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EQUIVALENCE OF VOLTERRA INTEGRAL EQUATIONS

MARKO ŠVEC, Bratislava

Dedicated to Professor Kurzweil on the occasion of his sixtieth birthday

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The purpose of this paper is to investigate the equivalence of the following two integral equations:

$$y(t) = f(t) + \int_0^t a(t, s)\, y(s)\, ds, \tag{1}$$

$$x(t) \in f(t) + \int_0^t a(t, s)\, x(s)\, ds + \int_0^t b(t, s)\, g(s, x(s))\, ds \tag{2}$$

where $x, y, f\colon J = \langle 0, \infty) \to \mathbb{R}^n$ are n-dimensional functions, $g(t, x)\colon J \times \mathbb{R}^n \to 2^{R^n}$ is a multifunction, $a, b\colon J \times \mathbb{R}^{n^2}$ are matrix functions.

We will also investigate the equivalence of the integro-differential equations

$$y'(t) = F(t) + A(t)\, y(t) + \int_0^t B(t, s)\, y(s)\, ds, \tag{3}$$

$$x'(t) \in F(t) + A(t)\, x(t) + \int_0^t B(t, s)\, x(s)\, ds + \int_0^t C(t, s)\, g(s, x(s))\, ds \tag{4}$$

where $x, y, F\colon J \to \mathbb{R}^n$ are n-dimensional vector functions, $A\colon J \to \mathbb{R}^{n^2}$, B and $C\colon J \times J \to \mathbb{R}^{n^2}$ are matrix functions and g as above is a multifunction. $|\cdot|$ will denote a suitable vector (matrix) norm.

Definition 1. Let $\psi\colon J \to \mathbb{R}$ be a positive continuous function. We say that (1) and (2) ((3) and (4)) are *ψ-asymptotically equivalent* on J if for each solution $y(t)$ of (1) (of (3)) existing on J there exists a solution $x(t)$ of (2) (of (4)) defined on J such that

$$\lim \psi^{-1}(t)\, |y(t) - x(t)| = 0 \quad \text{as} \quad t \to \infty \tag{5}$$

and conversely, for each solution $x(t)$ of (2) (of (4)) existing on J there exists a solution $y(t)$ of (1) (of (3)) defined on J such that (5) holds.

Definition 2. Let ψ be as in Definition 1 and let $p > 0$. We say that the equations (1) and (2) ((3) and (4)) are (ψ, p) – *integrally equivalent* on J if for each solution $y(t)$ of (1) (of (3)) existing on J there is a solution $x(t)$ of (2) (of (4)) existing on J such that

$$\psi^{-1}(t)\,|y(t) - x(t)| \in L_p(J) \tag{6}$$

and conversely, for each solution $x(t)$ of (2) (of (4)) existing on J there is a solution $y(t)$ of (1) (of (3)) defined on J such that (6) is true.

Definition 3. Let ψ be as in Definition 1. We say that a function $z\colon J \to \mathbb{R}^n$ is ψ-*bounded* on J if

$$\sup \psi^{-1}(t)\,|z(t)| < \infty\,. \tag{7}$$

Remark 1. The asymptotic equivalence of (1) and (2) (and of (3) and (4)) was studied e.g. by J. A. Nohel [1], [2] and by A. C. Lima [3] in the case that g is a real vector function.

1.

We start with the study of the equivalence between (1) and (2). We first proceed formally using the resolvent kernel $r(t, s)$ belonging to the kernel $a(t, s)$. That is, $r(t, s)$ is a solution of the equation

$$r(t, s) = -a(t, s) + \int_s^t a(t, u)\, r(u, s)\,\mathrm{d}u\,, \quad 0 \leqq s \leqq t\,. \tag{8}$$

Then the solution $y(t)$ of (1) is of the form

$$y(t) = f(t) - \int_0^t r(t, s)\, f(s)\,\mathrm{d}s\,. \tag{9}$$

Let $x(t)$ be a solution of (2). Then there exists a function

$$v(t) \in g(t, x(t)) \quad \text{a.e. on } J \tag{10}$$

which is measurable and locally integrable, such that

$$x(t) = f(t) + \int_0^t a(t, s)\, x(s)\,\mathrm{d}s + \int_0^t b(t, s)\, v(s)\,\mathrm{d}s\,, \quad t \in J\,. \tag{11}$$

Applying formula (9) we obtain

$$x(t) = y(t) + \int_0^t \left\{ b(t, s) - \int_s^t r(t, u)\, b(u, s)\,\mathrm{d}u \right\} v(s)\,\mathrm{d}s$$

or

$$x(t) = y(t) + \int_0^t R(t, s)\, v(s)\, \mathrm{d}s\,, \quad t \in J \tag{12}$$

where

$$R(t, s) = b(t, s) - \int_s^t r(t, u)\, b(u, s)\, \mathrm{d}u\,. \tag{13}$$

Because all operations used here are reversible we can get (11) from (12). The correctness of all steps used here is guaranteed e.g. by the assumption of local integrability of $a(t, s)$, $b(t, s)$ and f or of local integrability of $|a(t, s)|^p$, $|b(t, s)|^p$ and $|f(t)|^p$, $p > 1$. We shall always suppose the continuity of $f(t)$ on J.

Definition 4. Let $A \subset \mathbb{R}^n$. Then $|A| = \sup\{|a| : a \in A\}$.

Notations.

$\psi(t)$ and $\varphi(t)$ are positive continuous functions on J;

$\mathscr{C}(J, \mathbb{R}^n) = \{$the set of all continuous functions $\varphi\colon J \to \mathbb{R}^n$ topologized with the compact-open topology$\}$;

$B_\psi = B_\psi(J, \mathbb{R}^n) = \{$the set of all continuous functions $z\colon J \to \mathbb{R}^n$ such that $\sup_J \psi^{-1}(t)\, |z(t)| = \|z\|_\psi < \infty\}$;

$B_{\psi,\varrho} = \{z(t) \in B_\psi : \|z\|_\psi \leqq \varrho\}$;

$L_\infty = L_\infty(J, \mathbb{R}^n) = \{$the set of all measurable and essentially bounded functions on $J\}$, $\|z\|_\infty = \operatorname{ess\,sup}_J |z(t)|$;

$L_{p,\psi}(J, \mathbb{R}^n) = \{$the set of all $z\colon J \to \mathbb{R}^n$ such that $\psi^{-1}(t)\, z(t) \in L_p(J, \mathbb{R}^n)\}$, $\|z\|_{p,\psi} = \|\psi^{-1}(t)\, z(t)\|_p$;

$LL_p(J, \mathbb{R}^n) = \{$the set of all functions $z\colon J \to \mathbb{R}^n$ such that $z(t) \in L_p(I, \mathbb{R}^n)$ where I is any compact subinterval of $J\}$.

Let X be a linear topological space and let $A \subset X$. Then $\mathrm{cf}(A)$ denotes the family of all convex and closed subsets of A.

Let $g(t, x)\colon J \times \mathbb{R}^n \to \Omega(\mathbb{R}^n)$, where $\Omega(\mathbb{R}^n)$ denotes the set of all nonempty compact subsets of $\mathbb{R}^n$. Let $z(t) \in B_\psi$. Then by $M(z(t))$ we denote the set of all measurable selectors from $g(t, z(t))$.

Lemma 1. *Let $\psi(t)$, $\varphi(t)$ be positive continuous functions on J. Let the following assumptions be satisfied:*

$$p \in (1, \infty)\,, \quad p^{-1} + q^{-1} = 1\,,$$

a) *$|R(t, s)|^p$ is locally integrable on $0 \leqq s \leqq t < \infty$;*

b) *there exists a constant $K > 0$ such that*

$$\int_0^t |\psi^{-1}(t)|\, R(t, s)|\, \varphi(s)|^p\, \mathrm{d}s \leqq K^p/, \quad t \in J$$

and

$$\lim_{h\to 0}\left\{\left|\int_t^{t+h}|R(t+h,s)|^p\,\mathrm{d}s\right|^{1/p}+\left(\int_0^t|R(t+h,s)-R(t,s)|^p\,\mathrm{d}s\right)^{1/p}\right\}=0$$

uniformly with respect to $t\in J$;

c) *the function* $g(t,x)\colon J\times\mathbb{R}^n\to\Omega(\mathbb{R}^n)$ *satisfies the following conditions:*

(H_1) *for each* $(t,x)\in J\times\mathbb{R}^n$, $g(t,x)$ *is convex;*

(H_2) *for each* $t\in J$, $g(t,x)$ *is upper semicontinuous on* $\mathbb{R}^n$;

(H_3) *for each measurable function* $x\colon J\to\mathbb{R}^n$ *there exists a measurable function* $f_x\colon J\to\mathbb{R}^n$ *such that* $f_x(t)\in g(t,x(t))$ *a.e. on* J;

d) *the function* $F\colon J\times J\to J$ *is such that*

(i) $F(t,u)$ *is nondecreasing in* u *for each fixed* $t\in J$ *and integrable on compact subintervals of* J *for each fixed* $u\in J$;

(ii) $\int_0^\infty F^q(t,c)\,\mathrm{d}t<\infty$ *for every* $c\geqq 0$;

(iii) $\liminf_{u\to\infty}\frac{1}{u}\int_0^\infty F^q(t,u)\,\mathrm{d}t=0$;

e) $|g(t,x)|\leqq\varphi(t)\,F(t,\psi^{-1}(t)\,|x|)$;

f) *the function* $y\colon J\to\mathbb{R}^n$ *is continuous and* ψ*-bounded, i.e.* $\|y\|_\psi=\varrho<\infty$.

Then the operator T *defined for* $z(t)\in B_\psi$ *by the relation*

$$T(z(t))=\left\{y(t)+\int_0^t R(t,s)\,v(s)\,\mathrm{d}s\colon v(t)\in M(z(t))\right\}\tag{14}$$

maps B_ψ *into* 2^{B_ψ}, *is precompact and upper semicontinuous in* $\mathscr{C}(J,R^n)$ *and there exists such* B_{ψ,u_0} *that* T *maps* B_{ψ,u_0} *into* $\mathrm{cf}(B_{\psi,u_0})$.

Proof. Let $z(t)\in B_\psi$ and let $\|z\|_\psi=\varrho_0$. Then by (H_1) and (H_2) $M(z(t))$ is nonempty and convex. Respecting the assumptions e), d) (i) we get for $v(t)\in M(z(t))$ the inequalities

$$|v(t)|\leqq|M(z(t))|\leqq\varphi(t)\,F(t,\psi^{-1}(t)|\,z(t)|)\leqq\varphi(t)\,F(t,\varrho_0)$$

and

$$\|v\|_{q,\varphi}\leqq\left(\int_0^\infty F^q(t,\varrho_0)\,\mathrm{d}t\right)^{1/q}.$$

Thus $v(t)\in L_{q,\varphi}(J)$. Then it follows from the continuity of $\varphi(t)$ on J and from the assumption a) that $R(t,s)\,v(s)$ is locally integrable on J. It means that the operator T defined by (14) is well defined.

Denote

$$\xi(t)=y(t)+\int_0^t R(t,s)\,v(s)\,\mathrm{d}s\,,\quad v(t)\in M(z(t))\,.$$

Then

$$\psi^{-1}(t)\,|\xi(t)| \leqq \psi^{-1}(t)|\,y(t)| + \int_0^t \psi^{-1}(t)\,|R(t,s)|\,|v(s)|\,\mathrm{d}s \leqq \tag{15}$$

$$\leqq \varrho + \int_0^t \psi^{-1}(t)\,|R(t,s)|\,\varphi(s)\;F(s,\varrho_0)\,\mathrm{d}s \leqq$$

$$\leqq \varrho + \left(\int_0^t [\psi^{-1}(t)\,|R(t,s)|\varphi\,(s)]^p\,\mathrm{d}s\right)^{1/p}\left(\int_0^t F^q(s,\varrho_0)\,\mathrm{d}s\right)^{1/q} \leqq$$

$$\leqq \varrho + K\left(\int_0^\infty F^q(s,\varrho_0)\,\mathrm{d}s\right)^{1/q}$$

where we have used the Hölder inequality, assumptions b) and d) (ii). Thus $\xi(t)$ is a ψ-bounded function on J.

Let $0 \leqq t_1,\ t_2 = t_1 + h \geqq 0,\ |h| < 1$. Then

$$|\xi(t_2) - \xi(t_1)| \leqq |y(t_2) - y(t_1)| + \tag{16}$$

$$+ \int_0^{t_1} |R(t_2,s) - R(t_1,s)|\,|v(s)|\,\mathrm{d}s + \left|\int_{t_1}^{t_2} |R(t_2,s)|\,|v(s)|\,\mathrm{d}s\right| \leqq$$

$$\leqq |y(t_2) - y(t_1)| + \left|\int_0^{t_1} |R(t_2,s) - R(t_1,s)|^p\,\mathrm{d}s\right|^{1/p} . \left(\int_0^{t_1} [\varphi(s)\,F(s,\varrho_0)]^q\,\mathrm{d}s\right)^{1/q} +$$

$$+ \left|\int_{t_1}^{t_2} |R(t_2,s)|^p\,\mathrm{d}s\right|^{1/p} . \left|\int_{t_1}^{t_2} [\varphi(s)\,F(s,\varphi_0)]^q\,\mathrm{d}s\right|^{1/q} \leqq |y(t_2) - y(t_1)| +$$

$$+ \left\{\left(\int_0^{t_1} |R(t_2,s) - R(t_1,s)|^p\,\mathrm{d}s\right)^{1/p} + \left|\int_{t_1}^{t_2} |R(t_2,s)|^p\,\mathrm{d}s\right|^{1/p}\right\} .$$

$$. \left(\int_0^{t_2} [\varphi(s)\,F(s,\varrho_0)]^q\,\mathrm{d}s\right)^{1/q} .$$

This and the assumption b) imply the continuity of $\xi(t)$ at t_1. Sumarily, we conclude that all functions of $T(z(t))$ are continuous on J and ψ-bounded. Thus $T(z(t)) \subset B_\psi$ and T maps B_ψ into 2^{B_Ψ}. From the convexity of $M(z(t))$, which follows from (H_1), we get the convexity of $T(z(t))$.

Now we shall consider the set $B_{\psi,2\varrho}$. Let $z(t) \in B_{\psi,2\varrho}$. Then for $\xi(t) \in T(z(t))$ we have

$$\xi(t) = y(t) + \int_0^t R(t,s)\,v(s)\,\mathrm{d}s\,, \quad v(t) \in M(z(t))\,.$$

Repeating the same argument as in (15) we get

$$\psi^{-1}(t)\,|\xi(t)| \leqq \varrho + K\left(\int_0^\infty F^q(s,2\varrho)\,\mathrm{d}s\right)^{1/q} . \tag{17}$$

From the assumption d) (iii) we get that for $(2K)^{-1}$ there exists such $u_0 >$ $> \max\{2\varrho, 1\}$ that

$$\int_0^\infty F^q(t, u_0)\,\mathrm{d}t \leqq (2K)^{-q}\,u_0\,.$$

Therefore, taking into account the assumption b) and the monotonicity of F in u, we get

$$\psi^{-1}(t)\,|\xi(t)| \leqq \varrho + K\frac{1}{2K}\,u_0^{1/q}\,.$$

Because $0 < q^{-1} < 1$ and $u_0 > 1$, we have that $u_0^{q^{-1}} < u_0$ and

$$\psi^{-1}(t)\,|\xi(t)| \leqq u_0\,. \tag{18}$$

Thus $|T(z(t))| \leqq u_0$ and because u_0 is the same for all $z(t) \in B_{\psi,2\varrho}$ and all $2\varrho \leqq u_0$ we get

$$TB_{\psi,0} \subset B_{\psi,u_0}\,. \tag{19}$$

This means that all functions from TB_{ψ,u_0} are uniformly bounded on J.

We shall now prove the equicontinuity of all functions from TB_{ψ,u_0} on compact subintervals of J. Let $z(t) \in B_{\psi,u_0}$ and

$$\xi(t) = y(t) + \int_0^t R(t, s)\,v(s)\,\mathrm{d}s\,, \quad v(t) \in M(z(t))\,.$$

Then for $0 \leqq t_1 < t_2 < \infty$ we get the inequality (16) (ϱ substituted by u_0). The expression at the end of (16) is the same for all $z(t) \in B_{\psi,u_0}$. Therefore, from the inequality (16) we obtain the equicontinuity of all functions from TB_{ψ,u_0} on compact subintervals of J. Furthermore, the uniform boundedness and the equicontinuity on compact subintervals of J yield that TB_{ψ,u_0} is precompact in $\mathscr{C}(J, \mathbb{R}^n)$.

For each bounded set $A \subset B$ there exists a bounded ball $B_{\psi,u}$ such that $A \subset B_{\psi,u}$ and $TB_{\psi,u} \subset B_{\psi,u}$ hold. Therefore, we can conclude from the above considerations that T is precompact in $\mathscr{C}(J, \mathbb{R}^n)$.

Now we are going to prove that T is an upper semicontinuous multifunction on B_ψ.

Let $z_n(t), z(t) \in B_\psi$, $n = 1, 2, \ldots$ and let $z_n(t)$ converge to $z(t)$ in B_ψ, i.e. $\psi^{-1}(t)\,z_n(t)$ converges to $\psi^{-1}(t)\,z(t)$ uniformly on J. Therefore, the set $\{z_n(t), z(t), n = 1, 2, ..\}$ is bounded in B_ψ and there exists $u \geqq u_0$ such that $z_n(t) \in B_{\psi,u}$, $n = 1, 2, \ldots, z(t) \in$ $\in B_{\psi,u}$ and $TB_{\psi,u} \subset B_{\psi,u}$ and $TB_{\psi,u}$ is a precompact set in $\mathscr{C}(J, \mathbb{R}^n)$.

Let $h_n(t) \in T(z_n(t))$, $n = 1, 2, \ldots$. Then there exists $v_n(t) \in M(z_n(t))$, $n = 1, 2, \ldots$, such that

$$h_n(t) = y(t) + \int_0^\infty R(t, s)\,v_n(s)\,\mathrm{d}s$$

and

$$\|v_n\|_{q,\varphi}^q = \int_0^\infty [\varphi^{-1}(t)\,|v_n(t)|]^q\,\mathrm{d}t \leqq \int_0^\infty F^q(t, u)\,\mathrm{d}t\,.$$

Thus the sequence $\{\|v_n\|_{q,\varphi}\}$ is bounded and therefore the sequence $\{v_n(t)\}$ is weakly precompact in $L_{q,\varphi}(J, \mathbb{R}^n)$ (see [4], IV. 8.4) and there exists a subsequence $\{v_{n_j}\}$ of the sequence $\{v_n(t)\}$ which weakly converges to some $v(t) \in L_{q,\varphi}(J, \mathbb{R}^n)$.

From the fact that $\{z_{n_j}\}$ converges to z and from the assumption (H_2) it follows that for each $\varepsilon > 0$ and for almost every fixed $t \in J$ there exists $N = N(\varepsilon, t) > 0$ such that for $n_j \geqq N$,

$$g(t, z_{n_j}(t)) \in O_\varepsilon(g(t, z(t)))$$

where $O_\varepsilon(g(t, z(t))$ is the ε-neighborhood of the set $g(t, z(t))$. It means that for $n_j \geqq N$,

$$v_{n_j}(t) \in O_\varepsilon(g(t, z(t)))\,.$$

Further, by the Banach-Saks theorem there exists a subsequence of $\{v_{n_j}\}$, $n_j \geqq N + 1$, denote it by $\{v_{j_k}\}$, $j_k \geqq N$, such that

$$(20) \qquad \left\| \frac{1}{k} \sum_{s=1}^{k} v_{j_s} - v \right\|_{q,\varphi} \to 0 \quad \text{as} \quad k \to \infty\,.$$

Because $O_\varepsilon(g(t, z(t))$ is convex we have $\sigma_k(t) = \sum_{s=1}^{k} v_{j_s}(t) \in O_\varepsilon(g(t, z(t))$, $k = 1, 2, \ldots$. By the Riesz theorem we get from (20) the existence of such a subsequence of $\{\sigma_k(t)\}$ which converges to $v(t)$ a.e. on J. Putting $\varepsilon \to 0$ we can conclude that $v(t) \in \in g(t, z(t))$.

Furthermore, $v(t)$ being from $L_{q,\varphi}(J, \mathbb{R}^n)$, the function

$$h(t) = y(t) + \int_0^t R(t, s)\,v(s)\,\mathrm{d}s\,, \quad t \in J\,,$$

is well defined and $h(t) \in T(z(t))$. In view of the fact that $\{v_{n_j}(t)\}$ converges weakly to $v(t)$ and that $R(t, \cdot) \in LL_{p,\varphi}(J)$ we get that the sequence

$$h_{n_j}(t) = y(t) + \int_0^t R(t, s)\,v_{n_j}(s)\,\mathrm{d}s\,, \quad j = 1, 2, \ldots$$

converges to $h(t)$ on J. In fact, let $t_1 \in J$. Then put $\tilde{R}(t, s) = R(t, s)$ for $0 \leqq s \leqq t_1$ and $\tilde{R}(t_1, s) = 0$ for $s > t_1$. Evidently $\tilde{R}(t_1, s) \in L_{p,\varphi}(J)$ and

$$h_{n_j}(t_1) = y(t_1) + \int_0^{t_1} \tilde{R}(t_1, s)\,v_{n_j}(s)\,\mathrm{d}s\,,$$

$$h(t_1) = y(t_1) + \int_0^{t_1} \tilde{R}(t_1, s)\,v(s)\,\mathrm{d}s$$

and

$$h_{n_j}(t_1) - h(t_1) = \int_0^\infty \tilde{R}(t_1, s)\,(v_{n_j}(s) - v(s))\,\mathrm{d}s \to 0 \quad \text{as} \quad j \to \infty\,.$$

We have $h_{n_j}(t) \in TB_{\psi,u}$, $j = 1, 2, \ldots$, The set $TB_{\psi,u}$ being precompact there exists a subsequence of the sequence $\{h_{n_j}(t)\}$ which converges to a function $\tilde{h}(t) \in TB_{\psi,u}$ uniformly on every compact subinterval of J. Thus we get that $h(t) = \tilde{h}(t) \in T(z(t))$. This completes the proof of upper semicontinuity of T.

To end the proof of Lemma 1 we have only to prove its last statement. Consider the ball B_{ψ,u_0}. Let $z(t) \in B_{\psi,u_0}$ and $\xi(t) \in T(z(t))$. Then $\|\xi(t)\|_\psi \leqq u_0$ (see (18)) and by (19), $T(z(t)) \subset B_{\psi,u_0}$. By the hypotheses (H_1) and (H_2), $M(z(t))$ is nonempty and convex and therefore $T(z(t))$ is also nonempty and convex. Let $h_n(t) \in T(z(t))$, $n = 1, 2, \ldots$ and let the sequence $\{h_n(t)\}$ convergences in the norm $\|\cdot\|_\psi$. This means that $\{\psi^{-1}(t)\, h_n(t)\}$ converges uniformly on J to some continuous function $\psi^{-1}(t)\, h(t)$. Using the same argument as in the proof of upper semicontinuity of T we get that $h(t) \in T(z(t))$ which means that $T(z(t))$ is a closed set.

Lemma 2. ([5], Corollary 2.8.) *Let A be a closed, bounded and convex subset of a locally convex topological vector space X. If $T: A \to \mathrm{cf}(A)$ is an upper semicontinuous map and if $\overline{TA}$ is compact, then there exists $x \in A$ such that $x \in Tx$.*

Theorem 1. *Let all assumptions of Lemma* 1 *except* f) *be satisfied. Moreover, assume that*

1. $|r(t, s)|^p$ *is locally integrable on* $0 \leqq s \leqq t < \infty$;
2. *there exists a positive constant k such that*

$$\int_0^t |\psi^{-1}(t)\, r(t, s)\, \varphi(s)|^p\, \mathrm{d}s \leqq k^p\,, \quad t \in J\,,$$

and

$$\lim_{h \to 0} \left\{ \left| \int_t^{t+h} |r(t + h, s)|^p\, \mathrm{d}s \right|^{1/p} + \left(\int_0^t |r(t + h, s) - r(t, s)|^p\, \mathrm{d}s \right)^{1/p} \right\} = 0$$

for $t \in J$;

3. $\displaystyle\lim_{t \to \infty} \int_0^{t_0} |\psi^{-1}(t)\, R(t, s)\, \varphi(s)|^p\, \mathrm{d}s = 0$ *for every fixed* $t_0 > 0$.

Then there is a ψ-asymptotic equivalence between the ψ-bounded solutions $x(t)$ of (2) *and ψ-bounded solutions $y(t)$ of* (1). *If, instead of* 3, *the condition*

4. $\displaystyle\int_0^t |\psi^{-1}(t)\, R(t, s)\, \varphi(s)|^p\, \mathrm{d}s \in L_1(J)$

is satisfied, then there is a (ψ, p)-integral equivalence between the ψ-bounded solutions $x(t)$ of (2) *and ψ-bounded solutions $y(t)$ of* (1).

Proof. Let $y(t)$ be a ψ-bounded solution of (1). Then the assumptions 1 and 2 imply that $y(t)$ is a continuous function on J. Thus, $y(t)$ satisfies the assumption f) of Lemma 1.

Let T be the operator defined by (14). Let B_{ψ,u_0} be as in the proof of Lemma 1. Evidently $B_{\psi,u_0} \subset B_\psi \subset \mathscr{C}(J, \mathbb{R}^n)$ is a bounded, closed and convex subset of B_ψ as well as of $\mathscr{C}(J, \mathbb{R}^n)$. In view of Lemma 1 the operator T is upper semicontinuous, $TB_{\psi,u_0} \subset B_{\psi,u_0}$ and $\overline{TB}_{\psi u_0}$ is compact. Then by Lemma 2 there exists $x(t) \in B_{\psi,u_0}$ such that $x(t) \in T(x(t))$, i.e. $x(t)$ is a solution of (2) and there exists $v_0(t) \in M(x(t))$ such that

$$x(t) = y(t) + \int_0^t R(t, s)\, v_0(s)\, ds\,, \quad t \in J\,.$$

Then

$$\begin{aligned}\psi^{-1}(t)\, |x(t) - y(t)| &\leqq \int_0^t \psi^{-1}(t)\, |R(t, s)|\, |v_0(s)|\, ds \leqq \\ &\leqq \int_0^{t_0} \psi^{-1}(t)\, |R(t, s)|\, \varphi(s)\, F(s, u_0)\, ds + \\ &+ \int_{t_0}^t \psi^{-1}(t)\, |R(t, s)|\, \varphi(s)\, F(s, u_0)\, ds \leqq \\ &\leqq \left(\int_0^{t_0} |\psi^{-1}(t)\, R(t, s)\, \varphi(s)|^p\, ds\right)^{1/p} \left(\int_0^\infty F^q(s, u_0)\, ds\right)^{1/q} + \\ &+ \left(\int_{t_0}^t |\psi^{-1}(t)\, R(t, s)\, \varphi(s)|^p\, ds\right)^{1/p} \left(\int_{t_0}^\infty F^q(s, u_0)\, ds\right)^{1/q}.\end{aligned}$$

Finally, using the assumption b) from Lemma 1 we have

$$\begin{aligned}\psi^{-1}(t)\, |x(t) - y(t)| &\leqq \left(\int_0^{t_0} |\psi^{-1}(t)\, R(t, s)\, \varphi(s)|^p\, ds\right)^{1/p} \cdot \\ &\cdot \left(\int_0^\infty F^q(s, u_0)\, ds\right)^{1/q} + K\left(\int_{t_0}^\infty F^q(s, u_0)\, ds\right)^{1/q}.\end{aligned}$$

The first term on the right hand side tends to zero as $t \to \infty$ by assumption 3. The second term can be made arbitarily small if we take t_0 large enough. Thus, we conclude that $\lim \psi^{-1}(t)\, |x(t) - y(t)| = 0$ as $t \to \infty$, which means the asymptotic equivalence of $x(t)$ and $y(t)$.

Using the assumption d) (ii) of Lemma 1 and the assumption 4, we have

$$\begin{aligned}\psi^{-1}(t)\, |x(t) - y(t)| &\leqq \int_0^t \psi^{-1}(t)\, |R(t, s)|\, \varphi(s)\, F(s, u_0)\, ds \leqq \\ &\leqq \left(\int_0^t |\psi^{-1}(t)\, R(t, s)\, \varphi(s)|^p\, ds\right)^{1/p} \left(\int_0^\infty F^q(s, u_0)\, ds\right)^{1/q}.\end{aligned}$$

The first factor is from $L_p(J)$ due to 4. Thus $x(t)$ and $y(t)$ are (ψ, p)-integrally equivalent.

Let now $x(t)$ be a ψ-bounded solution of (2) and therefore of (11). It means that there exists such $v(t) \in M(x(t))$ that $x(t)$ satisfies also the equation (12) where $y(t)$ is a solution of (1). Thus we have

$$y(t) = x(t) - \int_0^t R(t, s)\, v(s)\, ds\,, \quad t \in J\,.$$

An easy calculation shows that $y(t)$ is a ψ-bounded solution of (1). The ψ-asymptotic equivalence and the (ψ, p)-integral equivalence of $y(t)$ and $x(t)$ can be proved as above.

Lemma 3. *Let $\psi(t)$, $\varphi(t)$ be positive continuous functions on J. Assume that*

α) *$R(t, s)$ is measurable and locally essentially bounded on $0 \leqq s \leqq t < \infty$;*

β) *there is a constant $K > 0$ such that*

$$\int_0^t \psi^{-1}(t)\, |R(t, s)|\, \varphi(s)\, ds \leqq K \quad \textit{for all} \quad t \in J$$

and

$$\lim_{h \to 0} \left\{ \left| \int_t^{t+h} |R(t + h, s)|\, ds \right| + \int_0^t |R(t + h, s) - R(t, s)|\, ds \right\} = 0$$

uniformly with respect to $t \in J$;

γ) *the function $g(t, x)$ satisfies the condition c) of Lemma 1;*

δ) *the function $F: J \times J \to J$ is such that*

(i) *$F(t, u)$ is nondecreasing in u for each fixed $t \in J$ and is measurable and bounded on J for each fixed $u \in J$;*

(ii) *$\lim F(t, c) = 0$ as $t \to \infty$ for each fixed $c \geqq 0$;*

(iii) $\limsup\limits_{u \to \infty} \dfrac{u}{F(t, u)} = \infty$ *uniformly for $t \in J$*

or

(iii′) $\limsup\limits_{u \to \infty} \dfrac{u}{F(t, u)} = d > 0$ *uniformly for $t \in J$ where $2K(d - \gamma)^{-1} < 1$ for some γ, $0 < \gamma < d$;*

(iv) $|g(t, x)| \leqq \varphi(t)\, F(t, \psi^{-1}(t)\, |x|)$ *a.e. on J;*

ε) *the function $y: J \to \mathbb{R}^n$ is continuous and ψ-bounded on J, i.e.*

$$\|y\|_\psi = \sup_J \psi^{-1}(t)\, y(t)| = \varrho < \infty\,.$$

Then all statements concerning the operator T defined by (14) in Lemma 1 hold true.

Proof. Let $z(t) \in B_\psi$ and $\|z\|_\psi = \varrho_0 < \infty$. For $v(t) \in M(z(t))$ we have

$$|v(t)| \leqq |M(z(t))| \leqq \varphi(t)\, F(t, \psi^{-1}(t)\, |z(t)|) \leqq \varphi(t)\, F(t, \varrho_0)\,.$$

Thus $v(t) \in LL_1(J)$. Furthermore, because $\varphi^{-1}(t)\,|v(t)| \leqq F(t, \varrho_0)$, the assumption δ) (i) yields that $\varphi^{-1}(t)\, v(t)$ is bounded on J and by δ) (ii) we get $\lim \varphi^{-1}(t)\, v(t) = 0$ as $t \to \infty$. The assumption α) guarantees that the operator T is well defined. For $v(t) \in M(z(t))$ we have

$$\left|y(t) + \int_0^t R(t, s)\, v(s)\, \mathrm{d}s\right| \leqq |y(t)| + \|F(t, \varrho_0)\|_\infty \,.$$

$$\cdot \int_0^t |R(t, s)|\, \varphi(s)\, \mathrm{d}s \leqq |y(t)| + \|F(t, \varrho_0)\|_\infty\, K\, \psi(t)$$

which means that

$$\xi(t) = y(t) + \int_0^t R(t, s)\, v(s)\, \mathrm{d}s \in LL_1(J)\,.$$

Further, we have

$$\psi^{-1}(t)\, |\xi(t)| \leqq \psi^{-1}(t)\, |y(t)| + HK \leqq \varrho + HK$$

where $H = \sup_J F(t, \varrho_0)$. Thus we get that all functions $\xi(t) \in T(z(t))$ are ψ-bounded by the same constant $\varrho + HK$.

Let $0 \leqq t_1$, $t_2 = t_1 + h \geqq 0$, $|h| < 1$. Then

$$\begin{aligned}(21)\qquad |\xi(t_2) - \xi(t_1)| &\leqq |y(t_2) - y(t_1)| + \int_0^{t_1} |R(t_2, s) - R(t_1, s)|\, |v(s)|\, \mathrm{d}s + \\ &+ \left|\int_{t_1}^{t_2} |R(t_2, s)|\, |v(s)|\, \mathrm{d}s\right| \leqq |y(t_2) - y(t_1)| + \\ &+ H \sup_{\langle 0, t_2\rangle} |\varphi(s)| \left\{\int_0^{t_1} |R(t_2, s) - R(t_1, s)|\, \mathrm{d}s + \left|\int_{t_1}^{t_2} |R(t_2, s)|\, \mathrm{d}s\right|\right\}.\end{aligned}$$

In virtue of ε) and β) we get that $\xi(t)$ is continuous at t_1. From this fact we conclude that all functions of $T(z(t))$ are continuous on J. Thus $T(z(t)) \subset B_\psi$ and T maps B_ψ into 2^{B_ψ}. The convexity of $M(z(t))$ implies the convexity of $T(z(t))$. It follows from the assumption δ) (iii) that there exists such $\varrho_1 > \varrho$ that $F(t, 2\varrho_1) < K^{-1}\varrho_1$ for all $t \in J$. We remark that ϱ_1 can be chosen arbittarily large. Consider the ball $B_{\psi, 2\varrho_1}$. If $z(t) \in B_{\psi, 2\varrho_1}$ then for $\xi(t) \in T(z(t))$ there exists $v(t) \in M(z(t))$ such that

$$\xi(t) = y(t) + \int_0^t R(t, s)\, v(s)\, \mathrm{d}s$$

and

$$\psi^{-1}(t)\, |\xi(t)| \leqq \varrho + \int_0^t \psi^{-1}(t)\, |R(t, s)|\, \varphi(s)\, F(s, 2\varrho_1)\, \mathrm{d}s \leqq \varrho + \varrho_1 \leqq 2\varrho_1\,.$$

Similarly, from δ) (iii′), for given γ, $0 < \gamma < d$, there exists $\varrho_1 > \varrho$ such that $F(t, 2\varrho_1) < (d - \gamma)^{-1} 2\varrho_1$ for every $t \in J$. Then

$$\sup_J F(t, 2\varrho_1) \leqq (d - \gamma)^{-1} 2\varrho_1$$

and

$$\psi^{-1}(t) |\xi(t)| \leqq \varrho + \int_0^t \psi^{-1}(t) |R(t, s)| \varphi(s) F(s, 2\varrho_1) \, ds \leqq$$

$$\leqq \varrho + (d - \gamma)^{-1} 2\varrho_1 K < 2\varrho_1 .$$

Thus we have that $TB_{\psi,2\varrho_1} \subset B_{\psi,2\varrho_1}$. It means that all functions from $TB_{\psi,2\varrho_1}$ are uniformly ψ-bounded. Let $0 \leqq t_1 < t_2 \leqq L < \infty$ and $\xi(t) \in TB_{\psi,2\varrho_1}$. Proceeding in the same way as in (21) we get

$$|\xi(t_2) - \xi(t_1)| \leqq |y(t_2) - y(t_1)| + \sup_{0,L} \varphi(t) \sup_J F(t, 2\varrho_1) \, .$$

$$. \left\{ \int_0^{t_1} |R(t_2, s) - R(t_1, s)| \, ds + \int_{t_1}^{t_2} |R(t_2, s)| \, ds \right\} .$$

Thus the functions $\xi(t)$ of $TB_{\psi,2\varrho_1}$ are equicontinuous on compact subintervals of J.

Now, the uniform ψ-boundedness and the equicontinuity on compact subintervals of J imply that $TB_{\psi,2\varrho_1}$ is precompact in $\mathscr{C}(J, \mathbb{R}^n)$.

From our considerations it follows that for any bounded set $A \subset B_\psi$ there exists a ball $B_{\psi,u}$ such that $A \subset B_{\psi,u}$ and $TB_{\psi,u} \subset B_{\psi,u}$ hold and $TB_{\psi,u}$ is precompct in $\mathscr{C}(J, \mathbb{R}^n)$. Thus T is precompact in $\mathscr{C}(J, \mathbb{R}^n)$.

Now we are going to prove that T is an upper semicontinuous function on B_ψ.

Let $z_n(t)$, $n = 1, 2, \ldots, z(t) \in B_\psi$ and let the sequence $\{z_n\}$ converge to z in B_ψ. Therefore, the set $\{z_n(t), n = 1, 2, \ldots, z(t)\}$ is bounded in B_ψ and there exists $u > 0$ such that $\{z_n(t), n = 1, 2, \ldots, z(t)\} \subset B_{\psi,u}$ and $TB_{\psi,u} \subset B_{\psi,u}$ and $TB_{\psi,u}$ is precompact in $\mathscr{C}(J, \mathbb{R}^n)$.

Let $h_n(t) \in T(z_n(t))$, $n = 1, 2, \ldots$. Then there exists $v_n(t) \in M(z(t))$ such that

$$h_n(t) = y(t) + \int_0^t R(t, s) \, v_n(s) \, ds \tag{22}$$

and $|v_n(t)| \leqq \varphi(t) F(t, u)$. Thus $v_n(t) \in LL_1(J)$ and the sequence $\{v_n(t)\}$ is bounded in $L_1(\langle 0, L \rangle)$ for every $L > 0$. If $\{E_k\}$, $E_k \subset \langle 0, L \rangle$ measurable, is a nonincreasing sequence such that $\bigcap_{k=1}^{\infty} E_k = \emptyset$, then

$$\lim_{k \to \infty} \left| \int_{E_k} v_n(s) \, ds \right| \leqq \lim_{k \to \infty} \int_{E_k} |v_n(s)| \, ds \leqq \lim_{k \to \infty} \int_{E_k} \varphi(s) F(s, u) \, ds = 0 \, .$$

Then (see [4], Th. IV. 8.9) it is possible to choose from the sequence $\{v_n(t)\}$ a subsequence $\{v_{n_k}(t)\}$ which weakly converges to a function $v(t) \in L_1(\langle 0, L \rangle)$.

From the fact that $v_{n_k}(t) \in g(t, z_{n_k}(t))$, $k = 1, 2, \dots$, and that $\{z_{n_k}(t)\}$ converges to $z(t)$ in B_ψ and from the hypothesis (H_2) it follows that for $\varepsilon > 0$ and for given $t \in J$ there exists $N = N(\varepsilon, t)$ such that for any $n_k \geqq N$ we have

$$g(t, z_{n_k}(t)) \subset O_\varepsilon(g(t, z(t)))$$

where $O_\varepsilon(g(t, z(t)))$ is ε-neighbourhood of the set $g(t, z(t)))$. It means that for all $n_k \geqq N$,

$$v_{n_k}(t) \in O_\varepsilon(g(t, z(t)))\,.$$

Consider the sequence $\{v_{n_k}\}$, $n_k \geqq N$. Then (see [4], Corollary V. 3.14) it is possible to construct such convex combination from v_{n_k}, $n_k \geqq N$, denote them $g_m(t)$, $m = 1, 2, \dots$, that the sequence $\{g_m(t)\}$ converges to $v(t)$ in $L_1(\langle 0, L\rangle)$. Then by the Riesz theorem there exists a subsequence $\{g_{m_i}(t)\}$ of $\{g_m(t)\}$ which converges to $v(t)$ a.e. on $\langle 0, L\rangle$. From the convexity of $O_\varepsilon(g(t, z(t)))$ and from the fact that $v_{n_k}(t) \in O_\varepsilon(g(t, z(t)))$ it follows that $g_{m_i}(t) \in O_\varepsilon(g(t, z(t)))$, $i = 1, 2, \dots$, and therefore $v(t) \in \overline{O_\varepsilon}(g(t, z(t)))$. If we let $\varepsilon \to 0$ we conclude that $v(t) \in g(t, z(t))$. We recall that in our consideration t was a fixed point and that $g(t, z(t))$ is a convex compact subset of $\mathbb{R}^n$. Thus the function

$$h(t) = y(t) + \int_0^t R(t, s)\, v(s)\, ds\,, \quad t \in \langle 0, L\rangle$$

is well defined and $h(t) \in T(z(t))$ for $t \in \langle 0, L\rangle$. Taking into account the fact that the sequence $\{v_{n_k}\}$ weakly converges to v on $\langle 0, L\rangle$ and the assumption $\alpha)$ we get that the sequence

$$\{h_{n_k}(t)\} = \left\{y(t) + \int_0^t R(t, s)\, v_{n_k}(s)\, ds\right\}, \quad k = 1,2, \dots$$

converges to $h(t)$ a.e. on $\langle 0, L\rangle$.

The functions $h_{n_k}(t)$, $k = 1, 2, \dots$, being uniformly bounded and equicontinuous on $\langle 0, L\rangle$, it is possible to choose a subsequence of the sequence $\{h_{n_k}(t)\}$ which converges on $\langle 0, L\rangle$ uniformly to a function $\tilde{h}(t)$. Hence $\tilde{h}(t) = h(t)$ a.e. on $\langle 0, L\rangle$.

The number $L > 0$ being chosen arbitrarily, we conclude that from the sequence (22) it is possible to choose a subsequence which converges to a function $h(t)$ uniformly on every compact subinterval of J and $h(t) \in T(z(t))$ for $t \in J$. This completes the proof of upper semicontinuity of T.

The proof of existence of such a ball B_{ψ,u_0} that T maps B_{ψ,u_0} into $\mathrm{cf}(B_{\psi,u_0})$ is similar to that in the proof of Lemma 1.

Theorem 2. *Let $\psi(t)$, $\varphi(t)$ be positive continuous functions on J. Assume that*
1. *$|r(t, s)|$ is locally integrable on $0 \leqq s \leqq t < \infty$;*
2. *there exists a constant $P > 0$ such that*

$$\int_0^t \psi^{-1}(t)\, |r(t, s)|\, \varphi(s)\, ds \leqq P \quad \textit{for all} \quad t \in J$$

and

$$\lim_{h\to 0}\left\{\left|\int_t^{t+h}|r(t+h,s)|\,\mathrm{d}s\right| + \int_0^t |r(t+h,s)-r(t,s)|\,\mathrm{d}s\right\} = 0$$

for $t \in J$;

3. *all assumptions of Lemma* 3 *are satisfied except* ε);

4. $\lim_{t\to\infty}\int_0^{t_0}\psi^{-1}(t)\,|R(t,s)|\,\varphi(s)\,\mathrm{d}s = 0$ *for every fixed* $t_0 > 0$.

Then there is a ψ*-asymptotic equivalence between the* ψ*-bounded solutions* $x(t)$ *of* (2) *and* ψ*-bounded solutions* $y(t)$ *of* (1). *If instead of* 4 *the condition*

5. $\int_0^{t_0}\psi^{-1}(t)\,|R(t,s)|\,\varphi(s)\,\mathrm{d}s \in L_p(J),\ p \in (0,\infty)$

is satisfied, then there is a (ψ, p)*-integral equivalence between the* ψ*-bounded solutions* $x(t)$ *of* (2) *and* ψ*-bounded solutions* $y(t)$ *of* (1).

Proof. Let $y(t)$ be a ψ-bounded solution of (1). Then the assumptions 1 and 2 imply that $y(t)$ is a continuous function on J. Thus the assumption ε) of Lemma 3 is satisfied.

Let T be the operator defined by (14) and let $B_{\psi,u_0} \subset B_\psi$ be such that $TB_{\psi,u_0} \subset B_{\psi,u_0}$. Such B_{ψ,u_0} exists (see the proof of Lemma 3). B_{ψ,u_0} is a bounded, closed and convex subset of B_ψ as well as of $\mathscr{C}(J, \mathbb{R}^n)$. Due to Lemma 3 T is upper semicontinuous in $\mathscr{C}(J, \mathbb{R}^n)$ and $\overline{TB_{\psi,u_0}}$ is compact. Then by Lemma 2 there exists $x(t) \in B_{\psi,u_0}$ such that $x(t) \in T(x(t))$, i.e. $x(t)$ is a solution of (2). Therefore there exists such $v(t) \in M(x(t))$ that

$$x(t) = y(t) + \int_0^t R(t,s)\,v(s)\,\mathrm{d}s\,, \quad t \in J\,.$$

Then

$$\psi^{-1}(t)\,|x(t)-y(t)| \leqq \int_0^t \psi^{-1}(t)\,|R(t,s)|\,|v(s)|\,\mathrm{d}s \leqq$$

$$\leqq \int_0^t \psi^{-1}(t)\,|R(t,s)|\,\varphi(s)\,F(s,u_0)\,\mathrm{d}s\,.$$

Using the assumption δ) (i) from Lemma 3 we get

$$\psi^{-1}(t)\,|x(t)-y(t)| \leqq \sup_J F(s,u_0)\int_0^{t_0}\psi^{-1}(t)\,|R(t,s)|\,\varphi(s)\,\mathrm{d}s\,+$$

$$+\sup_{t_0\leqq s<\infty} F(s,u_0)\int_{t_0}^t \psi^{-1}(t)\,|R(t,s)|\,\varphi(s)\,\mathrm{d}s\,.$$

The first term on the right hand side tends to zero as $t \to \infty$ by the assumption 4. The second term is not greater than $\sup\limits_{t_0\leqq s<\infty} F(s,u_0)\,K$. In the above considerations $t_0 > 0$ was an arbitrary number. Using the assumption δ) (ii) in Lemma 3 we get that for any $\varepsilon > 0$ it is possible to find such $t_0 \geqq 0$ that $\sup\limits_{t_0 \leqq s < \infty} F(s, u_0) K < \varepsilon$. From all these considerations we can conclude that $\lim \psi^{-1}(t) |x(t) - y(t)| = 0$ as $t \to \infty$.

On the other hand, using the assumption δ) (i) in Lemma 3, we get that

$$\psi^{-1}(t) |x(t) - y(t)| \leqq \sup_J F(s, u_0) \int_0^t \psi^{-1}(t) |R(t, s)| \varphi(s) \, ds .$$

By the assumption 5 of our theorem we see that

$$\psi^{-1}(t) |x(t) - y(t)| \in L_p(J) .$$

Now, let $x(t)$ be a ψ-bounded solution of (2). Then there exists $v(t) \in M(x(t))$ such that

$$x(t) = y(t) + \int_0^t R(t, s) v(s) \, ds , \quad t \in J$$

where $y(t)$ is a solution of (1). The proof of the ψ-asymptotic equivalence and (ψ, p)-integral equivalence of $x(t)$ and $y(t)$ is the same as above.

2.

Now, we will consider the equivalences of the equations (3) and (4). We suppose that $A(t) \in LL_1(J)$, $B(t, s) \in LL_1(D)$, $C(t, s) \in LL_1(D)$, where $D = \{0 \leqq s \leqq t < \infty\}$. Integrating (3) we get

$$y(t) = \xi + \int_0^t F(s) \, ds + \int_0^t \left[A(s) + \int_s^t B(u, s) \, du \right] y(s) \, ds , \tag{23}$$

$t \in J$.

Denote $f(t) = \xi + \int_0^t F(s) \, ds$, $a(t, s) = A(s) + \int_s^t B(u, s) \, du$, and let $\gamma(t, s)$ be the resolvent kernel belonging to $a(t, s)$. We see that $f(t)$ is a continuous function on J, $a(t, s)$ and $\gamma(t, s)$ are locally integrable on D. Then

$$y(t) = f(t) + \int_0^t a(t, s) y(s) \, ds \tag{24}$$

and

$$y(t) = f(t) - \int_0^t \gamma(t, s) f(s) \, ds . \tag{25}$$

Let $x(t)$, $x(0) = \xi$, be a solution of (4). Then there exists such $v(t) \in M(x(t))$ that

$$x'(t) = F(t) + A(t) x(t) + \int_0^t B(t, s) x(s) \, ds + \int_0^t C(t, s) v(s) \, ds . \tag{26}$$

Integrating this equation we have

$$x(t) = \xi + \int_0^t F(s)\,ds + \int_0^t \left[A(s) + \int_s^t B(u, s)\,du\right] x(s)\,ds + $$
$$+ \int_0^t \left(\int_s^t C(u, s)\,du\right) v(s)\,ds$$

or

$$x(t) = f(t) + \int_0^t a(t, s)\,x(s)\,ds + \int_0^t \left(\int_s^t C(u, s)\,du\right) v(s)\,ds\,. \tag{27}$$

Using (25) we get

$$x(t) = y(t) + \int_0^t \left\{\int_s^t \left[C(u, s) - \gamma(t, u)\int_s^u C(\tau, s)\,d\tau\right] du\right\} v(s)\,ds\,. \tag{28}$$

If we denote

$$\int_s^t \left[C(u, s) - \gamma(t, u)\int_s^u C(\tau, s)\,d\tau\right] du = \Gamma(t, s) \tag{29}$$

we have

$$x(t) = y(t) + \int_0^t \Gamma(t, s)\,v(s)\,ds\,, \quad x(0) = \xi\,. \tag{30}$$

Thus we have a situation similar to that represented by (9) and (12). Therefore, the following theorem holds true.

Theorem 3. *Let $\psi(t)$, $\varphi(t)$ be positive continuous functions on J. Let $\gamma(t, s)$ fulfil the same hypotheses as $r(t, s)$ and $\Gamma(t, s)$ as $R(t, s)$ in Theorem 1 (Theorem 2) and let $g(t, x)$ be the same as in Theorem 1 (Theorem 2). Then there is a ψ-asymptotic equivalence and a (ψ, p)-integral equivalence, respectively, between the set of all ψ-bounded solutions of (3) and the set of all ψ-bounded solutions of (4).*

References

[1] *Nohel J. A.:* Asymptotic relationships between systems of Volterra equations, Ann. di Mat. Pura ed Appl., (IV) T. XC, 1971, 149–165.

[2] *Nohel J. A.:* Asymptotic equivalence of Volterra equations, Ann. di Mat. Pura ed Appl., T. XCVI, 1973, 339–347.

[3] *Lima A. C.:* Asymptotic equivalence of Volterra integral equations, Acta FRNUC Mathematica, T. XXXIII, 1977, 19–33.

[4] *Dunford N., Schwartz J. T.:* Linear operators, General Theory, 1958, Interscience Publishers, New York, London.

[5] *Seah, Sek Wui:* Asymptotic equivalence of multivalued differential systems, Bolletino U.M.I., (5) 17-B, 1980, 1124–1145.

Author's address: 842 15 Bratislava, Mlynská dolina (Katedra matematickej analýzy MFF-UK).