Časopis pro pěstování matematiky

Jaromír Duda

Varieties having directly decomposable congruence classes

Časopis pro pěstování matematiky, Vol. 111 (1986), No. 4, 394--403

Persistent URL: `http://dml.cz/dmlcz/118287`

VARIETIES HAVING DIRECTLY DECOMPOSABLE CONGRUENCE CLASSES

JAROMÍR DUDA, Brno

(Received May 22, 1984)

A number of concepts was introduced on the Cartesian product of similar algebras. Among them the notion of *directly decomposable congruences* appears comparatively often in the literature. There is no doubt that this in particular follows from the fact that the varieties having directly decomposable congruences are *Mal'cev definable*, see [7]. In the present paper we study the *direct decomposability of congruence classes*. It is shown that also the varieties having directly decomposable congruence classes can be characterized by Mal'cev condition, moreover, the identities obtained are simpler than those describing the varieties with directly decomposable congruences. Recently, it was proved independently in [11], [6] that the varieties having *directly decomposable tolerances* and *directly decomposable compatible reflexive relations* are also definable by certain identities. Following these results we close the paper with Mal'cev characterizations of the varieties having *directly decomposable tolerance classes* and *directly decomposable relation classes* (see the definitions below for these concepts).

1. MAL'CEV CONDITION FOR VARIETIES HAVING DIRECTLY DECOMPOSABLE CONGRUENCE CLASSES

Definition 1. We say that a variety $\mathscr{V}$ has *directly decomposable congruence classes* if any congruence class C of the Cartesian product $\mathfrak{A} \times \mathfrak{B} \in \mathscr{V}$ can be written as a product $C = pr_{\mathfrak{A}}\, C \times pr_{\mathfrak{B}}\, C$.

The following auxiliary result will be useful in the sequel.

Lemma 1. *Let C be a subset of the Cartesian product $A \times B$. Then the following conditions are equivalent:*

(i) $C = pr_A\, C \times pr_B\, C$;

(ii) $\langle x, y\rangle, \langle u, v\rangle \in C$ *imply* $\langle x, v\rangle \in C$ *for any elements* $x, u \in A$ *and* $y, v \in B$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). It suffices to verify the inclusion $pr_A\, C \times pr_B\, C \subseteq C$. To this end take

$\langle a, b\rangle \in pr_A C \times pr_B C$. Then $\langle a, s\rangle \in C$ nad $\langle t, b\rangle \in C$ for suitable elements $t \in A$ and $s \in B$. Applying the hypothesis (ii) we find $\langle a, b\rangle \in C$ as required.

First we prove the announced Mal'cev condition for varieties having directly decomposable congruence classes.

Theorem 1. *For a variety $\mathscr{V}$ the following conditions are equivalent:*
(1) *$\mathscr{V}$ has directly decomposable congruence classes;*
(2) *there exist binary polynomials $s_1, \ldots, s_m$, $t_1, \ldots, t_m$ and $(2 + m)$-ary polynomials $c_1, \ldots, c_n$ such that*

$$x = c_1(x, y, s_1(x, y), \ldots, s_m(x, y)),$$
$$x = c_1(y, x, t_1(x, y), \ldots, t_m(x, y)),$$
$$c_k(y, x, s_1(x, y), \ldots, s_m(x, y)) = c_{k+1}(x, y, s_1(x, y), \ldots, s_m(x, y)),$$
$$c_k(x, y, t_1(x, y), \ldots, t_m(x, y)) = c_{k+1}(y, x, t_1(x, y), \ldots, t_m(x, y)), \quad 1 \leqq k < n,$$
$$x = c_n(y, x, s_1(x, y), \ldots, s_m(x, y)),$$
$$y = c_n(x, y, t_1(x, y), \ldots, t_m(x, y))$$

hold in $\mathscr{V}$.

Proof. (1) $\Rightarrow$ (2). Take $\mathfrak{A} = \mathfrak{B} = \mathfrak{F}_{\mathscr{V}}(x, y)$, the free algebra in $\mathscr{V}$ on free generators x and y. Denote by C the congruence class $[\langle x, y\rangle]\, \Theta(\langle x, y\rangle, \langle y, x\rangle)$ of $\mathfrak{A} \times \mathfrak{B}$. Since $\langle x, y\rangle \in C$ and $\langle y, x\rangle \in C$ the hypothesis of direct decomposability yields that $\langle x, x\rangle \in C$ or, equivalently, $\langle\langle x, x\rangle, \langle x, y\rangle\rangle \in \Theta(\langle x, y\rangle, \langle y, x\rangle)$. Applying the binary scheme, see [3], to the last statement we find that

$$\langle x, x\rangle = \gamma_1(\langle x, y\rangle, \langle y, x\rangle),$$
$$\gamma_k(\langle y, x\rangle, \langle x, y\rangle) = \gamma_{k+1}(\langle x, y\rangle, \langle y, x\rangle), \quad 1 \leqq k < n,$$
$$\langle x, y\rangle = \gamma_n(\langle y, x\rangle, \langle x, y)$$

for some binary algebraic functions $\gamma_1, \ldots, \gamma_n$ over the algebra $\mathfrak{A} \times \mathfrak{B}$. Using the fact that $\mathfrak{A} = \mathfrak{B} = \mathfrak{F}_{\mathscr{V}}(x, y)$ we can express the foregoing equalities in the form

$$\langle x, x\rangle = c_1(\langle x, y\rangle, \langle y, x\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_m, t_m\rangle),$$
$$c_k(\langle y, x\rangle, \langle x, y\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_m, t_m\rangle) =$$
$$= c_{k+1}(\langle x, y\rangle, \langle y, x\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_m, t_m\rangle), \quad 1 \leqq k < n,$$
$$\langle x, y\rangle = c_n(\langle y, x\rangle, (x, y\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_m, t_m\rangle),$$

where $s_i = s_i(x, y)$, $t_i = t_i(x, y)$, $i = 1, \ldots, m$, and $\gamma_k(\langle u, v\rangle, \langle w, z\rangle) = c_k(\langle u, v\rangle, \langle w, z\rangle, \langle s_1(x, y), t_1(x, y)\rangle, \ldots, \langle s_m(x, y), t_m(x, y)\rangle)$ for some $(2 + m)$-ary polynomials c_k, $k = 1, \ldots, n$, as follows from the definition of an algebraic function, see e.g. [8].

Writing these relations separately in each variable the desired identities of (2) readily follow.

(2) $\Rightarrow$ (1). Let C be an arbitrary congruence class (say of the congruence Ψ) on the Cartesian product $\mathfrak{A} \times \mathfrak{B} \in \mathscr{V}$. Take elements $\langle x, y\rangle$ and $\langle u, v\rangle$ from C. By writing u instead of y in identities of (2) containing polynomials $s_1(x, y), \ldots$ $\ldots, s_m(x, y)$ and v instead of x in the remaining ones we get

$$x = c_1(x, u, s_1(x, u), \ldots, s_m(x, u)),$$
$$v = c_1(y, v, t_1(v, y), \ldots, t_m(v, y)),$$
$$c_k(u, x, s_1(x, u), \ldots, s_m(x, u)) = c_{k+1}(x, u, s_1(x, u), \ldots, s_m(x, u)),$$
$$c_k(v, y, t_1(v, y), \ldots, t_m(v, y)) = c_{k+1}(y, v, t_1(v, y), \ldots, t_m(v, y)), \quad 1 \leqq k < n,$$
$$x = c_n(u, x, s_1(x, u), \ldots, s_m(x, u)),$$
$$y = c_n(v, y, t_1(v, y), \ldots, t_m(v, y)).$$

Simultaneously, we have

(*) $\langle\langle x, y\rangle, \langle u, v\rangle\rangle \in \Psi$, by hypothesis,
(**) $\langle\langle u, v\rangle, \langle x, y\rangle\rangle \in \Psi$, by the symmetry of Ψ, and
(***) $\langle\langle s_i(x, u), t_i(v, y)\rangle, \langle s_i(x, u), t_i(v, y)\rangle\rangle \in \Psi$, $i = 1, \ldots, m$, by the reflexity of Ψ.

Applying the $(2 + m)$-ary polynomials $c_1, \ldots, c_n$ to (*), (**), (***), and using the transitivity of Ψ, we find that $\langle\langle x, v\rangle, \langle x, y\rangle\rangle \in \Psi$ or, equivalently, $\langle x, v\rangle \in$ $\in [\langle x, y\rangle] \Psi = C$. Lemma 1 completes the proof.

2. DIRECTLY DECOMPOSABLE CONGRUENCE CLASSES ON n-PERMUTABLE AND ON MODULAR VARIETIES

In this section we first study the direct decomposability of congruence classes on *n-permutable varieties*. Identities obtained for $n = 2$ and $n = 3$ are of the greatest interest, however, in order to arrange this part conveniently we give here also Mal'cev characterizations of n-permutable varieties with directly decomposable congruence classes for arbitrary integers $n > 1$.

Theorem 2. *For a variety $\mathscr{V}$ and an integer $n > 1$ the following conditions are equivalent:*

(1) *$\mathscr{V}$ has n-permutable congruences and directly decomposable congruence classes;*

(2) *there exist binary polynomials $s_1, \ldots, s_m$, $t_1, \ldots, t_m$ and $(1 + m)$-ary polynomials $d_1, \ldots, d_{n-1}$ such that*

$$x = d_1(x, s_1(x, y), \ldots, s_m(x, y)),$$
$$x = d_1(y, t_1(x, y), \ldots, t_m(x, y)),$$
$$d_k(y, s_1(x, y), \ldots, s_m(x, y)) = d_{k+1}(x, s_1(x, y), \ldots, s_m(x, y)),$$
$$d_k(x, t_1(x, y), \ldots, t_m(x, y)) = d_{k+1}(y, t_1(x, y), \ldots, t_m(x, y)), \quad 1 \leqq k < n - 1,$$

$$x = d_{n-1}(y, s_1(x, y), \dots, s_m(x, y)),$$
$$y = d_{n-1}(x, t_1(x, y), \dots, t_m(x, y))$$

hold in $\mathscr{V}$.

The proof depends on a lemma. As usual, the symbol $R(a, b)$ denotes the smallest reflexive compatible relation containing the pair $\langle a, b\rangle$.

Lemma 2. *For a variety* $\mathscr{V}$ *and an integer* $n > 1$ *the following conditions are equivalent*:

(i) $\mathscr{V}$ *has n-permutable congruences, i.e.* $\Theta \vee \Psi = \Theta \circ \Psi \circ \dots$ *(n factors) for any* $\Theta, \Psi \in \operatorname{Con} \mathfrak{A}$, $\mathfrak{A} \in \mathscr{V}$;

(ii) $\Theta(a, b) = R(a, b) \circ \dots \circ R(a, b)$ *((n − 1) factors) for any* $a, b \in \mathfrak{A} \in \mathscr{V}$.

Proof. See [4].

Proof of Theorem 2. (1) ⇒ (2). Analogously as in the proof of Theorem 1 the hypothesis of direct decomposability of congruence classes gives that $\langle\langle x, x\rangle, \langle x, y\rangle\rangle \in \Theta(\langle x, y\rangle, \langle y, x\rangle)$ on the Cartesian product $\mathfrak{A} \times \mathfrak{A} = \mathfrak{F}_{\mathscr{V}}(x, y) \times \mathfrak{F}_{\mathscr{V}}(x, y)$. Combining this with Lemma 2 we have $\langle\langle x, x\rangle, \langle x, y\rangle\rangle \in R^{n-1}(\langle x, y\rangle, \langle y, x\rangle)$. Now using the definition of the relation product and the functional description of a reflexive compatible relation from [2], we obtain unary algebraic functions $\delta_1, \dots, \delta_{n-1}$ over $\mathfrak{A} \times \mathfrak{A}$ such that

$$\langle x, x\rangle = \delta_1(\langle x, y\rangle),$$
$$\delta_k(\langle y, x\rangle) = \delta_{k+1}(\langle x, y\rangle), \quad 1 \leqq k < n - 1,$$
$$\langle x, y\rangle = \delta_{n-1}(\langle y, x\rangle).$$

Since $\mathfrak{A} = \mathfrak{F}_{\mathscr{V}}(x, y)$ these equalities can be expressed in the form

$$\langle x, x\rangle = d_1(\langle x, y\rangle, \langle s_1, t_1\rangle, \dots, \langle s_m, t_m\rangle),$$
$$d_k(\langle y, x\rangle, \langle s_1, t_1\rangle, \dots, \langle s_m, t_m\rangle) = d_{k+1}(\langle x, y\rangle, \langle s_1, t_1\rangle, \dots, \langle s_m, t_m\rangle),$$
$$1 \leqq k < n - 1,$$
$$\langle x, y\rangle = d_{n-1}(\langle y, x\rangle, \langle s_1, t_1\rangle, \dots, \langle s_m, t_m\rangle)$$

for some binary polynomials $s_i = s_i(x, y)$, $t_i = t_i(x, y)$, $i = 1, \dots, m$, and suitable $(1 + m)$-ary polynomials $d_1, \dots, d_{n-1}$ of $\mathscr{V}$. Writing these relations separately in each variable we immediately get the identities of (2).

(2) ⇒ (1). Conversely, assume the identities (2). Then it is easy to check that the ternary polynomials $p_1, \dots, p_{n-1}$ given by $p_k(u, v, w) = d_k(v, t_1(u, w), \dots, t_m(u, v))$, $k = 1, \dots, n - 1$, satisfy the identities

$$u = p_1(u, w, w),$$
$$p_k(u, u, w) = p_{k-1}(u, w, w), \quad 1 \leqq k < n - 1,$$
$$w = p_{n-1}(u, u, w).$$

By [9], $\mathscr{V}$ has n-permutable congruences.

Finally, the direct decomposability of congruence classes is ensured by $(2 + m)$-ary polynomials $c_k(u, v, w_1, \ldots, w_m) = d_k(u, w_1, \ldots, w_m)$, $k = 1, \ldots, n - 1$, see Theorem 1 (2). The proof is complete.

Perhaps the most important consequence is

Corollary 1. *For a variety $\mathscr{V}$ the following conditions are equivalent:*

(1) *$\mathscr{V}$ has permutable and directly decomposable congruences;*

(2) *$\mathscr{V}$ has permutable congruences and directly decomposable congruence classes;*

(3) *there exist binary polynomials $s_1, \ldots, s_m$, $t_1, \ldots, t_m$ and a $(1 + m)$-ary polynomial d_1 such that*

$$\begin{aligned} x &= d_1(x, s_1(x, y), \ldots, s_m(x, y)), \\ x &= d_1(y, t_1(x, y), \ldots, t_m(x, y)), \\ x &= d_1(y, s_1(x, y), \ldots, s_m(x, y)), \\ y &= d_1(x, t_1(x, y), \ldots, t_m(x, y)) \end{aligned}$$

hold in $\mathscr{V}$.

Proof. The identities characterizing varieties with permutable and directly decomposable congruences, see [5], coincide with those of Corollary 1 (3), hence we have (1) $\Leftrightarrow$ (3). The equivalence (2) $\Leftrightarrow$ (3) follows directly from Theorem 2.

Similarly, Theorem 2 and [5] yield

Corollary 2. *For a variety $\mathscr{V}$ the following conditions are equivalent:*

(1) *$\mathscr{V}$ has 3-permutable and directly decomposable congruences;*

(2) *$\mathscr{V}$ has 3-permutable congruences and directly decomposable congruence classes;*

(3) *there exist binary polynomials $s_1, \ldots, s_m$, $t_1, \ldots, t_m$ and $(1 + m)$-ary polynomials d_1, d_2 such that*

$$\begin{aligned} & x = d_1(x, s_1(x, y), \ldots, s_m(x, y)), \\ & x = d_1(y, t_1(x, y), \ldots, t_m(x, y)), \\ & d_1(y, s_1(x, y), \ldots, s_m(x, y)) = d_2(x, s_1(x, y), \ldots, s_m(x, y)), \\ & d_1(x, t_1(x, y), \ldots, t_m(x, y)) = d_2(y, t_1(x, y), \ldots, t_m(x, y)), \\ & x = d_2(y, s_1(x, y), \ldots, s_m(x, y)), \\ & y = d_2(x, t_1(x, y), \ldots, t_m(x, y)) \end{aligned}$$

hold in $\mathscr{V}$.

B. Jónsson [10] has shown (see also [8; p. 30]) that any algebra with 3-permutable congruences is *congruence modular.* Hence the following Theorem 3 is a strengthening of Corollary 2.

Theorem 3. *Let V be a congruence modular variety. Then the following conditions are equivalent*:

(1) $\mathscr{V}$ *has directly decomposable congruences;*

(2) $\mathscr{V}$ *has directly decomposable congruence classes.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Take the congruence class $C = [\langle a, b\rangle]\, \Theta(\langle a, b\rangle, \langle a', b'\rangle)$ on the Cartesian product $\mathfrak{A} \times \mathfrak{B} \in \mathscr{V}$. Since $\langle a, b\rangle \in C$ and $\langle a', b'\rangle \in C$ the hypothesis of direct decomposability yields $\langle a, b'\rangle \in C$ and thus also $\langle\langle a, b\rangle, \langle a, b'\rangle\rangle \in \Theta(\langle a, b\rangle, \langle a', b'\rangle)$. It is known, see [12; Proposition 3, p. 100], that the last statement is equivalent to the direct decomposability of congruences on congruenc modular varieties.

We close this section with some

Examples. 1. The *variety of rings with* 1 has simple polynomials ensuring permutability and direct decomposability of congruences: Take $d_1(a, b, c) = a \,.\, b + c$ and $s_1 = 0$, $s_2 = x$, $t_1 = -1$, $t_2 = x + y$.

2. The *variety of implicative algebras* can be used as a suitable example of a 3-permutable variety with directly decomposable congruences (recall from [9] that an implicative algebra is a groupoid satisfying the identities $(xy)\,x = x$, $(xy)\,y = (yx)\,x$, $x(yz) = y(xz)$ and thus also $(xx)\,y = y$). In this case we take $d_1(a, b, c) = (ca)\,b$, $d_2(a, b, c) = (ba)\,c$ and $s_1 = s_2 = t_1 = x$, $t_2 = y$.

Then

$$\begin{aligned}
d_1(x, s_1, s_2) &= (xx)\,x = x\,,\\
d_1(y, t_1, t_2) &= (yy)\,x = x\,,\\
d_1(y, s_1, s_2) &= (xy)\,x = x = (xx)\,x = d_2(x, s_1, s_2)\,,\\
d_1(x, t_1, t_2) &= (yx)\,x = (xy)\,y = d_2(y, t_1, t_2)\,,\\
d_2(y, s_1, s_2) &= (xy)\,x = x\,,\\
d_2(x, t_1, t_2) &= (xx)\,y = y\,.
\end{aligned}$$

3. Further, the important *variety of commutative* BCK-*algebras* (see [1] and references there) has 3-permutable and directly decomposable congruences (recall from [1] that a commutative BCK-algebra is a groupoid with a distinguishing element 0 satisfying the identities $xx = 0$, $x0 = x$, $x(xy) = y(yx)$, $(xy)\,z = (xz)\,y$ and hence also $x(0y) = x$). Take $d_1(a, b, c) = b[(ac)\,(ab)]$, $d_2(a, b, c) = c[(ab)\,(ac)]$ and $s_1 = s_2 = t_1 = x$, $t_2 = y$. Then

$$\begin{aligned}
d_1(x, s_1, s_2) &= x[(xx)\,(xx)] = x\,,\\
d_1(y, t_1, t_2) &= x[(yy)\,(yx)] = x\,,\\
d_1(y, s_1, s_2) &= x[(yx)\,(yx)] = x = x[(xx)\,(xx)] = d_2(x, s_1, s_2)\,,\\
d_1(x, t_1, t_2) &= x[(xy)\,(xx)] = x(xy) = y(yx) = y[(yx)\,(yy)] = d_2(y, t_1, t_2)\,,
\end{aligned}$$

$$d_2(y, s_1, s_2) = x[(yx)(yx)] = x,$$
$$d_2(x, t_1, t_2) = y[(xx)(xy)] = y$$
is a concrete form of the identities from Corollary 2 (3).

3. TWO GENERALIZATIONS: DIRECT DECOMPOSABILITY OF RELATION CLASSES AND TOLERANCE CLASSES

The aim of this section is to show that the direct decomposability can be studied not only on congruence classes but also on classes of more general compatible binary relations. As a result two new Mal'cev conditions are obtained. First we need

Definition 2. Let R be a reflexvie compatible binary relation on an algebra $\mathfrak{A}$, $a \in A$. Then the subset $[a] R = \{x \in \mathfrak{A};\ \langle x, a \rangle \in R\}$ is called a *relation class.*

In particular, $[a]\, T$ is called a *tolerance class* provided T is a tolerance (i.e. a reflexive compatible and symmetric binary relation) on $\mathfrak{A}$.

Definition 3. A variety $\mathscr{V}$ has *directly decomposable relation (tolerance) classes* if any relation (tolerance, respectively) class C of the Cartesian product $\mathfrak{A} \times \mathfrak{B} \in \mathscr{V}$ is of the form $C = pr_{\mathfrak{A}}\, C \times pr_{\mathfrak{B}}\, C$.

Now we are ready to characterize varieties having directly decomposable tolerance classes.

Theorem 4. *For a variety $\mathscr{V}$ the following conditions are equivalent:*

(1) *$\mathscr{V}$ has directly decomposable tolerance classes;*

(2) *there exist ternary polynomials $s_1, \ldots, s_m$, $t_1, \ldots, t_m$ and a $(4 + m)$-ary polynomial f such that*

$$x = f(x, y, z, z, s_1(x, y, z), \ldots, s_m(x, y, z)),$$
$$y = f(x, y, z, z, t_1(x, y, z), \ldots, t_m(x, y, z)),$$
$$z = f(z, z, x, y, s_1(x, y, z), \ldots, s_m(x, y, z)),$$
$$z = f(z, z, x, y, t_1(x, y, z), \ldots, t_m(x, y, z))$$

hold in $\mathscr{V}$.

Proof. (1) $\Rightarrow$ (2). Denote by T the smallest tolerance on the Cartesian product $\mathfrak{F}_{\mathscr{V}}(x, y, z) \times \mathfrak{F}_{\mathscr{V}}(x, y, z)$ containing the pairs $\langle\langle x, x\rangle, \langle z, z\rangle\rangle$ and $\langle\langle y, y\rangle, \langle z, z\rangle\rangle$ (i.e. $T = T(\langle\langle x, x\rangle, \langle z, z\rangle\rangle, \langle\langle y, y\rangle, \langle z, z\rangle\rangle)$). Then the tolerance class ${}^dC = [\langle z, z\rangle]\, T$ contains the elements $\langle x, x\rangle$, $\langle y, y\rangle$ and thus, by hypothesis, also $\langle x, y\rangle \in C$. Equivalently, $\langle\langle x, y\rangle, \langle z, z\rangle\rangle \in T$ holds. Now applying the well-known functional description of tolerances, see e.g. [2], to T we get that

$$\langle x, y\rangle = \varphi(\langle x, x\rangle, \langle y, y\rangle, \langle z, z\rangle, \langle z, z\rangle),$$

$$\langle z, z\rangle = \varphi(\langle z, z\rangle, \langle z, z\rangle, \langle x, x\rangle, \langle y, y\rangle),$$

where φ is a suitable 4-ary algebraic function over $\mathfrak{F}_{\mathscr{V}}(x, y, z) \times \mathfrak{F}_{\mathscr{V}}(x, y, z)$. By the standard technique we obtain

$$\langle x, y\rangle = f(\langle x, x\rangle, \langle y, y\rangle, \langle z, z\rangle, \langle z, z\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_m, t_m\rangle),$$
$$\langle z, z\rangle = f(\langle z, z\rangle, \langle z, z\rangle, \langle x, x\rangle, \langle y, y\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_m, t_m\rangle)$$

for some ternary polynomials $s_i = s_i(x, y, z)$, $t_i = t_i(x, y, z)$, $i = 1, \ldots, m$, and a $(4 + m)$-ary polynomial f. Writing the above equalities componentwise we immediately get the desired identities from (2).

(2) $\Rightarrow$ (1). Conversely, consider an arbitrary tolerance class $D = [\langle z_1, z_2\rangle] S$ on the Cartesian product $\mathfrak{A} \times \mathfrak{B} \in \mathscr{V}$. Choose elements $\langle x, y\rangle, \langle u, v\rangle \in D$. By Lemma 1 it suffices to show that also $\langle x, v\rangle \in D$. To this end we write u instead of y, z_1 instead of z in the identities from (2) containing $s_1, \ldots, s_m$ and, further, v instead of y, y instead of x and z_2 instead of z in the remaining ones. In this way we get that

$$\begin{aligned}
x &= f(x, u, z_1, z_1, s_1(x, u, z_1), \ldots, s_m(x, u, z_1)),\\
v &= f(y, v, z_2, z_2, t_1(y, v, z_2), \ldots, t_m(y, v, z_2)),\\
z_1 &= f(z_1, z_1, x, u, s_1(x, u, z_1), \ldots, s_m(x, u, z_1)),\\
z_2 &= f(z_2, z_2, y, v, t_1(y, v, z_2), \ldots, t_m(y, v, z_2)).
\end{aligned}$$

Simultaneously, we have

(*) $\langle\langle x, y\rangle, \langle z_1, z_2\rangle\rangle \in S$,
$\langle\langle u, v\rangle, \langle z_1, z_2\rangle\rangle \in S$ by hypothesis;

(**) $\langle\langle z_1, z_2\rangle, \langle x, y\rangle\rangle \in S$,
$\langle\langle z_1, z_2\rangle, \langle u, v\rangle\rangle \in S$ by the symmetry of S and

(***) $\langle\langle s_i(x, u, z_1), t_i(y, v, z_2)\rangle, \langle s_i(x, u, z_1), t_i(y, v, z_2)\rangle\rangle \in S$, $i = 1, \ldots, m$, by the reflexivity of S.

Applying the $(4 + m)$-ary polynomial f to (*), (**) and (***) we conclude that $\langle\langle x, v\rangle, \langle z_1, z_2\rangle\rangle \in S$. Hence $\langle x, v\rangle \in D$ which was to be proved.

Our last theorem characterizes the varieties having directly decomposable relation classes. The proof of this statement follows the same line as that of Theorem 4 and is therefore omitted.

Theorem 5. *For a variety $\mathscr{V}$ the following conditions are equivalent:*

(1) *$\mathscr{V}$ has directly decomposable relation classes;*

(2) *there exist ternary polynomials $s_1, \ldots, s_m$, $t_1, \ldots, t_m$ and a $(2 + m)$-ary polynomial g such that*

$$x = g(x, y, s_1(x, y, z), \ldots, s_m(x, y, z)),$$

$$y = g(x, y, t_1(x, y, z), \ldots, t_m(x, y, z)),$$
$$z = g(z, z, s_1(x, y, z), \ldots, s_m(x, y, z)),$$
$$z = g(z, z, t_1(x, y, z), \ldots, t_m(x, y, z))$$

hold in $\mathscr{V}$.

For illustration we present

Example 4. The *variety of lattices* has directly decomposable relation classes: Take $g(a, b, c, d) = (a \wedge c) \vee (b \wedge d)$ and $s_1 = x \vee z$, $s_2 = x \wedge z$, $t_1 = y \wedge z$, $t_2 = y \vee z$.

Then

$$\begin{aligned} g(x, y, s_1, s_2) &= x \vee (x \wedge y \wedge z) = x, \\ g(x, y, t_1, t_2) &= (x \wedge y \wedge z) \vee y = y, \\ g(z, z, s_1, s_2) &= z \vee (x \wedge z) \quad = z, \\ g(z, z, t_1, t_2) &= (y \wedge z) \vee z \quad = z. \end{aligned}$$

Notice that also $s_1 = t_2 = x \vee y \vee z$, $s_2 = t_1 = x \wedge y \wedge z$ can be used.

4. CONCLUDING REMARKS

(i) The original Mal'cev condition characterizing the varieties with directly decomposable congruences, see [7], involves some ternary polynomials. Our Theorem 3 shows that in the case of congruence modular varieties binary polynomials are sufficient. This fact was for the first time observed by H. Werner [12].

(ii) Evidently, the direct decomposability of congruences, tolerances, etc. imply the direct decomposability of the corresponding relation classes. Using Mal'cev conditions, this fact is clearly visible: Mal'cev conditions from our Theorem 4 and Theorem 5 arise from those of [6, 11] by identifying $z = u$.

(iii) In a recent paper we have shown that regularity of tolerance implies permutability of congruences on a given variety. Unfortunately this phenomenon holds neither for direct decomposability of tolerance classes nor for relation classes. Counterexample: lattices.

References

[1] *W. H. Cornish:* A minimal equational base for commutative BCK-algebras. Math. Sem. Kobe Univ. *10* (1982), 251—259.

[2] *I. Chajda:* Lattices of compatible relations. Arch Math. (Brno) *13* (1977), 89—96.

[3] *J. Duda:* On two schemes applied to Mal'cev type theorems. Ann. Univ. Sci. Budapest, Sectio Math. *26* (1983), 39—45.

[4] *J. Duda:* A Mal'cev characterization of n-permutable varieties with directly decomposable congruences. Algebra Univ. *16* (1983), 269—274.

[5] *J. Duda:* 3-permutable and directly decomposable congruences. Glasnik Matematički (Zagreb), to appear.
[6] *J. Duda:* Varieties with directly decomposable diagonal subalgebras form Mal'cev classes. Ann. Univ. Sci. Budapest, Sectio Math., to appear.
[7] *G. A. Fraser, A. Horn:* Congruence relations in direct products. Proc. Amer. Math. Soc. *26* (1970), 390—394.
[8] *G. Grätzer:* Universal Algebra, Second Expanded Edition. Springer-Verlag, Berlin, Heidelberg and New York, 1979.
[9] *J. Hagemann, A. Mitsche:* On n-permutable congruences. Algebra Univ. *3* (1973), 8—12.
[10] *B. Jónsson:* On the representation of lattices. Math. Scand. *1* (1953), 193—206.
[11] *J. Niederle:* Decomposition conditions for compatible relations. Czech. Math. J. *33* (1983), 522—524.
[12] *H. Werner:* Congruences on products of algebras and functionally complete algebras. Algebra Univ. *4* (1974), 99—105.

Author's address: 616 00 Brno 16, Kroftova 21.