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ON SOME STABILITY PROPERTIES OF STOCHASTIC DIFFERENTIAL EQUATIONS OF ITÔ'S TYPE

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In the present paper we deal with Liapunov criteria for some stability properties of solutions of the n-dimensional stochastic differential equation

$$\text{(E)} \qquad \mathrm{d}\zeta_t = b(t, \zeta_t)\,\mathrm{d}t + \sigma(t, \zeta_t)\,\mathrm{d}w_t\,,$$

where w_t is an l-dimensional Wiener process, $w_0 = 0$.

The fundamental results in the stability theory of the equation (E) belong to R. Z. Khasminskii (cf. [2], [3]). In the monograph [3] (English translation, [4]) the basic statements on the stability of the trivial stationary solution of (E) can be found (cf. also [7]). Later the theory was developed in numerous works, for instance, the stability of general (compact) sets was treated in works of A. Friedman and M. A. Pinsky (see e.g. [5]). These results can be found in a self-contained form in Friedman's monograph [6], where also some applications in the theory of partial differential equations are given.

In the present paper we shall investigate a "pathwise stability" of general solutions of the equation (E). We say that the equation (E) is stable (in a certain sense), if – roughly speaking – all its solutions are Liapunov-like stable in the same sense (cf. Definitions 2.1 and 5.1). The method used here is the classical stochastic version of the first Liapunov method as used, e.g., in [3] or [6]. Most of the results are formulated in terms of the operator $\hat{L}$(cf. (1.1)).

The paper is divided into five sections. In Section 1 we give preliminaries and some notations. In Section 2 theorems on various types of stability in probability are stated. Some examples are given in Section 3. Section 4 contains an instability theorem with some examples. It is shown that in the one-dimensional case an ordinary differential equation can always be stabilized (Example 3.2) and for $n \geqq 3$ destabilized (Example 4.2) by adding an appropriate noise. Stability in the mean is treated in Section 5.

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1. PRELIMINARIES AND NOTATIONS

We consider the equation (E), where $b = (b_i)$ is an n-dimensional vector, $\sigma =$

$= (\sigma_{ij})$ is a matrix $n \times l$, b and σ are defined on $\langle 0, \infty) \times R_n$; w_t is an l-dimensional Wiener process. The solutions of (E) are considered in the strong sense. Throughout the paper it is supposed that the coefficients b and σ fulfil the assumptions of the following Theorem 1.1 and thus, existence and uniqueness of solutions of (E) (in the usual sense) are guaranteed. For the proof of Theorem 1.1 see e.g. [1] or [6], where also other fundamental statements from the theory of stochastic differential equations used in this paper can be found.

Theorem 1.1. *Let b, σ be measurable functions and assume that for all $N > 0$, $t \in \langle 0, N\rangle$, $x, y, z \in R_n$, $|x| \leqq N$, $|y| \leqq N$ we have*

$$|b(t, z)| + \|\sigma(t, z)\| \leqq K_N(1 + |z|),$$
$$|b(t, x) - b(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leqq K_N|x - y|,$$

where K_N are positive constants, $|\cdot|$ stands for the n-dimensional Euclidean norm and $\|\cdot\|$ for the $n \times l$-matrix norm. If X is an n-dimensional random variable stochastically independent of w_t and satisfying $\mathsf{E}|X|^2 < \infty$, and $s \geqq 0$, then there exists a solution ξ_t of the equation (E) *defined on $\langle s, \infty)$ such that $\xi_s = X$. If η_t is another solution with these properties, then*

$$\mathsf{P}[\sup_{t \in \langle s, \infty)} |\xi_t - \eta_t| > 0] = 0$$

holds.

We shall denote by $\xi^{s,X}$ (or $\xi_t^{s,X}$) the solution of (E) defined on $\langle s, \infty)$ such that $\xi_s^{s,X} = X$; $\xi^X = \xi^{0,X}$. For $s \geqq 0$, $\bar{x} = (x, y) \in R_n \times R_n$ we denote by $\xi^{s,\bar{x}}$ the couple $(\xi^{s,x}, \xi^{s,y})$. The sign $\rightharpoonup$ stands for the weak convergence of measures. R_+ stands for $\langle 0, \infty)$ and $C_{1,2}(M)$ (where $M \subset R_+ \times R_k$ is open) for the set of functions defined on M whose first time-derivatives as well as the first and second space-derivatives exist and are continuous. We recall that if X_t is a continuous k-dimensional random process, $G \subset R_k$ is open and $F \subset R_k$ is a closed set, $s \in R_+$, then the random variable

$$\tau = \inf\{t \geqq s;\ X_t \notin G\}$$

is called the exit time (after s) from G. The random variable

$$\tau' = \inf\{t \geqq s;\ X_t \in F\}$$

is called the first hitting time (after s) of F. For $(t, x, y) \in R_+ \times R_n \times R_n$ we denote by $\bar{A}(t, x, y) = (\bar{A}_{ij})$ the $2n \times 2n$ matrix

$$\bar{A}(t, x, y) = \begin{pmatrix} \sigma(t, x)\, \sigma^{\mathsf{T}}(t, x) & \sigma(t, x)\, \sigma^{\mathsf{T}}(t, y) \\ \sigma(t, y)\, \sigma^{\mathsf{T}}(t, x) & \sigma(t, y)\, \sigma^{\mathsf{T}}(t, y) \end{pmatrix}$$

(where σ^{T} is the transposed matrix to σ). We shall introduce the operator $\hat{L}$. If $G \subset R_+ \times R_n \times R_n$ is open, $V \subset C_{1,2}(G)$, set

$$\hat{L}V(t, x, y) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n}\left[b_i(t, x)\frac{\partial V}{\partial x_i} + b_i(t, y)\frac{\partial V}{\partial y_i}\right] + \tag{1.1}$$

$$+ \frac{1}{2}\sum_{i,j=1}^{n}\left[\bar{A}_{ij}\frac{\partial^2 V}{\partial x_i\,\partial x_j} + \right.$$

$$\left. + \bar{A}_{i,j+n}\left(\frac{\partial^2 V}{\partial x_i\,\partial y_j} + \frac{\partial^2 V}{\partial x_j\,\partial y_i}\right) + \bar{A}_{i+n,j+n}\frac{\partial^2 V}{\partial y_i\,\partial y_j}\right]$$

(the derivatives of V and the elements of $\bar{A}$ taken at the point (t, x, y)). To simplify the references, we conclude this section by the following theorem:

Theorem 1.2. *Let b, σ satisfy the assumptions of Theorem 1.1 and assume that there exists a funcion $V \in C_{1,2}(\mathbb{R}_+ \times (G \setminus F))$ for some $F \subset G \subset \mathbb{R}_{2n}$ (G open and F closed) such that*

$$\hat{L}V(t, x, y) \leqq 0 \quad \text{for} \quad (t, x, y) \in \mathbb{R}_+ \times (G \setminus F).$$

Furthermore, assume that for all $s \geqq 0$, $\bar{x} \in G \setminus F$ we have

$$\mathsf{P}[\xi_t^{s,\bar{x}} \in F \text{ for some } t > s] = 0.$$

Consider the process $\xi^{s,\bar{x}}$ and denote by τ the exit time from G. Then

$$\mathsf{E}V(\tau \wedge t,\ \xi^{s,\bar{x}}_{\tau \wedge t}) \leqq V(s, \bar{x})$$

for all $t \geqq s$.

The proof can be easily obtained by Itô's formula (cf. [1], [6]).

2. STABILITY IN PROBABILITY

We consider the equation

$$\mathrm{d}\xi_t = b(t, \xi_t)\,\mathrm{d}t + \sigma(t, \xi_t)\,\mathrm{d}w_t, \tag{E}$$

where b and σ satisfy the same conditions as in Theorem 1.1, w_t is an l-dimensional Wiener process defined on a certain probability space $(\Omega, \mathscr{A}, \mathsf{P})$, $w_0 = 0$. We denote by S the set of all random variables X defined on $(\Omega, \mathscr{A}, \mathsf{P})$, satisfying $\mathsf{E}|X|^2 < \infty$ and stochastically independent of w_t. Let $\varrho: \langle 0, \infty) \times \mathbb{R}_n \times \mathbb{R}_n \to \langle 0, \infty)$ be a Borel measurable function such that

$$\hat{\varrho}(x, y) \leqq \varrho(t, x, y) \leqq \tilde{\varrho}(x, y) \quad \text{for all} \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_n \times \mathbb{R}_n,$$

where $\hat{\varrho}, \tilde{\varrho}$ are continuous functions and $\tilde{\varrho}(x, x) = 0$, $\hat{\varrho}(x, y) > 0$ holds for all $x, y \in \mathbb{R}_n$, $x \neq y$.

Definition 2.1. We say that the equation (E) is:

- *Stable in probability*, if for every $s \geqq 0$, $\varepsilon > 0$ and $X \in S$ there exists $\delta = \delta(s, \varepsilon, X) > 0$ such that for all $Y \in S$,

$$\mathsf{P}[\varrho(s, X, Y) > \delta] < \delta \quad \text{implies}$$
$$\mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) > \varepsilon) > \varepsilon .$$

  If $\delta = \delta(s. \varepsilon)$ can be found independent of $X \in S$, then the equation (E) is said to be *uniformly stable in probability*.
- *Asymptotically stable in probability*, if it is stable in probability and for every $s \geqq 0$, $\varepsilon > 0$ and $X \in S$ there exists $\delta = \delta(s, \varepsilon, X)$ such that for all $Y \in S$,

$$\mathsf{P}[\varrho(s, X, Y) > \delta] < \delta \quad \text{implies}$$
$$\mathsf{P}[\varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) \to 0 \text{ for } t \to \infty] \geqq 1 - \varepsilon .$$

  If $\delta = \delta(s, \varepsilon)$ can be found independent of $X \in S$ and the equation (E) is uniformly stable in probability, then it is said to be *uniformly asymptotically stable in probability.*
- *Globally (uniformly) asymptotically stable in probability*, if it is (uniformly) stable in probability and for every $s \geqq 0$, $X \in S$, $Y \in S$

$$\mathsf{P}[\varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) \to 0 \text{ for } t \to \infty] = 1$$

  holds.

In this section we shall prove some theorems dealing with sufficient conditions for the types of stability of the equation (E) defined above. Before formulating the theorems let us introduce the following notations:

$$D = \{(x, y) \in \mathbb{R}_n \times \mathbb{R}_n \mid x = y\} ,$$
$$W_r = \{z \in \mathbb{R}_{2n} \mid d(z, D) < r\} \quad (d - \text{Euclidean metric}) ,$$
$$\tilde{U}_r = \{z \in \mathbb{R}_{2n} \mid \tilde{\varrho}(z) < r\} , \quad \hat{U}_r = \{z \in \mathbb{R}_{2n} \mid \hat{\varrho}(z) < r\} ,$$
$$\tilde{Q}_r = \mathbb{R}_+ \times (\tilde{U}_r \setminus D) , \quad \hat{Q}_r = \mathbb{R}_+ \times (\hat{U}_r \setminus D) ,$$
$$B_r = \{z \in \mathbb{R}_{2n} \mid |z| < r\} .$$

Lemma 2.2. *Let $s \geqq 0$, $x, y \in \mathbb{R}_n$, $x \neq y$. Then*

$$\tau = \inf\{t \geqq s;\ \xi_t^{s,x} = \xi_t^{s,y}\} = \infty$$

holds almost surely.

Proof. Suppose first that the coefficients b, σ are globally Lipschitzian in x. Setting $V(x, y) = |x - y|^p$ for $p \in \mathbb{R}$, $x, y \in \mathbb{R}_n$, $x \neq y$ it is easy to verify that

$$\hat{L}V(t, x, y) \leqq K_{s,T}|x - y|^p \tag{2.1}$$

holds for some $K_{s,T} > 0$ and all $(t, x, y) \in \langle s, T\rangle \times \mathbb{R}_n \times \mathbb{R}_n$, $x \neq y$. Let $\delta > 0$,

$|x - y| > \delta$ and consider the process $\xi^{s,\bar{x}}$, where $\bar{x} = (x, y)$. Denote by $\tau_{m,\delta}, \tau_\delta$ the first hitting time (after s) of

$$\overline{W}_\delta \cup \overline{R_{2n} \setminus B_m},\ \overline{W}_\delta ,$$

respectively. Using (2.1) we get by Itô's formula for $t > s$:

$$\mathsf{E}\left|\xi^{s,x}_{\tau_{m,\delta}\wedge t} - \xi^{s,y}_{\tau_{m,\delta}\wedge t}\right|^p - |x - y|^p \leqq K_{s,t} \int_s^t \mathsf{E}\left|\xi^{s,x}_{\tau_{m,\delta}\wedge u} - \xi^{s,y}_{\tau_{m,\delta}\wedge u}\right|^p \mathrm{d}u .$$

Hence, noting that $\tau_{m,\delta} \wedge t \to \tau_\delta \wedge t$ holds almost surely for $m \to \infty$, we get by Gronwall's and Fatou's lemmas

$$\mathsf{E}\left|\xi^{s,x}_{\tau_\delta\wedge t} - \xi^{s,y}_{\tau_\delta\wedge t}\right|^p \leqq |x - y|^p\, \mathrm{e}^{K_{s,t}(t-s)} .$$

Taking $p = -1$ and using Chebyshev's inequality we get

$$\mathsf{P}[\tau_\delta < t] \leqq \frac{\delta}{|x - y|}\, \mathrm{e}^{K_{s,t}(t-s)} .$$

It follows that $\mathsf{P}[\tau_\delta < t] \to 0$ for $\delta \to 0+$, which implies the assertion of the lemma. For general coefficients b, σ we proceed analogously, approximating b, σ by appropriate sequences of globally Lipschitzian coefficients b_k, σ_k, respectively.

Theorem 2.3. A. *Assume that there exists a function $V \in C_{1,2}(\tilde{Q}_\eta)$ for some $\eta > 0$ satisfying*

(i) $V(t, \bar{x}) \to 0$ *for* $\bar{x} \to D$ *(for all* $t > 0$*)*,
(ii) $V_r = \inf_{\tilde{Q}_\eta \setminus \tilde{Q}_r} V(t, \bar{x}) > 0$ *for all* $0 < r < \eta$,
(iii) $\hat{L}V(t, \bar{x}) \leqq 0$ *for all* $(t, \bar{x}) \in \tilde{Q}_\eta$;

then (E) *is stable in probability.*

B. *If, moreover,* $\tilde{\varrho} = K\hat{\varrho}$ *for some* $K > 0$ *and*

(iv) $\lim_{\varepsilon\to 0+} \sup_{\tilde{U}_\varepsilon} V(t, \bar{x}) = 0$ *for all* $t > 0$ *holds,*

then (E) *is uniformly stable in probability.*

Proof. Let $0 < \varepsilon < \eta$, $s \geqq 0$, $\bar{x} = (x, y) \in \tilde{U}_\varepsilon \setminus D$. Consider the process $\xi^{s,\bar{x}}$ and denote by τ_ε, τ^m the exit time (after s) from $\tilde{U}_\varepsilon$, B_m, respectively, and $\tau_\varepsilon(t) = \tau_\varepsilon \wedge t$, $\tau_{\varepsilon,m}(t) = \tau_\varepsilon \wedge \tau^m \wedge t$. Using Theorem 1.2 and Lemma 2.2 we get

$$\mathsf{E}V(\tau_{\varepsilon,m}(t), \xi^{s,\bar{x}}_{\tau_{\varepsilon,m}(t)}) \leqq V(s, \bar{x})$$

and hence, taking $m \to \infty$ we get by Fatou's lemma

$$\mathsf{E}V(\tau_\varepsilon(t), \xi^{s,\bar{x}}_{\tau_\varepsilon(t)}) \leqq V(s, \bar{x}) .$$

It follows that

$$\mathsf{P}[\sup_{t\geqq\lambda\geqq s} \varrho(\lambda, \xi^{s,x}_\lambda, \xi^{s,y}_\lambda) > \varepsilon] \leqq \frac{V(s, \bar{x})}{V_\varepsilon} .$$

and taking $t \to \infty$ we conclude

$$\text{(2.2)} \qquad \mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) > \varepsilon] \leqq \frac{V(s, \bar{x})}{V_\varepsilon}.$$

To prove part A we consider a sequence $X_m \in S$ and a random variable $X \in S$ satisfying

$$\text{(2.3)} \qquad \mathsf{P}[\varrho(s, X_m, X) \geqq 1/m] \leqq 1/m.$$

From (2.3) we obtain $(X_m, X) \to (X, X)$ in probability and hence, denoting by μ_m, μ the joint distribution of $(X_m, X), (X, X)$, respectively, we have $\mu_m \to \mu$. Now taking $\lambda > 0$ arbitrarily we find a compact set $K \subset \mathbb{R}_n$ such that $\mu_m((\mathbb{R}_n \setminus K) \times \mathbb{R}_n) < \lambda$ for all m, and a $\delta > 0$ such that

$$W_\delta \cap (K \times \mathbb{R}_n) \subset \tilde{U}_\eta$$

and $V(s, \bar{x}) \leqq \lambda$ for all $\bar{x} \in W_\delta \cap (K \times \mathbb{R}_n)$. Furthermore, we find m_0 such that $\mu_m(\mathbb{R}_{2n} \setminus W_\delta) \leqq \lambda$ for $m \geqq m_0$. Then, for $m \geqq m_0$ we get by (2.2):

$$\mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,X_m}, \xi_t^{s,X}) > \varepsilon] = \int \mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) > \varepsilon]\, \mu_m(dx, dy) \leqq$$

$$\leqq \int_{W_\delta \cap (K \times \mathbb{R}_n)} \mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) > \varepsilon]\, \mu_m(dx, dy) + 2\lambda \leqq$$

$$\leqq \int_{W_\delta \cap (K \times \mathbb{R}_n)} \frac{V(s, x, y)}{V_\varepsilon}\, \mu_m(dx, dy) + 2\lambda \leqq \lambda\left(\frac{1}{V_\varepsilon} + 2\right).$$

Hence,

$$\mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,X_m}, \xi_t^{s,X}) > \varepsilon] \to 0 \quad \text{for} \quad m \to \infty,$$

which completes the proof of the assertion A.

To prove part B we consider the sequences $X_m \in S$, $Y_m \in S$ such that

$$\mathsf{P}[\varrho(s, X_m, Y_m) \geqq 1/m] \leqq 1/m.$$

Denoting by ν_m the joint distribution of (X_m, Y_m), we have $\nu_m(\mathbb{R}_{2n} \setminus \hat{U}_{1/m}) \leqq 1/m$ and so for m sufficiently large (to fulfil $\hat{U}_{1/m} \subset \tilde{U}_\eta$) we get

$$\mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,X_m}, \xi_t^{s,Y_m}) > \varepsilon] = \int \mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) > \varepsilon]\, \nu_m(dx, dy) \leqq$$

$$\leqq \frac{1}{m} + \sup_{\bar{x} \in \hat{U}_{1/m}} \frac{V(s, \bar{x})}{V_\varepsilon}.$$

Hence

$$\mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,X_m}, \xi_t^{s,Y_m}) > \varepsilon] \to 0 \text{ for } m \to \infty$$

holds which completes the proof of part B.

Theorem 2.4. *Assume that the assumptions of Theorem* 2.3. A *are fulfilled and, moreover,* $\lim_{\varepsilon \to 0+} \sup_{\tilde{Q}_\varepsilon} V(t, x) = 0$ *and the following condition* (P) *holds:*

(P) *For every* $0 < \delta < \lambda < \eta$, $\bar{x} \in \tilde{U}_\lambda \setminus \tilde{U}_\delta$, $s \geqq 0$

$$\mathsf{P}[\xi_t^{s,\bar{x}} \notin \tilde{U}_\lambda \setminus \tilde{U}_\delta \textit{ for some } t > s] = 1 \quad \textit{holds}.$$

Then the equation (E) *is asymptotically stable in probability. Furthermore, if* $\tilde{\varrho} = K\hat{\varrho}$ *for some* $K > 0$, *then* (E) *is uniformly asymptotically stable in probability.*

Proof. Employing the same notation as in the previous proof, we need to prove that

$$(2.4) \qquad \mathsf{P}[\varrho(t, \xi_t^{s,X_m}, \xi_t^{s,X}) \to 0 \quad \text{for} \quad t \to \infty] \to 1 \quad \text{for} \quad m \to \infty .$$

For simplicity we can suppose that $(X_m, X) \in \tilde{U}_\varepsilon$ and that V is bounded on $\tilde{Q}_\varepsilon$. Then, using the assumption (iii) of Theorem 2.3, we can easily prove that

$$\eta_m(t) = V(\tau_\varepsilon(t), \xi_{\tau_\varepsilon(t)}^{s,X_m}, \xi_{\tau_\varepsilon(t)}^{s,X}), \quad t \geqq s$$

are (nonnegative) supermartingales. Thus, there exist

$$(2.5) \qquad \hat{\xi}_m = \lim_{t \to \infty} \eta_m(t) \quad \text{almost surely}.$$

In the same way as in the proof of Theorem 2.3 we get

$$\mathsf{P}[\xi_{\tau_\varepsilon(t)}^{s,(X_m,X)} \in \partial\tilde{U}_\varepsilon] \leqq \frac{\mathsf{E}V(s, X_m, X)}{V_\varepsilon} \to 0 \quad \text{for} \quad m \to \infty$$

and so, denoting by Ω_m the set of trajectories of the process $\xi^{s,(X_m,X)}$ never leaving $\tilde{U}_\varepsilon$, we obtain $\mathsf{P}(\Omega_m) \to 1$ for $m \to \infty$.

Now it suffices to show that

$$\varrho(t, \xi_t^{s,X_m}(\omega), \xi_t^{s,X}(\omega)) \to 0 \quad \text{for} \quad t \to \infty , \quad \text{for} \quad \omega \in \Omega_m .$$

The assumption (P) yields

$$(2.6) \qquad \liminf_{t \to \infty} V(t, \xi_t^{s,X_m}, \xi_t^{s,X})|_{\Omega_m} = 0 .$$

By (2.5) and (2.6) it follows that

$$\hat{\xi}_m|_{\Omega_m} = 0 \Rightarrow \lim_{t \to \infty} V(t, \xi_t^{s,X_m}, \xi_t^{s,X})|_{\Omega_m} = 0 \Rightarrow \varrho(t, \xi_t^{s,X_m}, \xi_t^{s,X})|_{\Omega_m} \xrightarrow{t \to \infty} 0 .$$

Thus, (2.4) holds, which completes the proof. The case of uniform stability can be treated similarly.

Theorem 2.5. *Assume that the assumptions of Theorem* 2.3.A *are fulfilled, moreover,* $\limsup_{\varepsilon \to 0+ \tilde{Q}_\varepsilon} V(t, x) = 0$ *and the following condition* (G) *holds:*

(G) *For every* $s \geqq 0$, $\eta > 0$, $\bar{x} \in R_{2n} \setminus \tilde{U}_\eta$

$$\mathsf{P}[\xi_t^{s,\bar{x}} \in \tilde{U}_\eta \text{ for some } t > s] = 1 \quad \textit{holds}.$$

Then the equation (E) *is globally asymptotically stable in probability. Furthermore, if* $\tilde{\varrho} = K\hat{\varrho}$ *holds for some* $K > 0$, *then it is globally uniformly asymptotically stable in probability.*

Proof. We employ the relation (2.2) to find a $\delta > 0$ such that

$$\sup_{s \geqq 0} \sup_{(x,y) \in \tilde{U}_\delta} \mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) > \varepsilon] < \lambda$$

holds for a given $\lambda > 0$.

Now, taking $s \geqq 0$, $\bar{x} = (x, y) \in R_{2n} \setminus \tilde{U}_\delta$, we consider the process $\xi^{s,\bar{x}}$. If τ is the first hitting time (after s) of the set Cl $\tilde{U}_\delta$ (the assumption (G) guarantees that $\tau < \infty$ a.s.), then by the strong Markov property we obtain

$$\mathsf{P}[\limsup_{t \to \infty} \varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) > \varepsilon] =$$

$$= \int_{u=s}^{\infty} \int_{(z,r) \in \partial \tilde{U}_\delta} \mathsf{P}[\limsup_{t \to \infty} \varrho(t, \xi_t^{u,z}, \xi_t^{u,r}) > \varepsilon] \; \mathsf{P}[\tau \in \mathrm{d}u, \; \xi_\tau^{s,\bar{x}} \in \mathrm{d}[z, r]] \leqq$$

$$\leqq \int_{u=s}^{\infty} \int_{(z,r) \in \partial \tilde{U}_\delta} \mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi_t^{u,z}, \xi_t^{u,r}) > \varepsilon] \; \mathsf{P}[\tau \in \mathrm{d}u, \; \xi_\tau^{s,\bar{x}} \in \mathrm{d}[z, r]] \leqq \lambda .$$

Since $\lambda > 0$ was chosen arbitrary, we obtain

$$\mathsf{P}[\varrho(t, \xi_t^{s,x}, \xi_t^{s,y}) \to 0 \text{ for } t \to \infty] = 1 .$$

Thus, if $X \in S$, $Y \in S$, we get

$$\mathsf{P}[\varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) \to 0 \text{ for } t \to \infty] = 1 .$$

In the following theorem we give a sufficient condition for the assumption (G) to be fulfilled. The Liapunov function H occuring there is analogous to the so called G-function (cf. Friedman [6]).

Theorem 2.6. *Assume that there exist a non-decreasing sequence* G_m *of open sets,* $\bigcup_m G_m = R_{2n}$, *and a function* $H \in C_{1,2}(R_{2n} \setminus D)$, $H \geqq 0$ *such that*

(i) $H_m = \inf_{R_+ \times (R_{2n} \setminus G_m)} H(t, x, y) \to \infty$ *for* $m \to \infty$,

(ii) *for every* m, $\delta > 0$ *the inequality*

$$\hat{L}H(t, x, y) \leqq -\alpha_{\delta,m} < 0$$

holds for all $(t, x, y) \in R_+ \times (G_m \setminus \tilde{U}_\delta)$. *Then* (G) *holds.*

Proof. Take $\eta > 0$, $(s, x, y) \in R_+ \times R_{2n} \setminus \mathrm{Cl}\, \tilde{U}_\eta$ and m_0 such that $\bar{x} = (x, y) \in G_m$ for $m \geqq m_0$. Consider the process $\xi^{s,\bar{x}}$ and let τ, τ_m be the exit times (after s) from the sets $\mathbb{R}_{2n} \setminus \mathrm{Cl}\, \tilde{U}_\eta$, $G_m \setminus \mathrm{Cl}\, \tilde{U}_\eta$, respectively. The condition (ii) implies that

$$\mathsf{P}[\tau_m < \infty] = 1$$

(see e.g. [3], Theorem 3.7.1). Then, by Itô's formula and Fatou's lemma we obtain similarly as in the previous proofs

$$\mathsf{E}H(\tau_m, \xi^{s,\bar{x}}_{\tau_m}) \leqq H(s, \bar{x}).$$

It follows that

$$\mathsf{P}[\tau_m < \tau] \leqq \frac{H(s, \bar{x})}{H_m} \to 0 \quad \text{for} \quad m \to \infty.$$

Thus, we have

$$\mathsf{P}[\tau < \infty] \geqq \mathsf{P}[\tau \leqq \tau_m] \to 1 \quad \text{for} \quad m \to \infty,$$

i.e., (G) holds.

For the sake of clarity we summarize the results of Theorems 2.5 and 2.6:

Corollary 2.7. *Let there exist a non-decreasing sequence G_m of open sets, $\bigcup_m G_m = \mathbb{R}_{2n}$, and a function $V \in C_{1,2}(\mathbb{R}_{2n} \setminus D)$ satisfying:*

(i) $\inf_{R_+ \times (\mathbb{R}_{2n} \setminus G_m)} V \to \infty$ *for* $m \to \infty$,

(ii) $\inf_{(R_+ \times \mathbb{R}_{2n}) \setminus \hat{Q}_r} V > 0$ *for all* $r > 0$ *sufficiently small,*

(iii) $\lim_{\varepsilon \to 0+} \sup_{\tilde{Q}_\varepsilon} V = 0$,

(iv) $\hat{L}V(t, x, y) \leqq -\alpha_{\delta,m} < 0$ *for all* $m, \delta > 0$ *and* $(t, x, y) \in G_m \setminus \tilde{U}_\delta$.

Then the equation (E) *is globally asymptotically stable in probability. If, moreover, $\tilde{\varrho} = K\hat{\varrho}$ holds for some $K > 0$, then* (E) *is globally uniformly asymptotically stable in probability.*

Remark 2.8. The assumption $\tilde{\varrho} = K\hat{\varrho}$ can be weakened in all theorems in the present section (e.g., we can assume that for every $\varepsilon > 0$ such $\delta > 0$ can be found that $\hat{U}_\delta \subset \tilde{U}_\varepsilon$ holds). Nevertheless, it cannot be omitted.

3. EXAMPLES

Example 3.1. Set $\varrho(t, x, y)$ $(= \tilde{\varrho}(x, y) = \hat{\varrho}(x, y)) = |x - y|$ and, for $p > 0$, set

$$F_p(t, x, y) = p|x - y|^{p-2} \left(b(t, x) - b(t, y), x - y\right) + \\ + \tfrac{1}{2} \sum_{i,j} \hat{A}_{ij}(t, x, y) \left[p|x - y|^{p-2} \delta_{ij} + \right. \\ \left. + p(p - 2)(x_i - y_i)(x_j - y_j)|x - y|^{p-4}\right],$$

where $\hat{A} = (\hat{A}_{ij}(t, x, y)) = (\sigma(t, x) - \sigma(t, y))(\sigma(t, x) - \sigma(t, y))^{\mathrm{T}}$, δ_{ij} – Kronecker's delta. Using the function $V(x, y) = |x - y|^p$ we see that

$$\hat{L}V(t, x, y) = F_p(t, x, y)$$

for all (t, x, y), $x \neq y$. Thus, as a result of the previous section, the following assertions are obtained:

If there is $\delta > 0$ and $p > 0$ such that

$$F_p(t, x, y) \leqq 0 \tag{3.1}$$

for all $(t, x, y) \in \mathbb{R}_+ \times (W_\delta \setminus D)$, then the equation (E) is uniformly stable in probability (cf. Theorem 2.3). If

$$F_p(t, x, y) \leqq -\alpha_{\delta,m} < 0 \tag{3.2}$$

for all $(t, x, y) \in \mathbb{R}_+ \times (W_m \setminus W_\delta)$, $m > 0$, $\delta > 0$, then (E) is globally uniformly asymptotically stable in probability. In particular, if σ is Lipschitz continuous in x with a constant $K > 0$ and

$$(b(t, x) - b(t, y), x - y) \leqq -L|x - y|^2$$

holds for some $L > nK^2/2 - M$, where $M \geqq 0$ is such that

$$\sum_{i,j} \hat{A}_{ij}(t, x, y)(x_i - y_i)(x_j - y_j) \geqq M|x - y|^4$$

holds for all (t, x, y) (n is the dimension), then (3.2) is valid. To prove the last assertion it suffices to note that we have

$$F_p(t, x, y) \leqq -pL|x - y|^p + \frac{npK^2}{2}|x - y|^p + \frac{p^2n^2K^2}{2}|x - y|^p - pM|x - y|^p,$$

so choosing $p > 0$ sufficiently small we obtain

$$F_p(t, x, y) \leqq -\alpha|x - y|^p$$

for some $\alpha > 0$ and all (t, x, y) and thus, (3.2) holds. Furthermore, if the dimension $n = 1$ and $p = 1$, we have

$$F_1(t, x, y) = (b(t, x) - b(t, y)) \operatorname{sign}(x - y).$$

Hence, if the function $b(t, \cdot)$ is non-increasing for all $t > 0$, then (E) is stable in probability (for all σ).

Example 3.2. Set $\varrho(t, x, y) = |x - y|$. If $\sigma(t, \cdot) \in C_2$, $b(t, \cdot) \in C_2$ for all $t \geqq 0$ and the corresponding first and second derivatives are bounded uniformly with respect to t, and if for some $a > 0$ and all vectors $v = (v_1, \ldots, v_n) \in \mathbb{R}_n$, $|v| = 1$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_n$ the inequality

(3.3) $$\sum_{i,j} \frac{\partial b_i}{\partial x_j}(t, x)\, v_i v_j +$$

$$+ \frac{1}{2} \sum_{i,j,k,r,l} \frac{\partial \sigma_{ir}}{\partial x_k}(t, x) \frac{\partial \sigma_{jr}}{\partial x_l}(t, x)\,(v_l v_k \delta_{ij} - 2 v_l v_k v_i v_j) \leqq -a$$

holds, then the equation (E) is uniformly asymptotically stable in probability. Denoting by $\boldsymbol{B}$ the matrix $(\partial b_i/\partial x_j(t, x))$ and by Λ_r, $r = 1, 2, \ldots, l$, the matrices $(\partial \sigma_{ir}/\partial x_j(t, x))$ we can express the condition (3.3) in the form

$$(\boldsymbol{B}v, v) + \tfrac{1}{2} \sum_{r=1}^{l} |\Lambda_r v|^2 - \sum_{r=1}^{l} (\Lambda_r v, v)^2 \leqq -a\,.$$

The proof of the assertion can be obtained by applying the Liapunov function $V(x, y) = |x - y|^p$, $0 < p < 1$. From (3.3) we obtain for some $R_1 > 0$, $R_2 > 0$ (independent of p) and all $(t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}_{2n} \setminus D)$:

$$\hat{L}V(t, x, y) \leqq p|x - y|^{p-2} \sum_{i,j} \frac{\partial b_i}{\partial x_j}(t, y)\,(x_j - y_j)(x_i - y_i) +$$

$$+ \frac{1}{2} p \sum_{i,j,k,r,l} \frac{\partial \sigma_{ir}}{\partial x_k}(t, y) \frac{\partial \sigma_{jr}}{\partial x_l}(t, y)\,(x_k - y_k)(x_l - y_l)\,[\delta_{ij}|x - y|^{p-2} +$$

$$(p - 2)(x_i - y_i)(x_j - y_j)\,|x - y|^{p-4}] +$$

$$+ R_1|x - y|^{p+1} + R_2|x - y|^{p+2} \leqq$$

$$\leqq p|x - y|^p \Big(-a + \frac{p}{2} \sum_{i,j,k,r,l} \frac{\partial \sigma_{ir}}{\partial x_k}(t, y) \frac{\partial \sigma_{jr}}{\partial x_l}(t, y)\,(x_k - y_k)(x_l - y_l)\,.$$

$$.\,(x_i - y_i)(x_j - y_j)\,|x - y|^{-4}\Big) + R_1|x - y|^{p+1} + R_2|x - y|^{p+2}\,.$$

Now, choosing p sufficiently small we have

(3.4) $$\hat{L}V(t, x, y) \leqq -L|x - y|^p$$

for some $L > 0$, $\Delta > 0$ and all $(t, x, y) \in \mathbb{R}_+ \times (W_\Delta \setminus D)$. Now, by Theorem 2.4 we conclude that the equation (E) is uniformly asymptotically stable in probability. (Indeed, (3.4) guarantees that the condition (P) is fulfilled – see e.g. [3], Theorem 3.7.1).

In the one-dimensional case for $l = 1$ the condition (3.3) has the form

$$\frac{\partial}{\partial x} b(t, x) - \tfrac{1}{2}\left(\frac{\partial}{\partial x} \sigma(t, x)\right)^2 \leqq -a\,.$$

In particular, we have obtained that every one-dimensional ordinary differential equation (with a sufficiently smooth right-hand side) can be "stabilized" by adding an appropriate diffusion coefficient – compare with Example 4.1 below.

Example 3.3. The notions of the global asymptotical stability and the asymptotical stability of the equation (E) introduced in Definition 2.1 above coincide in the deterministic case (i.e. for $\sigma \equiv 0$). This is not the case in general, as the following example shows: Set $\varrho(x, y) = |x - y|$, dimension $n = 1$ and consider the equation

$$d\xi_t = \big| |\xi_t| - 1 \big| \, dw_t \, . \tag{3.5}$$

The equation (3.5) is uniformly asymptotically stable in probability (we can obtain this result from Theorem 2.4 by using the function $V(x, y) = |x - y|$ and noting that the condition (P) is fulfilled, because the set $M = \{-1, 1\}$ is globally asymptotically stable in the sense defined in [6]). However, (3.5) is not globally asymptotically stable in probability as it has two distinct stationary points.

In the rest of this section we deal with the one-dimensional autonomous case, i.e., we consider the equation

$$d\xi_t = b(\xi_t) \, dt + \sigma(\xi_t) \, dw_t \, , \tag{3.6}$$

where b and σ are Lipschitz continuous scalar functions.

Example 3.4. Assume that $\sigma > 0$ and set

$$\varrho(x, y) = \left| \int_x^y \exp\left[-2 \int_0^z \frac{b}{\sigma^2} \right] dz \right| .$$

Then the equation (3.6) is uniformly stable in probability.

Proof. Setting $V(x, y) = \varrho(x, y) = (= \tilde{\varrho}(x, y) = \hat{\varrho}(x, y))$ we use Theorem 2.3 (noting that $\hat{L}V = 0$).

As a consequence we get the following result, which concerns a certain "relative stability" with respect to compact sets:

Example 3.5. Assume that $\sigma > 0$ and $X \in S$, $K \subset \mathbb{R}$ are arbitrary, K is compact. Then for every $\varepsilon > 0$ such a $\delta > 0$ can be found that for every $Y \in S$

$$\mathsf{P}[|X - Y| > \delta] < \delta \quad \text{implies}$$
$$\mathsf{P}[\sup_\theta |\xi_{t^-}^X - \xi_t^Y| > \varepsilon] < \varepsilon \, , \quad \text{where} \quad \theta = \{t \geqq 0, \ \xi_t^X \in K\} \, .$$

Proof. Set

$$\varrho(x, y) \, (= \tilde{\varrho}(x, y)) = \left| \int_x^y \exp\left[-2 \int_0^z \frac{b}{\sigma^2} \right] dz \right| .$$

Taking $\varepsilon > 0$ we find $\delta > 0$ such that

$$(\mathbb{R} \times K) \cap \tilde{U}_\delta \subset (\mathbb{R} \times K) \cap W_\varepsilon \tag{3.7}$$

holds. Now, consider a sequence $X_m \in S$ such that

$$\mathsf{P}[|X_m - X| \geqq 1/m] \leqq 1/m \, .$$

It follows that for every $\Delta > 0$,

$$P[\varrho(X_m, X) \geqq \Delta] \to 0 \quad \text{for} \quad m \to \infty .$$

Thus, by the previous example, we have

$$P[\sup_{t \geqq 0} \varrho(\xi_t^{X_m}, \xi_t^X) \geqq \delta] \to 0 \quad \text{for} \quad m \to \infty .$$

which, together with (3.7), concludes the proof.

Let us recall that if an equation admits a solution which is a stationary Markov process, then we say that the weak stochastic stability occurs.

We conclude this section by establishing the connection between the weak stochastic stability and the stability defined above in the one-dimensional case:

Theorem 3.6. *Let $\varrho(x, y) = |x - y|$ and assume that for some $K > 0$ the inequality*

$$|b(x) - b(y)| \leqq K|x - y|$$

holds for all $x, y \in \mathbb{R}$. Let $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_1 < \sigma(x) < \lambda_2$ for $x \in \mathbb{R}$. Then, if the equation (3.6) *admits a stationary solution, it is stable in probability.*

Proof. Setting

$$V(x, y) = \left| \int_x^y \exp\left[-2 \int_0^z \frac{b}{\sigma^2} \right] dz \right|$$

we see that

$$V(x, y) \to 0 \quad \text{for} \quad (x, y) \to D ,$$

$$\hat{L}V = 0 \quad \text{on the set} \quad \mathbb{R}_2 \setminus D .$$

To fulfil the assumptions of Theorem 2.3 it remains to prove that

$$\inf_{|x-y| \geqq \delta} V(x, y) > 0$$

holds for all (sufficiently small) $\delta > 0$. We have

$$\inf_{|x-y| \geqq \delta} V(x, y) \geqq \delta \inf_z \exp\left[-2 \int_0^z \frac{b}{\sigma^2} \right],$$

hence to conclude the proof it suffices to prove the following lemma:

Lemma 3.7. *Under the conditions of Theorem* 3.6 *the function*

$$\psi(x) = \int_0^x \frac{2b}{\sigma^2}$$

is bounded from above.

Proof. Assuming that the above assertion is false, we find a sequence $x_m \in \mathbb{R}$ such that $\psi(x_m) = m$. Let

$$\delta = \frac{\lambda_1}{\sqrt{(2K)}} \quad \text{and}$$

$$U_m = \begin{cases} \nearrow (x_m, x_m + \delta) & \text{if } \psi'(x_m) \geqq 0 \\ \searrow (x_m - \delta, x_m) & \text{if } \psi'(x_m) < 0 . \end{cases}$$

By the mean value theorem we get for $x \in U_m$

$$\psi(x) \geqq \psi(x_m) - \frac{2K\delta}{\lambda_1^2} |x - x_m| \geqq m - \frac{2K\delta^2}{\lambda_1^2} = m - 1 .$$

From (U_m) we choose an infinite subsequence U_{m_k} containing only disjoint intervals. Then

$$\int_{-\infty}^{\infty} \exp\left[2\int_0^x \frac{b}{\sigma^2}\right] dx \geqq \sum_{k=1}^{\infty} \int_{U_{m_k}} e^{\psi(x)}\, dx \geqq \delta \sum_{k=1}^{\infty} e^{m_k - 1} = \infty .$$

Thus, we have

$$\int_{-\infty}^{\infty} \sigma^{-2}(x)\, e^{\psi(x)}\, dx \geqq \lambda_2^{-2} \int_{-\infty}^{\infty} e^{\psi} = \infty ,$$

which cannot hold in view of the existence of a stationary solution of the equation (3.6) (the function $\sigma^{-2} e^{\psi}$ is the density of the stationary distribution).

4. INSTABILITY

In this section we keep the notation and the assumptions made at the beginning of Section 2. First we formulate the following theorem on instability:

Theorem 4.1. *Assume that for some $\eta > 0$ there exists a nonnegative function $V \in C_{1,2}(\hat{Q}_\eta)$ such that*

(i) $\liminf_{\varepsilon \to 0+\hat{Q}_\varepsilon} V(t, \bar{x}) = \infty$,

(ii) $\hat{L}V(t, x, y) \leqq 0$ *for* $(t, x, y) \in \hat{Q}_\eta$.

Furthermore, let the following condition (P′) *be fulfilled:*

(P′) *There exist $\delta_0 > 0$, $\lambda_0 > \delta_0$ such that for all $\eta > \lambda > \lambda_0$, $0 < \delta < \delta_0$, $s \geqq 0$, $\bar{x} \in \hat{U}_\lambda \setminus \hat{U}_\delta$*

$$\mathsf{P}[\xi_t^{s,\bar{x}} \notin \hat{U}_\lambda \setminus \hat{U}_\delta \textit{ for some } t > s] = 1 \quad \textit{holds} .$$

Then, for all $\lambda_0 < \lambda < \eta$, $X \in S$, $Y \in S$ such that $\mathsf{P}[X = Y] = 0$ *we have*

$$\mathsf{P}[\sup_{t \geqq 0} \varrho(t, \xi_t^X, \xi_t^Y) > \lambda] = 1 .$$

Remark. In particular, the assertion implies that the equation (E) is not stable in probability. However, the assertion is stronger; it is a kind of "strong instability".

Proof of Theorem 4.1. Let ε, λ be such that $0 < \varepsilon < \delta_0 < \lambda_0 < \lambda < \eta$, let $\bar{x} = (x, y) \in U_\lambda \setminus U_\varepsilon$ and consider the process $\xi^{\bar{x}}$. Let $\tau_\varepsilon, \tau_{\varepsilon,m}$ be the exit times from $\hat{U}_\lambda \setminus \hat{U}_\varepsilon, (\hat{U}_\lambda \setminus \hat{U}_\varepsilon) \cap B_m$, respectively, and let τ^ε be the first hitting time of $\mathrm{Cl}\, \hat{U}_\varepsilon$. Using Itô's formula we get

$$\mathsf{E}V(\tau_{\varepsilon,m} \wedge t, \xi^{\bar{x}}_{\tau_{\varepsilon,m} \wedge t}) \leqq V(0, \bar{x}).$$

Taking $m \to \infty$ we obtain by Fatou's lemma

$$\mathsf{E}V(\tau_\varepsilon \wedge t, \xi^{\bar{x}}_{\tau_\varepsilon \wedge t}) \leqq V(0, \bar{x}).$$

By the assumption (P') we have $\tau_\varepsilon \wedge t \to \tau_\varepsilon$ almost surely for $t \to \infty$. Thus, we get

$$\mathsf{E}V(\tau_\varepsilon, \xi^{\bar{x}}_{\tau_\varepsilon}) \leqq V(0, \bar{x}).$$

It follows that

$$\mathsf{P}[\xi^{\bar{x}}_t \notin \partial U_\lambda \text{ for } t \in \langle 0, \tau^\varepsilon \rangle] \leqq \frac{V(0, \bar{x})}{\inf\limits_{\hat{Q}_\varepsilon} V(t, x)}. \tag{4.1}$$

For $\varepsilon \to 0+$ the right hand side of (4.1) tends to zero. Furthermore, $\tau^\varepsilon \to \infty$ for $\varepsilon \to 0+$, since $\bigcap\limits_{0 \leqq \varepsilon < \lambda} \hat{U}_\varepsilon = D$ and D is nonattainable (cf. Lemma 2.2).

Therefore, from (4.1) we obtain

$$\mathsf{P}[\sup_{t \geqq s} \varrho(t, \xi^x_t, \xi^y_t) \leqq \lambda] \leqq \mathsf{P}[\xi^{\bar{x}}_t \notin \partial \hat{U}_\lambda \text{ for } t \in \langle 0, \infty)] = 0.$$

It is easily checked that this implies the assertion of Theorem 4.1.

Example 4.2. Suppose that $\varrho(t, x, y) = |x - y|$ and let

$$G(t, x, y) = -(b(t, x) - b(t, y), x - y)\,|x - y|^{-2} + \\ + \tfrac{1}{2} \sum_{i,j} \hat{A}_{ij}(t, x, y)\,[2(x_i - y_i)(x_j - y_j)\,|x - y|^{-4} - \delta_{ij}|x - y|^{-2}],$$

where $\hat{A}(t, x, y) = (\hat{A}_{ij}(t, x, y)) = (\sigma(t, x) - \sigma(t, y))\,(\sigma(t, x) - \sigma(t, y))^\mathsf{T}$. If for some $\alpha > 0$ $\beta \geqq 0$, $\eta > 0$,

$$G(t, x, y) \leqq -\alpha |x - y|^\beta \tag{4.2}$$

holds for all $(t, x, y) \in \mathbb{R}_+ \times W_\eta$, then the equation (E) is not stable in probability. Moreover, the assertion of Theorem 4.1 is valid with all $0 < \lambda_0 < \eta$. Indeed the function $V(x, y) = k - \log|x - y|$, where k is a suitable positive constant, satisfies the assumptions of Theorem 4.1 (note that $\hat{L}V = G$). Using the same Liapunov function we can also see that (4.2) implies (P') (cf. [3] Theorem 3.7.1).

In particular, set $l = n^2$ and consider the equation

$$d\xi_i = b_i(t, \xi)\, dt + \sigma \sum_{j=1}^{n} \xi_j\, dW_{(i-1)n+j}, \quad i = 1, 2, \ldots, n$$

(σ – constant). We have

$$\hat{A}_{ij} = \sigma^2 |x - y|^2\, \delta_{ij}, \quad i, j = 1, 2, \ldots, n.$$

Hence

$$G(t, x, y) = -(b(t, x) - b(t, y), x - y)\, |x - y|^{-2} + \sigma^2(1 - \tfrac{1}{2}n).$$

Thus if we assume that b is (globally) lipschitz continuous in x uniformly with respect to t and the dimension is $n \geqq 3$ then the ordinary differential equation $\dot{x} = b(t, x)$ can always be "destabilized" by adding an appropriate diffusion. Note that this is not valid in the one-dimensional case (cf. Example 3.1).

Example 4.3. Set $\varrho(t\ x\ y) = |x - y|$ and assume that $b(t, \cdot) \in C_2$, $\sigma(t, \cdot) \in C_2$ for every $t > 0$, the corresponding first and second derivatives being bounded uniformly with respect to $t > 0$. Denoting by $\boldsymbol{B}$ the matrix $(\partial b_i / \partial x_j\, (t, x))$ and by Λ_r, $r = 1, 2, \ldots, l$ the matrices $(\partial \sigma_{ir} / \partial x_j\, (t, x))$ suppose that for some $a > 0$ the inequality

$$(4.3) \qquad (\boldsymbol{B}v, v) + \tfrac{1}{2} \sum_{r=1}^{l} |\Lambda_r v|^2 - \sum_{r=1}^{l} (\Lambda_r v, v)^2 \geqq a$$

holds for all $v \in R_n$ with $|v| = 1$ and $(t, x) \in R_+ \times R_n$. Then the assertion of Theorem 4.1 holds for some $\lambda_0 > 0$. In particular, the equation (E) is not stable in probability. The proof proceeds similarly as in Example 3.2, using the function $V(x, y) = k - \log |x - y|$.

Example 4.4. Set $\varrho(t, x, y) = |x - y|$ and consider the one-dimensional autonomous equation

$$(4.4) \qquad d\xi_t = \tfrac{1}{2}\xi_t\, dt + \sigma(\xi_t)\, dw_t,$$

where $\sigma(x) = 1$ for $x \leqq 1$, $\sigma(x) = x$ otherwise. Then the assertion of Theorem 4.1 is valid with arbitrary $\lambda_0 > 0$.

Indeed, we can set $V(x, y) = k - \log |x - y|$ for a suitable $k > 0$. Setting $H(x, y) = M - |x - y|$ for a suitable $M > 0$ we obtain ([3], Theorem 3.7.1) that (P′) is fulfilled.

On the other hand, we have

$$\int_0^x \exp\left[-2 \int_0^z \frac{b}{\sigma^2}\right] dz \to \pm\infty \quad \text{for} \quad x \to \pm\infty,$$

and thus, any solution of (4.4) is a recurrent Markov process (cf. [3], Example 3.8.2) – compare with Example 3.5 above.

We use the same notation as in Section 2. Let b and σ satisfy the conditions of Theorem 1.1 and let us assume that for the function $\tilde{\varrho}$ introduced at the beginning of Section 2,

$$\tilde{\varrho}(x, y) \leqq K(1 + |x|^2 + |y|^2)$$

holds for some $K > 0$ and all $x, y \in \mathbb{R}_n$.

Definition 5.1. We say that the equation (E) is

- *Stable in the mean*, if for every $s \geqq 0$, $\varepsilon > 0$ there exists $\delta = \delta(s, \varepsilon) > 0$ such that for all $X, Y \in S$
$$\mathsf{E}\varrho(s, X, Y) < \delta \quad \text{implies} \quad \mathsf{E}\varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) < \varepsilon \quad \text{for all} \quad t > s\,.$$
- *Asymptotically stable in the mean*, if it is stable in the mean and for every $s \geqq 0$ there exists $\Delta = \Delta(s) > 0$ such that for all $X, Y \in S$,
$$\mathsf{E}\varrho(s, X, Y) < \Delta \quad \text{implies} \quad \mathsf{E}\varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) \to 0 \quad \text{for} \quad t \to \infty\,.$$
- *Exponentially stable*, if for some $K_1 > 0$, $K_2 > 0$ and all $X, Y \in S$, $t \geqq s$ the inequality
$$\mathsf{E}\varrho(t, \xi_t^{s,X}, \xi_t^{s,Y}) \leqq K_1 \exp\{-K_2(t - s)\}\, \mathsf{E}\varrho(s, X, Y)$$
holds.

Denoting $Q = \mathbb{R}_+ \times (\mathbb{R}_{2n} \setminus D)$ we have the following

Theorem 5.2. *Let there exist a continuous function $V \in C_{1,2}(Q)$ such that for some $M > 0$, $N > 0$ the inequalities*

$$(5.1) \qquad M\varrho \leqq V \leqq N\varrho$$

$$(5.2) \qquad \textit{and} \quad \hat{L}V(t, x, y) \leqq \varphi(t, V(t, x, y))$$

hold for all $(t, x, y) \in Q$, where $\varphi: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying

$$(5.3) \qquad |\varphi(t, \alpha) - \varphi(t, \beta)| \leqq K_T|\alpha - \beta|\,, \quad K_T > 0\,,$$

for all $t, \alpha, \beta \in \mathbb{R}_+$, $t \leqq T$, $\varphi(t, 0) = 0$ and $\varphi(t, \cdot)$ is a concave function for all $t \in \mathbb{R}_+$.

Then the equation (E) is stable in the mean provided the trivial solution $x \equiv 0$ of the ordinary differential equation

$$(\mathrm{D}) \qquad \dot{x} = \varphi(t, x)$$

is stable in the Liapunov sense. If the trivial solution of (D) is asymptotically stable, then the equation (E) is asymptotically stable in the mean. If $\varphi(t, x) = -cx$ for some $c > 0$, then (E) is exponentially stable.

Proof. Let $(s, \bar{x}) \in Q$ and consider the process $\xi^{s,\bar{x}}$. Denoting by $\tau_{\varepsilon,m}$ the exit time (after s) from $B_m \cap (\mathbb{R}_{2n} \setminus \mathrm{Cl}\, W_\varepsilon)$, $\tau_{m,\varepsilon}(t) = \tau_{m,\varepsilon} \wedge t$ and using Itô's formula and (5.2) we get for $t \geqq \theta \geqq s$:

$$(5.4) \qquad \mathsf{E}V(\tau_{m,\varepsilon}(t), \xi^{s,\bar{x}}_{\tau_{m,\varepsilon}(t)}) = \mathsf{E}V(\tau_{\varepsilon,m}(\theta), \xi^{s,\bar{x}}_{\tau_{m,\varepsilon}(\theta)}) +$$

$$+ \mathsf{E}\int_{\tau_{\varepsilon,m}(\theta)}^{\tau_{m,\varepsilon}(t)} \hat{L}V(u, \xi^{s,\bar{x}}_u)\, du \leqq \mathsf{E}V(\tau_{\varepsilon,m}(\theta), \xi^{s,\bar{x}}_{\tau_{m,\varepsilon}(\theta)}) + \mathsf{E}\int_{\tau_{\varepsilon,m}(\theta)}^{\tau_{\varepsilon,m}(t)} \varphi(u, V(u, \xi^{s,\bar{x}}_u))\, du\,.$$

Taking $m \to \infty$, $\varepsilon \to 0+$ and using Lemma 2.2 we obtain by the dominated convergence theorem

$$\mathsf{E}V(t, \xi^{s,\bar{x}}_t) \leqq \mathsf{E}V(\theta, \xi^{s,\bar{x}}_\theta) + \int_\theta^t \mathsf{E}\varphi(u, V(u, \xi^{s,\bar{x}}_u))\, du\,.$$

Jensen's inequality yields

$$\mathsf{E}V(t, \xi^{s,\bar{x}}_t) \leqq \mathsf{E}V(\theta, \xi^{s,\bar{x}}_\theta) + \int_\theta^t \varphi(u, \mathsf{E}V(u, \xi^{s,\bar{x}}_u))\, du\,.$$

Thus, setting $\psi_t = \mathsf{E}V(t, \xi^{s,\bar{x}}_t)$, $t \geqq s$, we have

$$(5.5) \qquad 0 \leqq \psi_t \leqq \psi_\theta + \int_\theta^t \varphi(u, \psi_u)\, du$$

for $s \leqq \theta \leqq t$. We can easily complete the proof by (5.5) and (5.1) noting that ψ is a continuous function.

Corollary 5.3. *In particular, if a function $V \in C_{1,2}(Q)$ satisfies* (5.1) *and $\hat{L}V \leqq 0$ holds, then* (E) *is stable in the mean.*

Theorem 5.4. *Let there exist a continuous function $V \in C_{1,2}(Q)$ such that for some $M > 0$, $N > 0$* (5.1) *holds and*

$$\hat{L}V(t, x, y) \geqq \eta(t, V(t, x, y))$$

is fulfilled for all $(t, x, y) \in Q$, where $\eta: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$|\eta(t, \alpha) - \eta(t, \beta)| \leqq K_T|\alpha - \beta|\,, \quad K_T > 0\,,$$

for all $t, \alpha, \beta \in \mathbb{R}_+$, $t \leqq T$, $\eta(t, 0) = 0$ and $\eta(t, \cdot)$ is a convex function for all $t \geqq 0$. Then the equation (E) *is not stable in the mean provided the trivial solution $x \equiv 0$ of the ordinary differential equation $\dot{x} = \eta(t, x)$ is not stable in the Liapunov sense.*

The proof is similar to the previous one and can be omitted.

Example 5.5. For $\varrho(x, y) = |x - y|^p$, $0 < p \leqq 2$, by taking $V = \varrho$ we can easily obtain the following results (using the same notation as in Example 3.1):

If, for all $(t, x, y) \in Q$,

A. $F_p(t, x, y) \leqq 0$, then (E) is stable in the mean;

B. $F_p(t, x, y) \leqq -k|x - y|^{2p}(1 + |x - y|^p)^{-1}$ for some $k > 0$, then (E) is asymptotically stable in the mean;
C. $F_p(t, x, y) \leqq -c|x - y|^p$ for some $c > 0$, then (E) is exponentially stable;
D. $F_p(t, x, y) \geqq k|x - y|^{2p}(1 + |x - y|^p)^{-1}$, then (E) is not stable in the mean.

Thus, for $p = 2$ the equation

$$\mathrm{d}\xi_t = (-\alpha\xi_t + k_1)\,\mathrm{d}t + (\sigma\xi_t + k_2)\,\mathrm{d}w_t\,,$$

where $\sigma^2 > 2\alpha > 0$, $k_1, k_2 \in \mathbb{R}$, is not stable in the mean, although (cf. Example 3.1) it is stable in probability.

Example 5.6. Set $\varrho(x, y) = |x - y|$. The equation

$$\mathrm{d}\xi_t = \xi_t\,\mathrm{d}t + 2(|\xi_t| + 1)\,\mathrm{d}w_t$$

is an example of an equation having a (nontrivial) stationary solution, which is not stable in the mean (after setting $V = \varrho$ we can use Example 5.5 D with $p = 1$).

We shall prove the following simple converse theorem:

Theorem 5.7. *Assume that $b(t, \cdot) \in C_2$, $\sigma(t, \cdot) \in C_2$, $\varrho(t, \cdot) \in C_2$ and the corresponding first and second derivatives are continuous in (t, x) and bounded. Furthermore, suppose that*

$$|\hat{L}\varrho(t, x, y)| \leqq k\varrho(t, x, y) \tag{5.6}$$

holds for some $k > 0$ and all $(t, x, y) \in Q$. Then there exists a function $V \in C_{1,2}(Q)$ such that

$$M\varrho \leqq V \leqq N\varrho \tag{5.7}$$

$$\textit{and} \quad \hat{L}V(t, x, y) \leqq -cV(t, x, y) \tag{5.8}$$

hold for some $M > 0$, $N > 0$, $c > 0$ and all $(t, x, y) \in Q$ provided the equation (E) is exponentially stable.

Proof. Set

$$T = -\frac{1}{K_2}\log\frac{1}{2K_1} \quad (K_1, K_2 \text{ from Definition 5.1})$$

and

$$V(t, x, y) = \int_t^{t+T} \mathrm{E}\varrho(u, \xi_u^{t,x}, \xi_u^{t,y})\,\mathrm{d}u\,, \quad (t, x, y) \in Q\,.$$

We have

$$V(t, x, y) \leqq \int_t^{t+T} K_1\varrho(t, x, y)\,\mathrm{e}^{-K_2(u-t)}\,\mathrm{d}u \leqq N\varrho(t, x, y) \tag{5.9}$$

for a suitable $N > 0$. Furthermore, from (5.6) we get

$$V(t, x, y) \geqq -\frac{1}{k}\int_t^{t+T} \mathrm{E}\hat{L}\varrho(u, \xi_u^{t,x}, \xi_u^{t,y})\,\mathrm{d}u =$$

$$= -\frac{1}{k}\left(\mathsf{E}\varrho(t + T, \xi^{t,x}_{t+T}, \xi^{t,y}_{t+T}) - \varrho(t, x, y)\right) \geqq \frac{1}{2k}\, \varrho(t, x, y),$$

and so (5.7) holds. To complete the proof it remains to note (cf. [1]) that $V \in C_{1,2}(Q)$ and

$$\hat{L}V(t, x, y) = \mathsf{E}\varrho(t + T, \xi^{t,x}_{t+T}, \xi^{t,y}_{t+T}) - \varrho(t, x, y) +$$

$$+ \int_t^{t+T} \hat{L}\mathsf{E}\varrho(u, \xi^{t,x}_u, \xi^{t,y}_u)\, du = \mathsf{E}\varrho(t + T, \xi^{t,x}_{t+T}, \xi^{t,y}_{t+T}) -$$

$$- \varrho(t, x, y) \leqq -\tfrac{1}{2}\varrho(t, x, y) \leqq -\frac{1}{2N}\, V(t, x, y).$$

Remark 5.8. Assume that the first derivatives of the functions $b(t, \cdot)$, $\sigma(t, \cdot)$ are bounded unfiromly with respect to t. Then for $\varrho(x, y) = |x - y|^p$, $p > 0$, (5.6) holds with some $k > 0$.

Remark 5.9. Combining Theorems 5.7 and 2.5 we obtain for sufficiently smooth b, σ, ϱ that in the case $\tilde{\varrho} = K\hat{\varrho}$ (cf. Remark 2.8) the exponential stability implies the global uniform asymptotical stability in probability ((G) is guaranteed by (5.8)).

We can easily establish the relation between the various concepts of stability introduced in this paper for $\varrho(x, y) = |x - y|^p$, $p > 0$, in the linear case, i.e., for the equation

$$\text{(5.10)} \qquad d\xi_t = (B\xi_t + m)\, dt + \sum_{r=1}^{l} (\sigma_r \xi_t + k_r)\, dw_r(t),$$

where B and σ_r are constant $n \times n$ matrices and $m, k_r \in \mathbb{R}_n$. From Lemmas 6.4.1, 6.4.2 and 6.4.3 in [3] we obtain that if the equation (5.10) is asymptotically stable in the mean, then it is exponentially stable (and thus, globally uniformly asymptotically stable in probability). On the other hand, if (5.10) is asymptotically stable in probability with $\varrho(x, y) = |x - y|^p$, $p > 0$, then it is asymptotically stable in the mean with $\varrho(x, y) = |x - y|^q$, $q > 0$ (possibly $q < p$).

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