_1} + (k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\eta)\varphi_1} = (k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\xi)\varphi_1+_\circ(\eta)\varphi_1}\,.$$

In this calculation we have used the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$, $(\boldsymbol{D}_\beta)$, $(\boldsymbol{dg})$, $(\mathbf{R})$ of the structure $\mathscr{A}'$, the relations (9) and (7) from [4] and the assumption $\varphi_2 \in \mathrm{N}(\Sigma^*)$.

Since any point $(\beta, y) \in \mathbf{P}_\mathbf{L}$ satisfies

$$(\beta, y)\,\mathbf{f}_0 = ((\beta)\,\varphi_1\,, \ (y)\,\varphi_2) = (\beta, (y)\,\varphi_2)\,,$$

β is a neutral element of the group $(\boldsymbol{J}', +_\circ)$ and φ_1 is an automorphism of this group, hence $(\mathbf{U})\,\mathbf{f}_0 = \boldsymbol{U}$ is true for component $\mathbf{U} = (\mathbf{P}_\mathbf{L}, ++)$ of the partition $\mathscr{P}$. This means that the condition (a) of the eqiuvalence $\mathbf{f}_0$ is fulfilled iff (7) is true.

It remains to verify whether the automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$, $\varphi_2 \in \mathrm{N}(\Sigma^*)$ of the group $(\mathbf{P}, ++)$ fulfilling (7) satisfies also the condition (b) of the definition of the equivalence. Let $\mathbf{U} = (\mathbf{P}_{(k)}, ++)$, $k \in S$, be an arbitrary component of $\mathscr{P}$ and let $(\mathbf{U})\,\mathbf{f}_0 = (\mathbf{P}_{(k')}, ++)$, $k' \in S$ be its image under the automorphism $\mathbf{f}_0$, i.e. $k' = (k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}$. Further, let $(\eta_0, y_0) \in \mathbf{P}$ be an arbitrary proper point and $(\eta_0, y_0)\,\mathbf{f}_0 = ((\eta_0)\,\varphi_1, (y_0)\,\varphi_2)$ its image under the automorphism $\mathbf{f}_0$. Then

$$\mathbf{U} ++ (\eta_0, y_0) = \{(\eta_0 +_\circ \xi, -(k)\,\sigma_\xi + y_0) \mid \xi \in J'\};$$

$$(\mathbf{U})\,\mathbf{f}_0 = \{((\xi)\,\varphi_1, -(k')\,\sigma_{(\xi)\varphi_1}) \mid \xi \in J'\} = \{((\xi)\,\varphi_1, -(k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\sigma_{(\xi)\varphi_1}) \mid \xi \in J'\},$$

and also

$$\sigma_{(\xi)\varphi_1} = \sigma_{(\gamma)\varphi_1}\varphi_2^{-1}\sigma_\xi\varphi_2 .$$

Hence

$$(\mathbf{U})\,\mathbf{f}_0 = \{((\xi)\,\varphi_1, -(k)\,\sigma_\xi\varphi_2) \mid \xi \in J'\};$$

$$(\mathbf{U})\,\mathbf{f}_0 ++ (\eta_0, y_0)\,\mathbf{f}_0 = \{(\eta_0)\,\varphi_1 +_\circ (\xi)\,\varphi_1, -(k)\,\sigma_\xi\varphi_2 + (y_0)\,\varphi_2) \mid \xi \in J'\} =$$
$$= \{((\eta_0 +_\circ \xi)\,\varphi_1, (-(k)\,\sigma_\xi + y_0)\,\varphi_2 \mid \xi \in J'\} = (\mathbf{U} ++ (\eta_0, y_0))\,\mathbf{f}_0 .$$

In the case $\mathbf{U} = (\mathbf{P_L}, ++)$ the previous considerations become simpler. Thus we have proved

Theorem 4.1. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\boldsymbol{rs}), (\mathbf{D}_\infty), (\mathbf{D}_\beta), (\mathbf{dg}), (\mathbf{R})$. Then the automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$, $\varphi_1 \in \mathbf{Aut}(J', +_\circ)$, $\varphi_2 \in \mathbf{Aut}(S, +)$ of the group $(\mathbf{P}, ++)$ is simultaneously an automorphism of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ defined by (5) if and only if $\varphi_2 \in \mathrm{N}(\Sigma^*)$ and φ_1 is determined by (7).*

Remark 4.1. The set of all translations of the structure $\mathscr{A}'$ from Theorem 4.1 is the support of a group $(\mathbf{T}, \,)$ where $\mathbf{T} = \mathbf{T}'$ (see Theorem 1.3). According to the remarks made in Introduction the group $(\mathbf{T}, \,)$ is a normal subgroup of the group $(\mathbf{\Gamma}, \,)$ of all collineations of $\mathscr{A}'$. By virtue of Theorem 2.4 and Lemma 2.5 in [5] the set of all automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$ from Theorem 4.1 coincides with the set of all equivalences of the partition $\mathscr{P}$ in which the component $(\mathbf{P_L}, ++)$ and $(\mathbf{P}_{(0)}, ++)$ maps onto itselves, respectively.

5. THE GROUP OF ALL COLLINEATIONS OF THE EXTENSION $\mathscr{A}'$ OF AN ANTI-NET $\mathscr{A}$ FIXING THE IMPROPER POINT (0)

Let $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota \in I \cup \{\infty\}})$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties $(\boldsymbol{rs}), (\mathbf{D}_\infty), (\mathbf{D}_\beta), (\mathbf{dg})$ and $(\mathbf{R})$. In accordance with Theorem 1.3 all translations of the structure $\mathscr{A}'$ form a group $(\mathbf{T}, \,)$ where $\mathbf{T} = \mathbf{T}'$ denotes the set of all translations defined by (3). According to the remarks in Introduction the group $(\mathbf{T}, \,)$ is a normal subgroup of the group $(\mathbf{\Gamma}, \,)$ of all collineations of the structure $\mathscr{A}'$.

The relation (4) defines an operation "+" onto the set $\mathbf{P}$ of all proper points of the structure $\mathscr{A}'$. In virtue of Theorem 2.1, $(\mathbf{P}, ++)$ is a group isomorphic to the translation group $(\mathbf{T}', \,)$. The set of all proper points of lines of $\mathscr{A}'$ going through the point $(\beta, 0) \in \mathbf{P}$ together with the operation "++" is a proper subgroup of the group $(\mathbf{P}, ++)$. Theorem 2.3 implies that all such subgroups form the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ defined by (5).

Any equivalence $\mathbf{f}_0$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ – cf. Sec. 4 – is a collineation of $\mathscr{A}'$ fixing the point $(\beta, 0) \in \mathbf{P}$ (as follows from Theorem 2.3 in [5]), and obviously also the point $\mathsf{L} \in \mathbf{h}_\infty$. All collineations of the structure $\mathscr{A}'$ which fix the points $(\beta, 0) \in \mathbf{P}$ and $\mathsf{L} \in \mathbf{h}_\infty$ form a group – a subgroup of $(\Gamma, \ )$. Any automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$ described in Theorem 4.1 is a collineation of $\mathscr{A}'$ fixing the points $(\beta, 0) \in \mathbf{P}$, $\mathsf{L} \in \mathbf{h}_\infty$ and $(0) \in \mathbf{h}_\infty$. Conversely, any collineation of the structure $\mathscr{A}'$ with fixed points $(\beta, 0) \in \mathbf{P}$, $\mathsf{L} \in \mathbf{h}_\infty$ and $(0) \in \mathbf{h}_\infty$ is an automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$. Let us denote by $\mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ the group of all automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$. According to Theorem 4.1 we have

$$\text{(8)} \qquad \mathbf{Aut}_0(\mathscr{P}, \mathbf{P}) = \{\mathbf{f}_0 = (\varphi_1, \varphi_2) \mid \varphi_1 \in \mathbf{Aut}(J', +_\circ),\ \varphi_2 \in \mathrm{N}(\Sigma^*)\,, \\ \sigma_{(\xi)\varphi_1} = \sigma_{(\gamma)\varphi_1}\varphi_2^{-1}\sigma_\xi\varphi_2\}\,.$$

Let $\Gamma_{(0)}$ denote the set of all collineations of the structure $\mathscr{A}'$ fixing the improper points L, $(0) \in \mathbf{h}_\infty$. It may be easily verified that the translation group $(\mathbf{T}', \ )$ of $\mathscr{A}'$ is a normal subgroup of the group $(\Gamma_{(0)}, \ )$. Part c) of Theorem 2.4 in [5] gives

$$\Gamma_{(0)} = \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})\,.\,\mathbf{T}'\,;$$

i.e., the group $(\Gamma_{(0)}, \ )$ of all collineations of $\mathscr{A}'$ fixing the points L, $(0) \in \mathbf{h}_\infty$ is the subdirect product of the group $\mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ and the group $(\mathbf{T}', \ )$. (The notion of the subdirect product of groups is known from the group theory, see e.g. [3].) Summarizing these results we get

Theorem 5.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then any collineation $\delta \in \Gamma_{(0)}$ of $\mathscr{A}'$ with fixed improper points L, $(0) \in \mathbf{h}_\infty$ may be uniquely expressed in the form*

$$\delta = \mathbf{f}_0\tau$$

where $\mathbf{f}_0 = (\varphi_1, \varphi_2) \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ is an automorphism of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$, $\mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ is given in (8) and $\tau \in \mathbf{T}'$ is a translation defined by (3).

Proof. The correctness of the assertion of the theorem follows from the previous consideration and some results in [5]. ∎

6. HOMOTHETIES OF AN ANTI-NET

According to remarks made in Introduction, any homothety of the extension $\mathscr{A}'$ of the anti-net $\mathscr{A}$ is an axial collineation with the improper axis $\mathbf{h}_\infty$ and the proper centre Z. We may consider the homothety with the centre $\mathsf{Z} = (\beta, 0)$ of $\mathscr{A}'$ also as an equivalence of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ mapping any component of $\mathscr{P}$ onto itself.

Any component $\mathbf{U} = (\mathbf{P}_{(k)}, ++)$ of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$ is mapped onto itself by the equivalence $\mathbf{f}_0 = (\varphi_1, \varphi_2)$, $\varphi_1 \in \mathbf{Aut}(\boldsymbol{J}', +_\circ)$, $\varphi_2 \in \mathbf{N}(\Sigma^*)$, i.e.

$$(\mathbf{U})\,\mathbf{f}_0 = (\mathbf{P}_{(k')}, ++) = (\mathbf{P}_{(k)}, ++) = \mathbf{U}$$

if and only if $k' = k$, othervise

$$(k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1} = k\,, \quad \text{i.e.} \quad \varphi_2 = \sigma_{(\gamma)\varphi_1}\,.$$

Using (7) we get

$$\sigma_{(\xi)\varphi_1} = \sigma_{(\gamma)\varphi_1}\varphi_2^{-1}\sigma_\xi\varphi_2 = \sigma_\xi\varphi_2 = \sigma_\xi\sigma_{(\gamma)\varphi_1} = \sigma_{\xi\circ(\gamma)\varphi_1}\,,$$

therefore $(\xi)\,\varphi_1 = \xi \circ (\gamma)\,\varphi_1$. If for example $(\gamma)\,\varphi_1 = \lambda$, then

$$(\xi)\,\varphi_1 = \xi \circ \lambda = (\xi)\,\mathrm{R}_\lambda\,, \quad \varphi_2 = \sigma_\lambda\,.$$

If the element $(\gamma)\,\varphi_1 = \lambda$ runs through all elements of the index set $\boldsymbol{J}$, we get in this way all homotheties of $\mathscr{A}'$ with the centre $\mathrm{Z} = (\beta, 0)$. The relevant transformation formulas for these homotheties $\varkappa_\lambda$, $\lambda \in \boldsymbol{J}$ have the form

$$(9)\qquad \begin{array}{lll}
(\xi, y) \mapsto (\xi \circ \lambda, (y)\,\sigma_\lambda) & \forall \xi \in \boldsymbol{I}, & \forall y \in \mathrm{S}\,; \\
(k) \mapsto (k) & \forall k \in \mathrm{S}\,; & \\
\mathsf{L} \mapsto \mathsf{L}\,; & & \\
\{k, q\} \mapsto \{k, (q)\,\sigma_\lambda\} & \forall k, q \in \mathrm{S}\,; & \\
\mathbf{h}_\beta \mapsto \mathbf{h}_\beta\,; & & \\
\mathbf{h}_\infty \mapsto \mathbf{h}_\infty\,; & & \\
\mathbf{h}_\iota \mapsto \mathbf{h}_{\iota\circ\lambda} & \forall_\iota \in \boldsymbol{J}\,. &
\end{array}$$

All homotheties $\varkappa_\lambda$, $\lambda \in \boldsymbol{J}$ determined by (9) form a group $(\boldsymbol{\Pi}_{(\beta,0)},\ )$ with the neutral element $\varkappa_\gamma = \boldsymbol{\varepsilon}$. Thus we have proved the following theorem:

Theorem 6.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then any mapping $\varkappa_\lambda$, $\lambda \in \boldsymbol{J}$ defined by (9) is a homothety of the structure $\mathscr{A}'$ with the centre $(\beta, 0)$. The relations (9) determine all homotheties of $\mathscr{A}'$ with the centre $(\beta, 0)$. These homotheties form a group $(\boldsymbol{\Pi}_{(\beta,0)},\ )$ with the neutral element $\boldsymbol{\varepsilon}$.*

Remark 6.1. Any homothety $\varkappa$ of the structure $\mathscr{A}'$ with the centre $\mathrm{Z} = (\iota, x) \in \mathbf{P}$ may be uniquely expressed as the product

$$\varkappa = \boldsymbol{\tau}^{-1}_{(\iota,x)}\varkappa_\lambda\boldsymbol{\tau}_{(\iota,x)}$$

where $\boldsymbol{\tau}_{(\iota,x)} \in \mathbf{T}'$ and $\varkappa_\lambda$, $\lambda \in \boldsymbol{J}$ is a certain homothety from $\boldsymbol{\Pi}_{(\beta,0)}$. Hence the set of all homotheties of the structure $\mathscr{A}'$ with the centre Z may be described by

$$\boldsymbol{\Pi}_{(\iota,x)} = \boldsymbol{\tau}^{-1}_{(\iota,x)}\boldsymbol{\Pi}_{(\beta,0)}\boldsymbol{\tau}_{(\iota,x)}\,.$$

Let the extension $\mathscr{A}' = (\mathbf{P} \cup \mathbf{h}_\infty, \mathscr{L} \cup \{\mathbf{h}_\infty\}, (\mathbf{h}_\iota)_{\iota\in I\cup\{\infty\}})$ of an anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it possess the properties (***rs***), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$), (**dg**) and (**R**). Theorem 4.1 determines all automorphisms $\mathbf{f}_0$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ of $\mathscr{A}'$. If, moreover, the automorphism $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ fulfils the condition $\varphi_2 = \mathrm{id}_S$ then according to (7),

$$\sigma_{(\xi)\varphi_1} = \sigma_{(\gamma)\varphi_1}\sigma_\xi = \sigma_{(\gamma)\varphi_1\circ\xi},$$

i.e. $(\xi)\,\varphi_1 = (\gamma)\,\varphi_1 \circ \xi$ is true for any element $\xi \in \boldsymbol{J}'$. If for instance $(\gamma)\,\varphi_1 = \lambda$, $\lambda \in \boldsymbol{J}$ then $(\xi)\,\varphi_1 = \lambda \circ \xi = (\xi)\,\mathrm{L}_\lambda$.

We will denote by $\boldsymbol{\alpha}_{(\lambda,0)}$ the automorphism $\mathbf{f}_0$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ with this property. We easily verify by direct calculation that the transformation formulas for the mapping $\boldsymbol{\alpha}_{(\lambda,0)}$ have the form

$$(10)\qquad \begin{array}{lll} (\xi, y) & \mapsto (\lambda \circ \xi, y) & \forall \xi \in \boldsymbol{J}, \quad \forall y \in S\,; \\ (\beta, y) & \mapsto (\beta, y) & \forall k \in S\,; \\ (k) & \mapsto ((k)\,\sigma_\lambda^{-1}) & \forall k \in S\,; \\ \mathsf{L} & \mapsto \mathsf{L}\,; & \\ \{k, q\} & \mapsto \{(k)\,\sigma_\lambda^{-1}, q\} & \forall k, q \in S\,; \\ \mathbf{h}_\beta & \mapsto \mathbf{h}_\beta\,; & \\ \mathbf{h}_\infty & \mapsto \mathbf{h}_\infty\,; & \\ \mathbf{h}_\iota & \mapsto \mathbf{h}_{\lambda\circ\iota} & \forall \iota \in \boldsymbol{J}\,. \end{array}$$

It is immediately seen from (10) that any mapping $\boldsymbol{\alpha}_{(\lambda,0)}$, $\lambda \in \boldsymbol{J}$ fixes any point of the principal line $\mathbf{h}_\beta$ and that it reproduces any line $\{0, q\} \in \mathscr{L}$, $q \in S$. Thus any mapping $\boldsymbol{\alpha}_{(\lambda,0)}$, $\lambda \in \boldsymbol{J}$ determined by (10) is an axial collineation with the axis $\mathbf{h}_\beta$ and simultaneously a central collineation of $\mathscr{A}'$ with the centre $(0) \in \mathbf{h}_\infty$ — a perspective affinity of $\mathscr{A}'$. The image of the point $(\gamma, 0) \in \mathbf{P}$ under the perspective affinity $\boldsymbol{\alpha}_{(\lambda,0)}$, $\lambda \in \boldsymbol{J}$ is the point $(\lambda, 0) \in \mathbf{P}$.

Let us consider the set $\mathrm{A}_{(0)}$ of all perspective affinities $\boldsymbol{\alpha}_{(\lambda,0)}$, $\lambda \in \boldsymbol{J}$ determined by the relations (10). Let $\boldsymbol{\alpha}_{(\lambda,0)}$, $\boldsymbol{\alpha}_{(\mu,0)}$ be two elements of $\mathrm{A}_{(0)}$, then their composition

$$\boldsymbol{\alpha}_{(\lambda,0)}\boldsymbol{\alpha}_{(\mu,0)} = \boldsymbol{\alpha}_{(\mu\circ\lambda,0)} \in \mathrm{A}_{(0)}$$

is a perspective affinity. Evidently the affinity $\boldsymbol{\alpha}_{(\gamma,0)} = \boldsymbol{\varepsilon}$ is the neutral element of $\mathrm{A}_{(0)}$. Finally, the inverse element to the perspective affinity $\boldsymbol{\alpha}_{(\lambda,0)} \in \mathrm{A}_{(0)}$ is the perspective affinity $\boldsymbol{\alpha}_{(\lambda^{-1},0)} \in \mathrm{A}_{(0)}$. Therefore the set $\mathrm{A}_{(0)}$ together with the composition of mappings is a group $(\mathrm{A}_{(0)}, \,)$.

Lemma 7.1. *Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties (**rs**), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$), (**dg**) and (**R**). Then any mapping $\boldsymbol{\alpha}_{(\lambda,0)}$, $\lambda \in \boldsymbol{J}$ defined by (10) is a perspective affinity of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta \in \mathscr{L}$ and with the centre*

$(0) \in \mathbf{h}_\infty$. *All such perspective affinities form a group* $(\mathrm{A}_{(0)},)$ *with the group operation "the composition of mappings" and with the neutral element* $\alpha_{(\gamma,0)}$. *The relations* (10) *define exactly all perspective affinities of the structure* $\mathscr{A}'$ *with the axis* $\mathbf{h}_\beta$ *and the centre* $(0) \in \mathbf{h}_\infty$.

Proof. We shall only prove the last assertion of the lemma. The other assertions were established by the considerations before Lemma 7.1. Thus, let α be a non-identical perspective affinity of the structure $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $(0) \in \mathbf{h}_\infty$. Then α fulfils the second, fourth, sixth and seventh relations from (10). According to the remarks made in Introduction the perspective affinity α is uniquely determined by its action on one proper point which does not lie on the axis $\mathbf{h}_\beta$. Let us suppose that $(\gamma, 0)^\alpha = (\lambda, 0)$, $\lambda \in \boldsymbol{J} \setminus \{\gamma\}$. It is a routine matter to verify that α fulfils also the other relations of (10). Hence any non-identical perspective affinity α is a mapping $\alpha_{(\lambda,0)}$ for some $\lambda \in \boldsymbol{J} \setminus \{\gamma\}$. If α is the identical collineation then $\alpha = \alpha_{(\gamma,0)}$. ■

Now let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have a degree $\geqq 3$ and let it possess the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$ and $(\mathbf{D}_\beta)$. Our aim is to describe all perspective affinities of this extension $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} = \mathbf{h}_\beta \cap \mathbf{h}_\infty$. It has been repeated several times in this work that any perspective affinity of $\mathscr{A}'$ is uniquely determined by its action on one proper point of $\mathscr{A}'$ that does not lie on the axis of this affinity. Let $\alpha_{(\gamma,a)}$, $a \in S \setminus \{0\}$ denote the non-identical perspective affinity of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta \in \mathscr{L}$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ sending the point $(\gamma, 0) \in \mathbf{P}$ onto the point $(\gamma, a) \in \mathbf{P}$. The perspective affinity $\alpha_{(\gamma,a)}$ maps any line $\{k, k\} \in \mathscr{L}$, $k \in S$ going through the points $(\gamma, 0)$ and $(\beta, k) \in \mathbf{h}_\beta$ onto the line $\{k', k\}$, $k' \in S$ containing the points (γ, a) and $(\beta, k) \in \mathbf{h}_\beta$. Therefore, according to Lemma 2.1 in [4] we get

$$(k')\,\sigma_\gamma + a = k \quad \text{i.e.} \quad k' + a = k \quad \text{so that} \quad k' = k - a\,.$$

If the point $(\xi, y) \in \mathbf{P}$ lies on the line $\{k, k\}$, i.e. $(k)\,\sigma_\xi + y = k$, then its image $(\xi, y') \in \mathbf{P}$ in the perspective affinity $\alpha_{(\gamma,a)}$ lies on the line $\{k - a, k\}$. Hence

$$(k - a)\,\sigma_\xi + y' = k = (k)\,\sigma_\xi + y$$

and

$$y' = (a)\,\sigma_\xi + y\,.$$

Thus we have proved the transformation formulas for the perspective affinity $\alpha_{(\gamma,a)}$, $a \in S$:

$$(11) \qquad \begin{array}{lll} (\xi, y) & \mapsto (\xi, (a)\,\sigma_\xi + y) & \forall \xi \in \boldsymbol{J}, \quad \forall y \in S\,; \\ (\beta, y) & \mapsto (\beta, y) & \forall y \in S\,; \\ (k) & \mapsto (k - a) & \forall k \in S\,; \\ \mathsf{L} & \mapsto \mathsf{L}\,; & \\ \{k, q\} & \mapsto \{k - a, q\} & \forall k, q \in S\,; \\ \mathbf{h}_\iota & \mapsto \mathbf{h}_\iota & \forall \iota \in \boldsymbol{I} \cup \{\infty\}\,. \end{array}$$

Using the relations (11) we immediately obtain that for $a = 0$ the perspective affinity $\alpha_{(\gamma,0)}$ is an identical collineation.

Reversing the processes described above we may prove that any mapping $\alpha_{(\gamma,a)}$, $a \in S$ defined by (11) is a perspective affinity of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ sending the point $(\gamma, 0)$ onto the point (γ, a).

Let us denote by $\mathbf{A_L}$ the set of all perspective affinities $\alpha_{(\gamma,a)}$, $a \in S$ determined by the relations (11). The composition of two perspective affinities $\alpha_{(\gamma,a)}, \alpha_{(\gamma,b)} \in \mathbf{A_L}$ is a perspective affinity again, more exactly,

$$\alpha_{(\gamma,a)}\alpha_{(\gamma,b)} = \alpha_{(\gamma,b+a)} \in \mathbf{A_L} .$$

The neutral element of this composition is the perspective affinity $\alpha_{(\gamma,0)} \in \mathbf{A_L}$ and the inverse mapping to the perspective affinity $\alpha_{(\gamma,a)} \in \mathbf{A_L}$ is the perspective affinity $\alpha_{(\gamma,-a)} \in \mathbf{A_L}$. Thus we have proved

Lemma 7.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 3$ have the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$ and $(\mathbf{D}_\beta)$. Then any mapping $\alpha_{(\gamma,a)}$, $a \in S$ determined by the relations (11) is a perspective affinity of the structure $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$. By the relations (11) all such perspective affinities are determined. The set $\mathbf{A_L}$ of all perspective affinities together with the operation "the composition of mappings" is a group $(\mathbf{A_L}, \ )$ in which the neutral element is $\alpha_{(\gamma,0)} = \varepsilon$.*

Remark 7.1. Let us consider the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ with a degree $\geqq 3$, having the property $(\boldsymbol{rs})$. In section 1 we have proved that all possible perspective collineations with the axis $\mathbf{h}_\infty$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ – the central translations $\tau_{(\beta,a)}$, $a \in S$ – exist in $\mathscr{A}'$ if and only if $\mathscr{A}'$ has the property $(\mathbf{D}_\infty)$ (see Lemma 1.1). Similarly, all possible perspective collineations with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ – the perspective affinities $\alpha_{(\gamma,a)}$, $a \in S$ – exist in $\mathscr{A}'$ if and only if $\mathscr{A}'$ has the properties $(\mathbf{D}_\infty)$ and $(\mathbf{D}_\beta)$. Finally, it may be verified by a simple calculation that all possible perspective affinities with the axis $\mathbf{h}_\gamma$ (or with the axis $\mathbf{h}_\iota$, $\iota \in \boldsymbol{I} \cup \{\infty\}$) and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ exist in $\mathscr{A}'$ if and only if $\mathscr{A}'$ has the properties $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$ and $(\mathbf{D}_\gamma)$ ($\mathscr{A}'$ has the property $(\mathbf{D})$, respectively). About the properties $(\mathbf{D}_\gamma)$ and $(\mathbf{D})$ see Section 3 in [4].

It follows from the previous considerations that the validity of the minor Desargues condition of a certain type in $\mathscr{A}'$ is equivalent to the existence of all possible perspective collineations whose axis is the corresponding principal line and whose centre is the common point L of all principal lines.

Now let the extension $\mathscr{A}'$ of an anti-net have a degree $\geqq 4$ and let it possess the properties $(\boldsymbol{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\boldsymbol{dg})$ and $(\mathbf{R})$. Let $\alpha_{(\lambda,0)} \in \mathbf{A}_{(0)}$, $\lambda \in J$, and $\alpha_{(\gamma,a)} \in \mathbf{A_L}$, $a \in S$ be the perspective affinities with the axis $\mathbf{h}_\beta$ and with the centres $(0) \in \mathbf{h}_\infty$ and $\mathsf{L} \in \mathbf{h}_\infty$ defined by (10) and (11), respectively. Composing these mappings (in the given order) we get the mapping $\alpha_{(\lambda,0)}\alpha_{(\gamma,a)} = \alpha_{(\lambda,a)}$ in which

$$(12)\qquad \begin{array}{lll} (\xi, y) & \mapsto (\lambda \circ \xi, (a)\,\sigma_{\lambda\circ\xi} + y) & \forall \xi \in \boldsymbol{J}, \quad \forall y \in S\,; \\ (\beta, y) & \mapsto (\beta, y) & \forall y \in S\,; \\ (k) & \mapsto ((k)\,\sigma_\lambda^{-1} - a) & \forall k \in S\,; \\ \mathsf{L} & \mapsto \mathsf{L}\,; & \\ \{k, q\} & \mapsto \{(k)\,\sigma_\lambda^{-1} - a, q\} & \forall k, q \in S\,; \\ \mathbf{h}_\beta & \mapsto \mathbf{h}_\beta\,; & \\ \mathbf{h}_\infty & \mapsto \mathbf{h}_\infty\,; & \\ \mathbf{h}_\iota & \mapsto \mathbf{h}_{\lambda\circ\iota} & \forall \iota \in \boldsymbol{J} \end{array}$$

is true.

Composing the same mapping in the reversed order we get the mapping

$$\boldsymbol{\alpha}_{(\gamma,a)}\boldsymbol{\alpha}_{(\lambda,0)} = \boldsymbol{\alpha}'_{(\lambda,a)}\,;$$

such a mapping satisfies e.g. $(\xi, y) \mapsto (\lambda \circ \xi,\ (a)\,\sigma_\xi + y)$ $\forall \xi \in \boldsymbol{J}$, $\forall y \in S$. Hence $\boldsymbol{\alpha}'_{(\lambda,a)} \neq \boldsymbol{\alpha}_{(\lambda,a)}$ and consequently the composition of the given perspective affinities is not commutative.

The mapping $\boldsymbol{\alpha}_{(\lambda,a)}$, $\lambda \in \boldsymbol{J}$, $a \in S$ determined in (12) is obviously an axial collineation of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$. The improper point $(k_0) \in \mathbf{h}_\infty$, $k_0 \in S$ is a fixed point of the collineation $\boldsymbol{\alpha}_{(\lambda,a)}$ if and only if $(k_0)\,\sigma_\lambda^{-1} - a = k_0$ or

$$(13)\qquad (k_0)\,\sigma_\lambda + (a)\,\sigma_\lambda = k_0\,.$$

The axiom $(\alpha 4)$ of Definition 2.1 (the definition of the admissible algebra) implies the existence of the element $k_0 \in S$ fulfilling the relation (13). According to Construction 2.1 in [4], we may determine it as the intersection point

$$(13')\qquad k_0 = \mathbf{h}_\beta \cap (\gamma, 0)\,(\lambda, (a)\,\sigma_\lambda)\,.$$

The collineation $\boldsymbol{\alpha}_{(\lambda,a)}$ reproduces any line $\{k_0, q\} \in \mathscr{L}$, $q \in S$ so that the improper point $(k_0) \in \mathbf{h}_\infty$ is its centre. Consequently, any collineation $\boldsymbol{\alpha}_{(\lambda,a)}$, $\lambda \in \boldsymbol{J}$, $a \in S$ is a perspective affinity of the structure $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{Z} = (k_0)$. The element $k_0 \in S$ fulfilling (13) is determined by (13'). The perspective affinity $\boldsymbol{\alpha}_{(\lambda,a)}$ sends the point $(\gamma, 0) \in \mathbf{P}$ to the point $(\lambda, (a)\,\sigma_\lambda)$.

Theorem 7.1. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it posses the properties* (**rs**), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$), (**dg**) *and* (**R**). *Then the mappings $\boldsymbol{\alpha}_{(\lambda,a)}$, $\lambda \in \boldsymbol{J}$, $a \in S$ defined by the relations (12) are exactly all perspective affinities of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{Z} \in \mathbf{h}_\infty$. At the same time, we have*

a) *if $\lambda = \gamma$ then the centre of the perspective affinity $\boldsymbol{\alpha}_{(\gamma,a)}$, $a \in S$ is the improper point* L;

b) *if $a = 0$ then the centre of the perspective affinity $\boldsymbol{\alpha}_{(\lambda,0)}$, $\lambda \in \boldsymbol{J}$ is the improper point $\mathsf{Z} = (0)$;*

c) *if $a \neq 0$ and $\lambda \neq \gamma$ then the centre of the perspective affinity $\alpha_{(\lambda,a)}$, $a \in S$, $\lambda \in \boldsymbol{J}$ is the improper point $\mathsf{Z} = (k_0)$, $k_0 \in S$ determined by (13').*

Proof. We will prove that all perspective affinities of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and the centre $Z = (k_0)$ are determined by relations (12). The other statements follow from Lemma 7.1 and 7.2.

Let us consider a perspective affinity $\boldsymbol{\alpha}$ of the structure $\mathscr{A}'$ with the proper axis $\mathbf{h}_\beta$ and with the improper centre $Z = (k_0)$, $k_0 \in S \setminus \{0\}$. As any perspective collineation of $\mathscr{A}'$ with a given axis and with a given centre is uniquely determined by its action on one point different from the centre and not lying on the axis of this collineation, hence also the perspective affinity $\boldsymbol{\alpha}$ is uniquely determined e.g. by the point $(\gamma, 0)$ and its image $(\gamma, 0)^{\boldsymbol{\alpha}} = (\lambda, (a)\sigma_\lambda)$, $\lambda \in J \setminus \{\gamma\}$, $a \in S \setminus \{0\}$. The cases $\lambda = \gamma$, $a = 0$ are discussed in Lemmas 1.2, 1.1, respectively. The points $(\gamma, 0)$, $(\lambda, (a)\sigma_\lambda)$, (k_0) are collinear, thus $(k_0)\sigma_\lambda + (a)\sigma_\lambda = k_0$ is true, that is, $k_0 + a = (k_0)\sigma_\lambda^{-1}$.

The perspective affinity $\boldsymbol{\alpha}$ determined in this way sends the line $\{0, 0\} \in \mathscr{L}$ (containing the points $(\beta, 0), (\gamma, 0) \in \mathbf{P}$) onto the line $\{-a, 0\} \in \mathscr{L}$ (going through the points $(\beta, 0)$, $(\lambda, (a)\sigma_\lambda)$). For this reason the image of the improper point $(0) \in \mathbf{h}_\infty$ under $\boldsymbol{\alpha}$ is the improper point $(-a) \in \mathbf{h}_\infty$.

The perspective affinity $\boldsymbol{\alpha}_{(\gamma,-a)} \in \mathbf{A_L}$ given in (11) maps the point $(\lambda, (a)\sigma_\lambda)$ onto the point $(\lambda, (-a)\sigma_\lambda + (a)\sigma_\lambda) = (\lambda, 0)$ and sends any line $\{-a, q\} \in \mathscr{L}$, $q \in S$ onto the line $\{-a - (-a), q\} = \{0, q\}$. Therefore the image of the improper point $(-a) \in \mathbf{h}_\infty$ under the perspective affinity $\boldsymbol{\alpha}_{(\gamma,-a)}$ is the improper point $(-a - (-a)) = (0)$.

The composition of the perspective affinity $\boldsymbol{\alpha}$ with the perspective affinity $\boldsymbol{\alpha}_{(\gamma,-a)} \in \mathbf{A_L}$ (in this order) is the axial collineation $\boldsymbol{\alpha\alpha}_{(\gamma,-a)}$ of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$. Such an axial collineation maps the point $(\gamma, 0)$ to the point $(\lambda, 0)$ and reproduces any line $\{0, q\} \in \mathscr{L}$, $q \in S$. It means that $\boldsymbol{\alpha\alpha}_{(\gamma,-a)}$ is a perspective affinity of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $(0) \in \mathbf{h}_\infty$ mapping the point $(\gamma, 0)$ to the point $(\lambda, 0)$, that is

$$\boldsymbol{\alpha\alpha}_{(\gamma,-a)} = \boldsymbol{\alpha}_{(\lambda,0)} \in \mathbf{A}_{(0)} .$$

This yields

$$\boldsymbol{\alpha\alpha}_{(\gamma,-a)}\boldsymbol{\alpha}_{(\gamma,-a)}^{-1} = \boldsymbol{\alpha}_{(\lambda,0)}\boldsymbol{\alpha}_{(\gamma,-a)}^{-1} ,$$

hence

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{(\lambda,0)}\boldsymbol{\alpha}_{(\gamma,a)}$$

as $(\mathbf{A_L}, \ )$ is a group and $\boldsymbol{\alpha}_{(\gamma,-a)}^{-1} = \boldsymbol{\alpha}_{(\gamma,a)}$.

We may conclude: There exist $\lambda \in J$ and $a \in S$ such that the relations (12) hold for any perspective affinity $\boldsymbol{\alpha}$ of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$. ■

Theorem 7.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it possess the properties* $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ *and* $(\mathbf{R})$. *Then the set $\mathbf{A}_{\mathbf{h}_\beta}$ of all perspective affinities of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre Z lying on the improper principal line $\mathbf{h}_\infty$ together with the operation "the composition of mappings" is a group $(\mathbf{A}_{\mathbf{h}_\beta}, \ )$, in which the neutral element is $\boldsymbol{\varepsilon}$.*

Proof. According to Theorem 7.1 any perspective affinity α of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre Z lying on the improper principal line $\mathbf{h}_\infty$ is determined by the relations (12) for suitable $\lambda \in J$, $a \in S$, i.e. $\alpha = \alpha_{(\lambda,a)}$.

Let $\alpha_{(\lambda,a)}, \alpha_{(\mu,b)} \in A_{h_\beta}$ be two perspective affinities of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centres $Z_1 = (k_1)$, $k_1 \in S$, $Z_2 = (k_2)$, $k_2 \in S$, and let $(k_1)\sigma_\lambda^{-1} - a = k_1$, $(k_2)\sigma_\mu^{-1} - b = k_2$, respectively.

Composing such two perspective affinities (in the given order) we obtain the mapping

$$\alpha_{(\rho,c)} = \alpha_{(\lambda,a)}\alpha_{(\mu,b)} \in A_{h_\beta}$$

where $\varrho = \mu \circ \lambda$, $c = b + (a)\sigma_\mu^{-1}$, i.e. a perspective affinity from A_{h_β} again. Its centre $Z = (k_3) \in \mathbf{h}_\infty$ fulfils the equation

$$(k_3)\sigma_{\mu\circ\lambda}^{-1} - (b + (a)\sigma_\mu^{-1}) = k_3$$

or equivalently

$$(k_3)\sigma_\mu + (b)\sigma_\mu = (k_3)\sigma_\lambda^{-1} - a\,.$$

Let us remark that the element $k_3 \in S$ fulfilling the last equation is uniquely determined. This statement follows from the axiom (α4) of Definition 2.1 in [4].

The neutral element with respect to such composition is the identical collineation $\varepsilon = \alpha_{(\gamma,0)}$. The inverse element of the perspective affinity $\alpha_{(\lambda,a)} \in A_{h_\beta}$ is the perspective affinity $\alpha_{(\mu,b)} \in A_{h_\beta}$ with $\mu = \lambda^{-1}$, $b = -(a)\sigma_\lambda$. The composition of such perspective affinities of $\mathscr{A}'$ is not commutative, therefore $(A_{h_\beta},)$ is not an abelian group. ∎

Remark 7.2. Let $\mathscr{A}'$ be the extension of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties (**rs**), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$), (**dg**) and (**R**). Any perspective affinity α of $\mathscr{A}'$ with the axis $\mathbf{h}_\iota$, $\iota \in I$ and with the centre $Z = (k_0) \in \mathbf{h}_\infty$ may be uniquely expressed in the form

$$\alpha = \tau_{(\iota,0)}^{-1}\alpha_{(\lambda,a)}\tau_{(\iota,0)}$$

where $\tau_{(\iota,0)} \in T'$ and $\alpha_{(\lambda,a)}$ is a suitable perspective affinity of A_{h_β}. Thus the set A_{h_ι} of all perspective affinities of $\mathscr{A}'$ with the axis $\mathbf{h}_\iota$, $\iota \in I$ satisfies

$$A_{h_\iota} = \tau_{(\iota,0)}^{-1}A_{h_\beta}\tau_{(\iota,0)}\,.$$

8. THE GROUP OF ALL COLLINEATIONS OF AN ANTI-NET

In Section 4 we have investigated equivalences and automorphisms of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ of all proper points of the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ having the properties (**rs**), ($\mathbf{D}_\infty$), ($\mathbf{D}_\beta$), (**dg**) and (**R**). It follows from the results of Section 5 that the automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2) \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ are exactly all collineations of $\mathscr{A}'$ fixing the points $(\beta, 0) \in \mathbf{P}$, $(0) \in \mathbf{h}_\infty$ and obviously also the point $L \in \mathbf{h}_\infty$ (the point $L \in \mathbf{h}_\infty$ is a fixed point with respect to any collineation of $\mathscr{A}'$).

Now we will characterize the group $\mathbf{Aut}(\mathscr{P}, \mathbf{P})$ of all automorphisms $\mathbf{f}$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$, i.e. the group of all collineations of the structure $\mathscr{A}'$ fixing the points $(\beta, 0) \in \mathbf{P}$ and $\mathsf{L} \in \mathbf{h}_\infty$.

Lemma 8.1. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then any mapping $\alpha_{(\gamma,a)} \in \mathbf{A_L}$, $a \in S$ defined by the relations (11) – i.e. any perspective affinity of $\mathscr{A}'$ with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ – is an automorphism of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$. Moreover, $\alpha_{(\gamma,a)} \notin \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$.*

Proof. Let $a \in S$ be an arbitrary element of S and let (ξ, x), (η, y) be two arbitrary points of $\mathbf{P}$. Then the mapping $\alpha_{(\gamma,a)}$ satisfies

$$((\xi, x) ++ (\eta, y))^{\alpha(\gamma,a)} = (\eta +_\circ \xi,\ x + y)^{\alpha(\gamma,a)} =$$
$$= (\eta +_\circ \xi,\ (a)\, \sigma_{\eta +_\circ \xi} + (x + y)) = (\eta +_\circ \xi, (a)(\sigma_\eta \mathbin{\mathring{+}} \sigma_\xi) + x + y) =$$
$$= (\eta +_\circ \xi, (a)\, \sigma_\eta + (a)\, \sigma_\xi + x + y) = (\eta +_\circ \xi, (a)\, \sigma_\xi + x + (a)\, \sigma_\eta + y) =$$
$$= (\xi, (a)\, \sigma_\xi + x) ++ (\eta, (a)\, \sigma_\eta + y) = (\xi, x)^{\alpha(\gamma,a)} ++ (\eta, y)^{\alpha(\gamma,a)}$$

which means that $\alpha_{(\gamma,a)}$ is an automorphism of the group $(\mathbf{P}, ++)$.

Moreover, the mapping $\alpha_{(\gamma,a)}$ is a perspective affinity with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$. This implies that the point $(\beta, 0) \in \mathbf{h}_\beta$ is a fixed point with respect to $\alpha_{(\gamma,a)}$ and any component $\mathbf{U} = (\mathbf{P}_{(k)}, ++)$ will be sent onto the component $\mathbf{U}' = (\mathbf{P}_{(k-a)}, ++)$ under the mapping $\alpha_{(\gamma,a)}$. Thus the automorphism $\alpha_{(\gamma,a)}$, $a \in S$ fulfils the condition a) from the definition of equivalence of the partition of a group (see Sec. 4). Applying Theorem 2.3 we get that $\alpha_{(\gamma,a)}$, $a \in S$ fulfils also the condition b). Hence any mapping $\alpha_{(\gamma,a)}$, $a \in S$ is also an automorphism of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$. The image of the component $(\mathbf{P}_{(0)}, ++) \in \mathscr{P}$ under the mapping $\alpha_{(\gamma,a)}$ $a \in S \setminus \{0\}$ is the component $(\mathbf{P}_{(-a)}, ++) \in \mathscr{P}$, consequently $\alpha_{(\gamma,a)} \notin \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$. ∎

Lemma 8.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $\mathbf{R}()$. Then the result of composition of automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2) \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ and $\alpha_{(\gamma,a)} \in \mathbf{A_L}$, $a \in S$ is again an automorphism $\mathbf{f} = \mathbf{f}_0\, \alpha_{(\gamma,a)}$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ of all proper points of $\mathscr{A}'$. The automorphisms $\mathbf{f} = \mathbf{f}_0 \alpha_{(\gamma,a)}$ fixes the points $(\beta, 0) \in \mathbf{P}$ and $\mathsf{L} \in \mathbf{h}_\infty$; the image of the point $(0) \in \mathbf{h}_\infty$ is the point $(-a) \in \mathbf{h}_\infty$.*

Proof. Composing the automorphisms $\mathbf{f}_0$ and $\alpha_{(\gamma,a)}$ of the group $(\mathbf{P}, ++)$ we get an automorphism of $(\mathbf{P}, ++)$ again. Any equivalence of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ is a collineation of the structure $\mathscr{A}'$ fixing the points $(\beta, 0) \in \mathbf{P}$, $\mathsf{L} \in \mathbf{h}_\infty$. For this reason, composing two equivalences of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ we get an equivalence of the partition $\mathscr{P}$ again. This equivalence fixes the points $(\beta, 0) \in \mathbf{P}$, $\mathsf{L} \in \mathbf{h}_\infty$. The automorphism $\mathbf{f}_0 \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ maps the point $(0) \in \mathbf{h}_\infty$

onto itself while the automorphism $\alpha_{(\gamma,a)} \in \mathbf{A_L}$ maps this point onto the point $(-a) \in \mathbf{h}_\infty$. ∎

Lemma 8.3. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties* $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ *and* $(\mathbf{R})$ *and let the automorphisms* $\mathbf{f}_0 = (\varphi_1, \varphi_2) \in \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$, $\alpha_{(\gamma,a)} \in \mathbf{A_L}$, $a \in S$ *of the partition* $\mathscr{P}$ *of the group* $(\mathbf{P}, ++)$ *be given. Then there exists exactly one automorphism* $\alpha_{(\gamma,b)} \in \mathbf{A_L}$, $b \in S$ *such that*

$$\alpha_{(\gamma,a)}\mathbf{f}_0 = \mathbf{f}_0\alpha_{(\gamma,b)} \,.$$

Proof. Composing the automorphisms $\alpha_{(\gamma,a)} \in \mathbf{A_L}$, $a \in S$ and $\mathbf{f}_0 = (\varphi_1, \varphi_2) \in \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ we get an automorphism $\alpha_{(\gamma,a)}\mathbf{f}_0$ for which $(\xi, y)^{\alpha_{(\gamma,a)}\mathbf{f}_0} = (\xi, (a)\,\sigma_\xi + + y)^{\mathbf{f}_0} = ((\xi)\,\varphi_1, (a)\,\sigma_\xi\varphi_2 + (y)\,\varphi_2)$ is true for any proper point $(\xi, y) \in \mathbf{P}$. If we put

$$b = (a)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}\,, \quad \text{i.e.} \quad a = (b)\,\sigma_{(\gamma)\varphi_1}\varphi_2^{-1}\,,$$

then

$$\begin{aligned}(\xi, y)^{\alpha_{(\gamma,a)}\mathbf{f}_0} &= ((\xi)\,\varphi_1, (b)\,\sigma_{(\gamma)\varphi_1}\varphi_2^{-1}\sigma_\xi\varphi_2 + (y)\,\varphi_2) = \\ &= ((\xi)\,\varphi_1, (b)\,\sigma_{(\xi)\varphi_1} + (y)\,\varphi_2) = ((\xi)\,\varphi_1, (y)\,\varphi_2)^{\alpha_{(\gamma,b)}} = (\xi, y)^{\mathbf{f}_0\alpha_{(\gamma,b)}}\,.\end{aligned}$$

It is sufficient to use relation (7) for the proof of these equalities.

If we consider the automorphism $\alpha_{(\gamma,a)}\mathbf{f}_0$ as a collineation of the structure $\mathscr{A}'$ fixing the points $(\beta, 0) \in \mathbf{P}$, $\mathbf{L} \in \mathbf{h}_\infty$ then for any improper point $(k) \in \mathbf{h}_\infty$, $k \in S$ we have

$$\begin{aligned}(k)^{\alpha_{(\gamma,a)}\mathbf{f}_0} &= (k - a)^{\mathbf{f}_0} = ((k - a)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}) = \\ &= ((k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1} - (a)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1}) = ((k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1} - b) = \\ &= ((k)\,\varphi_2\sigma^{-1}_{(\gamma)\varphi_1})^{\alpha_{(\gamma,b)}} = (k)^{\mathbf{f}_0\alpha_{(\gamma,b)}}\end{aligned}$$

(we have used the results of Section 4). ∎

Let $\mathbf{F} = \{\mathbf{f} = \mathbf{f}_0\alpha_{(\gamma,a)} \mid \mathbf{f}_0 \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P}),\ \alpha_{(\gamma,a)} \in \mathbf{A_L}\}$
be the set of all automorphisms $\mathbf{f} = \mathbf{f}_0\alpha_{(\gamma,a)}$ of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$ which are products of automorphisms $\mathbf{f}_0 = (\varphi_1, \varphi_2) \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ and $\alpha_{(\gamma,a)} \in \mathbf{A_L}$, $a \in S$ of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$.

Lemma 8.4. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it possess the properties* $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ *and* $(\mathbf{R})$. *Then* $(\mathbf{F},\ )$ *is a group with the neutral element* $\boldsymbol{\varepsilon}$.

Proof. First of all we shall prove that for any two elements $\mathbf{f}' = \mathbf{f}'_0\alpha_{(\gamma,a)}$, $\mathbf{f}'' = = \mathbf{f}''_0\alpha_{(\gamma,b)}$, $a, b \in S$ of the set $\mathbf{F}$ their product $\mathbf{f}'\mathbf{f}''$ belongs to the set $\mathbf{F}$, too. Obviously

$$\begin{aligned}\mathbf{f}'\mathbf{f}'' &= (\mathbf{f}'_0\alpha_{(\gamma,a)})\,(\mathbf{f}''_0\alpha_{(\gamma,b)}) = \mathbf{f}'_0(\alpha_{(\gamma,a)}\mathbf{f}''_0)\,\alpha_{(\gamma,b)} = \mathbf{f}'_0(\mathbf{f}''_0\alpha_{(\gamma,a')})\alpha_{(\gamma,b)} = \\ &= (\mathbf{f}'_0\mathbf{f}''_0)\,(\alpha_{(\gamma,a')}\alpha_{(\gamma,b)}) = \mathbf{f}_0\alpha_{(\gamma,c)} \in \mathbf{F}\end{aligned}$$

(we have used the associativity of the composition of all automorphisms of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$, Lemmas 7.3 and 7.2 and the fact that $\mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ is a group).

Now we shall prove: If $\mathbf{f} = \mathbf{f}_0\alpha_{(\gamma,a)} \in \mathbf{F}$ then also $\mathbf{f}^{-1} \in \mathbf{F}$. Using Lemmas 7.2 and 8.3 and the fact that $\mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ is a group we obtain

$$\mathbf{f}^{-1} = (\mathbf{f}_0\alpha_{(\gamma,a)})^{-1} = \alpha_{(\gamma,a)}^{-1}\mathbf{f}_0^{-1} = \alpha_{(\gamma,-a)}\mathbf{f}'_0 = \mathbf{f}'_0\alpha_{(\gamma,a')} \in \mathbf{F}\,.$$

Thus $(\mathbf{F}, \ )$ is a subgroup of the group $\mathbf{Aut}(\mathscr{P}, \mathbf{P})$ of all automorphisms of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$ and ε is its neutral element. ∎

Theorem 8.1. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ have a degree $\geqq 4$ and let it have the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then $(\mathbf{F}, \ )$ is a group of all possible automorphisms of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$, i.e.*

$$(\mathbf{F}, \ ) = \mathbf{Aut}(\mathscr{P}, \mathbf{P})\,.$$

Proof. It is sufficient to prove that any automorphisms $\mathbf{f} \in \mathbf{Aut}(\mathscr{P}, \mathbf{P})$ may be expressed in the form $\mathbf{f} = \mathbf{f}_0\alpha_{(\gamma,a)}$ where $\mathbf{f}_0 \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$ and $\alpha_{(\gamma,a)} \in \mathbf{A_L}$.

Let $\mathbf{f} \in \mathbf{Aut}(\mathscr{P}, \mathbf{P})$ be an automorphism of the partition $\mathscr{P}$ of $(\mathbf{P}, ++)$ mapping the component $(\mathbf{P_L}, ++) \in \mathscr{P}$ onto itself and the component $(\mathbf{P}_{(0)}, ++) \in \mathscr{P}$ onto the component $(\mathbf{P}_{(a)}, ++) \in \mathscr{P}$, $a \in S$. If we consider the automorphism $\mathbf{f}$ as a collineation of $\mathscr{A}'$ then $\mathbf{f}$ fixes the points $(\beta, 0) \in \mathbf{P}$, $\mathsf{L} \in \mathbf{h}_\infty$ and the point $(0) \in \mathbf{h}_\infty$will be sent ot the point $(a) \in \mathbf{h}_\infty$. The mapping – the product $\mathbf{f}\alpha_{(\gamma,a)}$, where $\alpha_{(\gamma,a)} \in \mathbf{A_L}$ – is also a collineation of $\mathscr{A}'$ having $(\beta, 0)$, L and (0) as its fixes points. With respect to the results of Section 5 we have

$$\mathbf{f}\alpha_{(\gamma,a)} = \mathbf{f}_0 \in \mathbf{Aut}_0(\mathscr{P}, \mathbf{P})$$

as $(\mathbf{A_L}, \ )$ is a group, hence $\mathbf{f} = \mathbf{f}_0\alpha_{(\gamma,a)}^{-1}$. But Lemma 7.2 gives $\alpha_{(\gamma,a)}^{-1} = \alpha_{(\gamma,-a)}$, hence $\mathbf{f} = \mathbf{f}_0\alpha_{(\gamma,-a)} \in \mathbf{F}$.

We have proved that $\mathbf{Aut}(\mathscr{P}, \mathbf{P}) \subseteq \mathbf{F}$; according to Lemma 8.4 we get that $\mathbf{F} \subseteq \subseteq \mathbf{Aut}(\mathscr{P}, \mathbf{P})$, therefore $\mathbf{Aut}(\mathscr{P}, \mathbf{P}) = \mathbf{F}$. ∎

Let again $\mathscr{A}'$ be the structure from Theorem 8.1. According to this theorem any collineation of $\mathscr{A}'$ fixing the points $(\beta, 0) \in \mathbf{P}$ and $\mathsf{L} \in \mathbf{h}_\infty$ is a product of two collineations, the first of which fixes the points $(\beta, 0)$, L, (0) while the second is a perspective affinity $\alpha_{(\gamma,a)}$, $a \in S$ with the axis $\mathbf{h}_\beta$ and the centre $\mathsf{L} \in \mathbf{h}_\infty$. Using Theorem 2.4 in [5] we obtain the following

Theorem 8.2. *Let the extension $\mathscr{A}'$ of an anti-net $\mathscr{A}$ of a degree $\geqq 4$ have the properties $(\mathbf{rs})$, $(\mathbf{D}_\infty)$, $(\mathbf{D}_\beta)$, $(\mathbf{dg})$ and $(\mathbf{R})$. Then any collineation $\gamma \in \Gamma$ of $\mathscr{A}'$ may be expressed in the form*

$$\gamma = \mathbf{f}_0\alpha_{(\gamma,a)}\tau$$

where $\mathbf{f}_0 = (\varphi_1, \varphi_2)$ is an automorphism of the partition $\mathscr{P}$ of the group $(\mathbf{P}, ++)$, $\alpha_{(\gamma,a)} \in \mathbf{A_L}$ is a perspective affinity with the axis $\mathbf{h}_\beta$ and with the centre $\mathsf{L} \in \mathbf{h}_\infty$ and $\tau \in \mathbf{T}'$ is a translation from the translation group. The group $(\Gamma, \ )$ of all collineations of $\mathscr{A}'$ is a subdirect product of the group $\mathbf{Aut}(\mathscr{P}, \mathbf{P})$ of all automorphisms *of the partition $\mathscr{P}$ of* $(\mathbf{P}, ++)$ *and of the translation group* $(\mathrm{T}',)$, *i.e.* $(\Gamma,) = \operatorname{Aut}(\mathscr{P}, \mathbf{P}) . (\mathrm{T}',)$.

Proof. The correctness of the theorem follows from Theorems 8.1 and 2.4 in [5] and from the remarks made in Introduction. ∎

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Súhrn

ANTI-NETS II

(Collineations of anti-nets)

JAROSLAV LETTRICH

Tento článok bezprostredne nadväzuje na článok „Anti-nets I". Je venovaný vyšetrovaniu kolinácií antisiete. Sú tu popísané najprv grupy špeciálnych kolineácií (translačná grupa, grupa homotétií s daným stredom a grupa perspektívnych afinit s danou osou). Potom pomocou automorfizmov rozdelenia translačnej grupy antisiete je popísaná aj grupa všetkých možných kolineácií tejto antisiete. Podmienky existencie jednotlivých grúp kolineácií sú vyjadrené pomocou uzáverových podmienok a ich algebraických ekvivalentov, vyšetrovaných v „Anti-nets I".

Резюме

АНТИСЕТИ II

(Коллинеации антисети)

JAROSLAV LETTRICH

В статье, непосредственно продолжающей статью „Антисети I", рассматриваются коллинеации антисетей. В ней описаны прежде всего группы специальных коллинеаций (группа переносов, группа гомотетий с данным центром и группа перспективных аффинит с данной осью). После того с помощью автоморфизмов разделения группы переносов антисети описана также группа всех коллинеаций антисети. Условия существования этих групп коллинеаций выражены условиями замыкания и их алгебраическими эквивалентами, приведенными в „Антисети I".

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