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GLOBAL ASYMPTOTIC STABILITY FOR PLANE POLYNOMIAL FLOWS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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The question of the global asymptotic stability of a stationary point (which we may assume to be at the origin) of the two-dimensional C^1-system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y), \tag{1}$$

with Jacobian matrix J, under the hypotheses that

$$\operatorname{trace} J \equiv \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) < 0 \quad \text{on} \quad \mathbb{R}^2, \tag{I}$$

and

$$\det J \equiv \left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right) > 0 \quad \text{on} \quad \mathbb{R}^2, \tag{II}$$

was studied by Markus and Yamabe in [4], Hartman in [2], Hartman and Olech in [3], and Olech in [6]. These authors gave an affirmative answer to this question under some additional hypotheses of one kind or another. (See [5] for a brief exposition of these earlier results.) However, to the present time, no complete solution to this question has been given. In this paper we answer this question in the affirmative for two new special cases: In Theorem A where the vector field (1) is diffeomorphic to one whose flow is polynomial in the initial conditions x_0 and y_0; and in Theorem B where the components X, Y of the vector field (1) are quadratic polynomials in x and y. The following theorem in [6] includes most of the earlier results.

Theorem C. *If the system* (1) *has a stationary point at the origin and satisfies conditions* (I) *and* (II), *then the origin is globally asymptotically stable under any one of the following additional conditions.*

(01) *There are two positive constants p and r such that $X^2 + Y^2 \geqq p^2$ whenever $x^2 + y^2 \geqq r^2$.*

(02) *The mapping $u = X(x, y)$, $v = Y(x, y)$, is globally one-to-one.*

(03) *At least one of the products* $(\partial X/\partial x)(\partial Y/\partial y)$ *or* $(\partial X/\partial y)(\partial Y/\partial x)$ *never vanishes on* $\mathbb{R}^2$.

For example, conditions (I), (II), and (03) are satisfield if $J + J^*$ is negative definite at all points of $\mathbb{R}^2$. This is Hartman's condition in [2].

To obtain our first theorem, Theorem A, we use the results of [1] on polynomial flows. A C^1 vector field $X: \mathbb{R}^n \to \mathbb{R}^n$ is said to have a *polynomial flow* if the components of the local solution

$$x = \varphi(t, z) = \varphi_t(z) \tag{2}$$

to the n-dimensional autonomous system

$$\dot{x} = X(x), \quad x \in \mathbb{R}^n, \tag{3}$$

are (for each fixed value of t) *polynomials* in the components of the initial condition vector $z \in \mathbb{R}^n$. In [1] it is shown that such a vector field X must itself be polynomial (and not merely C^1), and must have constant divergence

$$\operatorname{div} X \equiv \text{constant} \equiv \alpha \in \mathbb{R}. \tag{4}$$

Furthermore, a polynomial flow must be complete (i.e., defined for all real t) and have bounded degree (as t varies over $\mathbb{R}$); and therefore it can be written as a polynomial in z with coefficients $a_j(t)$ which are analytic functions of t:

$$x = \varphi(t, z) = \sum_{|j| \leq d} a_j(t)\, z^j, \tag{5}$$

where $j = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n$, $z^j = z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}$, and each a_j maps $\mathbb{R}$ into $\mathbb{R}$. The integer d is the degree of φ, and $|j| = j_1 + j_2 + \ldots + j_n$. It follows from (4) that for each $n \times n$ matrix L with trace $L = \alpha \equiv \operatorname{div} X$, there exists an $n \times n$ skew-symmetric matrix function $H(x)$ such that

$$X(x) = Lx + (\partial H)^{\mathrm{T}} \tag{6}$$

where $\partial = (\partial_1, \partial_2, \ldots, \partial_n)$ is the row-vector of partial differential coordinate operators.

In two dimensions much more can be proved. It is shown in [1] that each polynomial flow $x = \varphi(t, z)$ is conjugate to one of the six (inequivalent) types. This conjugacy is effected by a change of coordinates $u = P(x)$, where P is a diffeomorphism of $\mathbb{R}^2$ whose components, as well as those of P^{-1}, are polynomials (in two variables).

Now it is clear that a vector field U conjugate to a vector field X will have exactly the same sets of stationary points. So our hypotheses (I) and (II) on the Jacobian matrix X' of X eliminate the cases, when the origin $(0, 0)$ is not an isolated stationary point. Thus the classification given in [1] implies that a two dimenional vector field (satisfying (I) and (II)) whose flow is polynomial with respect to initial conditions is conjugate to a linear asymptotically stable system or to the system $\dot{u} = au$, $\dot{v} = amv + u^m$, where $a < 0$ and $m = 1, 2, \ldots$. In all those cases $(0, 0)$ is the only stationary point and it is globally asymptotically stable. Thus we have proved the following theorem in the case $n = 2$.

Theorem A. *If a polynomial flow $\varphi(t, x)$ on $\mathbb{R}^n$ has the origin as a stationary point and if its vector field $X(x) = \dot{\varphi}(0, x)$ has a stable Jacobian matrix $X'(x)$, then the origin is globally asymptotically stable.*

Proof (for an arbitrary n). Stability assumption of the matrix $X'(x)$ implies local asymptotic stability of the origin. This means, taking into account the representation (5), that $\sum_{|j| \leqq d} a_j(t) z^j$ converges uniformly to zero as $t \to \infty$ for z from a neighbourhood of the origin. It then follows that $a_j(t)$ must tend to zero for each j. Hence for each fixed $z \in \mathbb{R}^n$, $\varphi(t, z) \to 0$ as $t \to \infty$, which means that the origin is globally asymptotically stable.

Let us now state our second result.

Theorem B. *If in the autonomous system (1) the functions $X(x, y)$ and $Y(x, y)$ are polynomials of degree 2 satisfying $X(0, 0) = Y(0, 0) = 0$ and also satisfying conditions (I) and (II), then both the determinant and the trace of the Jacobian matrix*

$$J = \begin{bmatrix} \partial X/\partial x & \partial X/\partial y \\ \partial Y/\partial x & \partial Y/\partial y \end{bmatrix}$$

are constant, and the origin is globally asymptotically stable.

Proof. Let

$$X(x, y) = a_1 x^2 + 2a_2 xy + a_3 y^2 + c_1 x + c_2 y\,,$$
$$Y(x, y) = b_1 x^2 + 2b_2 xy + b_3 y^2 + d_1 x + d_2 y\,,$$

then

$$\operatorname{trace} J = 2a_1 x + 2a_2 y + 2b_2 x + 2b_3 y + c_1 + d_2$$

and

$$\operatorname{trace} J < 0 \quad \text{implies} \quad a_1 + b_2 = a_2 + b_3 = 0\,.$$

Hence $\operatorname{tr} J \equiv \text{const.}$ Thus

$$Y(x, y) = b_1 x^2 - 2a_1 xy - a_2 y^2 + d_1 x + d_2 y\,,$$

and

$$J = \begin{bmatrix} 2(a_1 x + a_2 y) + c_1, & 2(a_2 x + a_3 y) + c_2 \\ 2(b_1 x - a_1 y) + d_1, & -2(a_1 x + a_2 y) + d_2 \end{bmatrix},$$

$$\det J = -4(a_1 x + a_2 y)^2 - 4(a_2 x + a_3 y)(b_1 x - a_1 y) + \text{lower degree terms}\,.$$

Therefore $\det J > 0$ implies

$$-(a_1 x + a_2 y)^2 - (a_2 x + a_3 y)(b_1 x - a_1 y) \geqq 0\,.$$

This holds only if the rank of

$$\begin{pmatrix} a_1 & a_2 & b_1 \\ a_2 & a_3 & -a_1 \end{pmatrix}$$

is at most one, hence only if there are λ and μ such that

$$a_2 = \lambda a_1 , \qquad b_1 = \mu a_1 ,$$
$$a_3 = \lambda a_2 , \quad -a_1 = \mu a_2 .$$

Hence, if $a_1 \neq 0$, then $\lambda\mu = -1$ and $a_2 \neq 0$, because $-a_1 = \mu a_2 = \lambda\mu a_1$. Consequently, the second degree part of det J vanishes.

Therefore det J is of degree $\leqq 1$. But since det $J > 0$ for each point (x, y), it follows that det J = constant = $c_1 d_2 - c_2 d_1$.

But it is known (a special case of the Jacobian Conjecture) that a quadratic polynomial mapping with constant nonzero Jacobian determinant is necessarily a one-to-one mapping. It then follows from Theorem C part (02) that the origin is globally asymptotically stable. This completes the proof of Theorem B.

We are grateful to H. W. Knobloch for pointing out to us that Theorem A can be proved just as easily in n dimensions (for $n \geqq 2$) as it can for the two-dimensional case and that its proof need not depend on the classification of 2-dimensional polynomial flows.

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