Richard C. Brown; Don B. Hinton
Sufficient conditions for weighted Gabushin inequalities

Časopis pro pěstování matematiky, Vol. 111 (1986), No. 2, 113--122

Persistent URL: http://dml.cz/dmlcz/118268

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

Vydává Matematický ústav ČSAV, Praha

SVAZEK 111 * PRAHA 30. 4. 1986 * ČÍSLO 2

SUFFICIENT CONDITIONS FOR WEIGHTED GABUSHIN INEQUALITIES

RICHARD C. BROWN, University, DON B. HINTON, Knoxville

Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

(Received May 25, 1985)

Let $I = [a, \infty)$ be a ray. The purpose of this paper is to develop sufficient conditions for the "product" inequality,

$$\int_I N|y^{(j)}|^p \leqq K\left[\int_I W|y|^q\right]^{\beta p/q}\left[\int_I P|y^{(n)}|^r\right]^{\alpha p/r} \tag{1}$$

or the equivalent "sum" inequality (with ε arbitrary in $(0, \infty)$),

$$\int_I N|y^{(j)}|^p \leqq K_1\left\{\varepsilon^{-p(j+1/q-1/p)}\left(\int_I W|y|^q\right)^{p/q} + \varepsilon^{p(n-j-1/r+1/p)}\left(\int_I P|y^{(n)}|^r\right)^{p/r}\right\}, \tag{2}$$

to hold. Here n is a positive integer, $0 \leqq j \leqq n - 1$,

$$1 \leqq p, q, r < \infty\,, \tag{3}$$

$$\frac{n}{p} \leqq \frac{n-j}{q} + \frac{j}{r}\,, \tag{4}$$

$$\beta = \beta(p) = (n - j - 1/r + 1/p)/(n - 1/r + 1/q)\,, \quad \alpha = 1 - \beta\,, \tag{5}$$

and N, W, P are positive Lebesgue measurable functions satisfying additional properties stated below. The (interrelated) constants K and K_1 are independent of the functions y in a domain $\mathscr{D}$ on which the inequalities are defined, but they may depend on N, W, P as well as on the numbers p, q, r, n, j. Concerning $y \in \mathscr{D}$ we require only that the integrals involving W and P exist and that $y^{(n-1)}$ be locally absolutely continuous on I. Thus (1)–(2) assert that $\int_I N|y^{(j)}|^p$ exists when the integrals on the right are finite. Further, although our method allows explicit calculation of K in (1), we do not concern ourselves with the determination of the least such K.

Many special cases of inequalities (1)–(2) are well known. If $N = W = P = 1$ and $p = q = r = 2$ or ∞, then (1) is a function inequality of Landau type, see e.g.

[8]. The weighted case with $p = q = r$ has recently been studied by Kwong and Zettl [8, 9], Goldstein, Kwong, and Zettl [6], and the present authors [2]. When p, q, r may be unequal but $N = P = W \equiv 1$, (1) was established by Gabushin [3] in 1967 under the condition (4). Further results on (1) with unequal p, q, r and nonconstant weights were given by Kwong and Zettl [10] under the condition of equality in (4).

In this paper we extend certain portions of [10] and [2] to allow inequality in (4) and a new class of weights N, W, P. In fact the weights we consider satisfy generalizations of the point and integral bound conditions in [2] for the case $p = q = r$. In particular an immediate consequence of Theorem 2 below is an easy derivation of Gabushin's original inequality.

Gabushin's inequality is closely related in form though distinct from a family of one variable inequalities established by Nirenberg [5] (especially (2.2), p. 125; (2.7), p. 130) in preparation for a theory of multidimensional interpolation inequalities. See also Adams [1], Gagliardo [4, 5], Kufner, John, and Fucik [11], Triebel [16], Miranda [13], and Henry [7].

We use the notation $\mathscr{L}_{\text{loc}}(I)$ to denote the Lebesgue measurable, complex-valued functions on I which are Lebesgue integrable over all compact subsets of I.

In addition to the above we assume

(6) $N \in \mathscr{L}_{\text{loc}}(I)$; for $q > 1$, $W^{-q'/q} \in \mathscr{L}_{\text{loc}}(I)$ where $1/q + 1/q' = 1$ and for $q = 1$, W^{-1} is bounded on I; for $r > 1$, $P^{-r'/r} \in \mathscr{L}_{\text{loc}}(I)$ where $I/r + 1/r' = 1$ and for $r = 1$, P^{-1} is bounded on I.

(7) f is a positive, continuous, nondecreasing function on I.

We define

$$\mathscr{A}_n = \{y \colon y^{(n-1)} \text{ is locally absolutely continuous on } I\},$$

$$\mathscr{B}_q = \left\{y \colon y \text{ is measurable and } \int_I W|y|^q < \infty\right\},$$

$$\mathscr{C}_r = \left\{y \in \mathscr{A}_n \colon \int_I P|y^{(n)}|^r < \infty\right\},$$

and

$$\mathscr{D} = \mathscr{B}_q \cap \mathscr{C}_r .$$

Additionally we establish the following notation for a positive function z:

$$T^{u,v}_{t,\varepsilon}(z) = \begin{cases} [\sup\{z(s)^{-1} \colon t \leqq s \leqq t + \varepsilon f(t)\}]^u & \text{if } v = 1, \\ \left[(\varepsilon f(t))^{-1} \int_t^{t+\varepsilon f(t)} z^{-v'/v}\right]^{u/v'} & \text{if } v > 1 \quad (1/v + 1/v' = 1). \end{cases}$$

First we recall a result of [2].

Lemma 1. *For $0 \leqq j \leqq n-1$ there is a constant M, depending only on n and j such that if $J = [c, d] \subset I$, $t \in J$, and $y \in \mathscr{A}_n$, then*

$$(8) \qquad |y^{(j)}(t)| \leqq ML^{-j-1} \int_J |y| + L^{n-j-1} \int_J |y^{(n)}|,$$

where $L = d - c$.

Lemma 2. *Suppose* (3), (6), *and* (7) *hold and M is as in Lemma* 1. *Then for $t \in I$, $\varepsilon > 0$, $y \in \mathscr{D}$, and $s \in J_t := [t, t + \varepsilon f(t)]$,*

$$(9) \qquad |y^{(j)}(s)|^p \leqq 2^{p-1} \left\{ M^p L_t^{-p(j+1/q)} T_{t,\varepsilon}^{p,q}(W) \left(\int_{J_t} W|y|^q \right)^{p/q} + \right.$$
$$\left. + L_t^{p(n-j-1/r)} T_{t,\varepsilon}^{p,r}(P) \left(\int_{J_t} P|y^{(n)}|^r \right)^{p/r} \right\},$$

where $L_t = \varepsilon f(t)$.

Proof. Inequality (9) follows from (8) by applying Hölder's inequality and the inequality $(u + v)^p \leqq 2^{p-1}(u^p + v^p)$ which holds for $u, v \geqq 0$ and $1 \leqq p < \infty$.

Theorem 1. *Suppose $1 \leqq r, q$,* (5)–(7) *hold, $p \geqq \max\{r, q\}$ (this implies* (4)), *and*

$$(10) \qquad R_1 := \sup_{t \in I, 0 < \varepsilon < \infty} \{ f(t)^{-p(j+1/q-1/p)} N(t) T_{t,\varepsilon}^{p,q}(W) \} < \infty,$$

$$(11) \qquad R_2 := \sup_{t \in I, 0 < \varepsilon < \infty} \{ f(t)^{p(n-j-1/r+1/p)} N(t) T_{t,\varepsilon}^{p,r}(P) \} < \infty.$$

Then (1) *holds for $y \in \mathscr{D}$ if $\int_I P|y^{(n)}|^r \neq 0$ with*

$$(12) \qquad K = K_2 := 2^p \max\{2^{p/q} M^p R_1, 2^{p/r} R_2\}$$

Proof. Fix $\varepsilon > 0$ and set $t_0 = a$, $t_{i+1} = t_i + \varepsilon f(t_i)$ for $i = 1, 2, \ldots$. Then by (9) with $s = t$,

$$N(t) |y^{(j)}(t)|^p \leqq 2^{p-1} \left\{ M^p \varepsilon^{-p(j+1/q)} f(t)^{-1} R_1 \left(\int_{J_t} W|y|^q \right)^{p/q} + \right.$$
$$\left. + \varepsilon^{p(n-j-1/r)} f(t)^{-1} R_2 \left(\int_{J_t} P|y^{(n)}|^r \right)^{p/r} \right\}.$$

Next we integrate this inequality over $[t_i, t_{i+1}]$ and use the fact that f nondecreasing implies

$$\int_{t_i}^{t_{i+1}} f^{-1} \leqq f(t_i)^{-1} (t_{i+1} - t_i) = \varepsilon$$

to conclude that (note $t \in [t_i, t_{i+1}]$ implies $J_t \subset [t_i, t_{i+2}]$)

$$(13)\qquad \int_{t_i}^{t_{i+1}} N|y^{(j)}|^p \leqq (K/2)\left\{2^{-p/q}\,\varepsilon^{-p(j+1/q-1/p)}\left(\int_{t_i}^{t_{i+2}} W|y|^q\right)^{p/q} + \right.$$

$$\left. + \,2^{-p/r}\,\varepsilon^{p(n-j-1/r+/p)}\left(\int_{t_i}^{t_{i+2}} P|y^{(n)}|^r\right)^{p/r}\right\}.$$

Summing (13) over i and using the inequality $\sum a_i^R \leqq (\sum a_i)^R$ which is valid for $a_i \geqq 0$ and $R \geqq 1$, yields that

$$(14)\qquad \int_I N|y^{(j)}|^p \leqq (K/2)\left\{2^{-p/q}\,\varepsilon^{-p(j+1/q-1/p)}\left(2\int_I W|y|^q\right)^{p/q} + \right.$$

$$\left. + \,2^{-p/r}\,\varepsilon^{p(n-j-1/r+1/p)}\left(2\int_I P|y^{(n)}|^r\right)^{p/r}\right\}.$$

Now choose ε so that the two terms on the right of (14) are equal; this gives

$$(15)\qquad \varepsilon^{p(n-1/r+1/q)} = \left(\int_I W|y|^q\right)^{p/q}\left(\int_I P|y^{(n)}|^r\right)^{p/r}.$$

Substitution of (15) into (14) and simplifying gives (1).

Corollary 1. *If in Theorem 1 W, P, and $P^{1/r}/W^{1/q}$ are nondecreasing on I and $N W^{-p\beta(p)/q} P^{-p(1-\beta(p))/r}$ is nonincreasing, then defining f by $f^{n-1/r+1/q} = P^{1/r}/W^{1/q}$ yields*

$$(16)\qquad R_1 = R_2 = N(a)\,W(a)^{-p\beta(p)/q}\,P(a)^{-p(1-\beta(p))/r} < \infty\,.$$

Proof. Since W, P are nondecreasing the proof is immediate from $T^{p,q}_{t,\varepsilon}(W) \leqq \leqq W(t)^{-p/q}$, $T^{p,r}_{t,\varepsilon}(P) \leqq P(t)^{-p/r}$, and simplification in (10) and (11).

Another case in which (10) and (11) hold is when W and P are nondecreasing, $f(t) \equiv 1$, and $NW^{-p/q}$ and $NP^{-p/r}$ are nonincreasing, e.g., if $W(t) = t^\gamma$, $P(t) = t^\alpha$, and $N(t) = t^\phi$, then these conditions reduce to (when $a > 0$), $\gamma > 0$, $\alpha \geqq 0$, and $\phi/p \leqq \min(\alpha/r, \gamma/q)$. The conditions of Corollary 1 require that $\gamma \geqq 0$, $\alpha \geqq 0$, $\alpha/r \geqq \gamma/q$, and $\phi/p \leqq \gamma\,\beta(p)/q + \alpha(1 - \beta(p))/r$.

Theorem 2. *Suppose* (3)–(7) *hold and*

$$(17)\quad S_1 := \sup_{t\in I, 0<\varepsilon<\infty}\left\{f(t)^{-p(j+1/q-1/p)}\left([\varepsilon f(t)]^{-1}\int_t^{t+\varepsilon f(t)} N\right) T^{p,q}_{t,\varepsilon}(W)\right\} < \infty\,,$$

$$(18)\quad S_2 := \sup_{t\in I, 0<\varepsilon<\infty}\left\{f(t)^{p(n-j-1/r+1/p)}\left([\varepsilon f(t)]^{-1}\int_t^{t+\varepsilon f(t)} N\right) T^{p,r}_{t,\varepsilon}(P)\right\} < \infty\,.$$

Then (1) *holds for $y \in \mathscr{D}$ if $\int_I P|y^{(n)}|^r \neq 0$ with*

$$(19)\qquad K = K_3 := 2^p\{\max M^p S_1, S_2\}\,.$$

Proof. With t fixed, multiply (9) by $N(s)$ and integrate over J_t; using (17)–(18) this gives

$$(20)\qquad \int_{J_t} N|y^{(j)}|^p \leqq 2^{p-1}\left\{M^p \varepsilon^{-p(j+1/q-1/p)} S_1 \left(\int_{J_t} W|y|^q\right)^{p/q} + \right.$$
$$\left. + \varepsilon^{p(n-j-1/r+1/p)} S_2 \left(\int_{J_t} P|y^{(n)}|^r\right)^{p/r}\right\} \leqq$$
$$\leqq (K/2)\left\{\varepsilon^{-p(j+1/q-1/p)} \left(\int_{J_t} W|y|^q\right)^{p/q} + \right.$$
$$\left. + \varepsilon^{p(n-j-1/r+1/p)} \left(\int_{J_t} P|y^{(n)}|^r\right)^{p/r}\right\}.$$

Fix a compact interval $[a, c]$. We want to cover $[a, c]$ with intervals J_t chosen so that the two terms on the right of (20) are equal. To make this possible, let $\delta > 0$ and $h(t)$ be a positive continuous function such that $\int_I h < \infty$. From (20) we have

$$(21)\qquad \int_{J_t} N|y^{(j)}|^p \leqq (K/2)\left\{\varepsilon^{-p(j+1/q-1/p)} \left(\int_{J_t} [W|y|^q + \delta h]\right)^{p/q} + \right.$$
$$\left. + \varepsilon^{p(n-j-1/r+1/p)} \left(\int_{J_t} P|y^{(n)}|^r\right)^{p/r}\right\}.$$

Set $t_0 = a$. Choose $t_1 = t_0 + \varepsilon_1 f(t_0)$ so that with $t = t_0$, $\varepsilon = \varepsilon_1$, the two terms on the right of (21) are equal. This is possible since the second term varies from 0 to ∞ as ε varies from 0 to ∞ (recall $p \geqq \min(r, q)$ by (4)); the first term goes to 0 as $\varepsilon \to \infty$, and as $\varepsilon \to 0$, it is bounded below by $c\varepsilon^{-pj+1}$ where c is a positive constant. The term ε^{-pj+1} either does not tend to 0 as $\varepsilon \to 0$ $(j > 0)$ or tends to zero more slowly than the second term which is $o(\varepsilon^{p(n-j-1/r+1/p)})$. With this choice of ε, (21) becomes after simplifying

$$(22)\qquad \int_{t_0}^{t_1} N|y^{(j)}|^p \leqq K\left(\int_{t_0}^{t_1} [W|y|^q + \delta h]\right)^{p\beta(p)/q} \left(\int_{t_0}^{t_1} P|y^{(n)}|^r\right)^{p(1-\beta(p))/r}.$$

Now choose $\varepsilon = \varepsilon_2$ so that with $t = t_1$ and $t_2 = t_1 + \varepsilon_2 f(t_1)$, the two terms on the right of (21) are equal; inequality (22) results with $[t_0, t_1]$ replaced by $[t_1, t_2]$. Continue this process. Calculation of equality of the two terms on the right of (21) for $t = t_i$ shows that

$$(23)\qquad \varepsilon_i^{p(n-1/r+1/q)} = \left(\int_{J_{t_i}} [W|y|^q + \delta h]\right)^{p/q} \Big/ \left(\int_{J_{t_i}} P|y^{(n)}|^r\right)^{p/r} \geqq$$
$$\geqq \left(\int_{J_{t_i}} \delta h\right)^{p/q} \Big/ \left(\int_{J_{t_i}} P|y^{(n)}|^r\right)^{p/r}.$$

If the sequence $\{t_i\}$ constructed above satisfies $t_i < c$ for all i, then (23) yields a contradiction since the right of (23) when divided by $\varepsilon_i^{p/q}$ tends to ∞ as $i \to \infty$ while the left side remains bounded. Thus there is an n such that $t_n \geqq c$.

Summing for $i = 1, \ldots, n$ we get

$$(24)\quad \int_a^c N|y^{(j)}|^p \leqq K \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} [W|y|^q + \delta h] \right)^{p\beta(p)/q} \left(\int_{t_{i-1}}^{t_i} P|y^{(n)}|^r \right)^{p(1-\beta(p))/r}.$$

A calculation using (4) and (5) shows that

$$p\,\beta(p)/q + p(1 - \beta(p))/r \geqq 1\,;$$

hence by Jensen's generalization of Holder's inequality [14, p. 52],

$$(25)\quad \int_a^c N|y^{(j)}|^p \leqq K \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} [W|y|^q + \delta h] \right)^{p\beta(p)/q} \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} P|y^{(n)}|^r \right)^{p(1-\beta(p))/r} \leqq$$

$$\leqq K \left(\int_I [W|y|^q + \delta h] \right)^{p\beta(p)/q} \left(\int_I P|y^{(n)}|^r \right)^{p(1-\beta(p))/r}.$$

Since c and δ are arbitrary in (25), the proof is now complete.

Note that with $N = W = P \equiv 1$ in Theorem 2 we may take $f(t) \equiv 1$ to obtain $S_1 = S_2 = 1$; thus an alternate proof of Gabushin's inequality is obtained.

We remark that if (1) holds on rays $[a, \infty)$ and $(-\infty, a]$, then application of Jensen's inequality as in the above proof shows that (1) holds on $(-\infty, \infty)$. Thus the $(-\infty, \infty)$ case is subsumed in the case of rays.

We recall the following result of Kwong and Zettl [10, Theorem 3] which will required for our final weighted generalization of Gabushin's inequality.

Lemma 3. *Suppose* $-\infty \leqq a < b \leqq \infty$, p', q' *satisfy* $1 < p', q'$ *and* $1/p' + 1/q' = 1$, *and* s *is a non-negative function such that* $s^{p'}$ *and* $s^{-q'}$ *are integrable on* $[0, T]$ *for all* $T > 0$. *Define*

$$u(t) = \int_0^t s^{p'}, \quad v(t) = \int_0^t s^{-q'} \quad \text{for} \quad t \geqq 0\,.$$

If for some non-negative functions f, g, h *there is a constant* C *such that*

$$\int_c^b g \leqq C \left(\int_c^b f \right)^{1/p'} \left(\int_c^b h \, dt \right)^{1/q'}$$

for all $c \in (a, b)$, *then for all* $c \in [a, b)$,

$$\int_c^b g\mu \leqq C \left(\int_c^b f\, u(\mu) \right)^{1/p'} \left(\int_c^b h\, v(\mu) \right)^{1/q'}$$

for any nondecreasing non-negative function μ *on* (a, b).

Theorem 3. *Suppose* (3)–(7) *hold,* $p \leqq \max\{q, r\}$,

(i) $W, P, N, P^{1/r}/W^{1/q}$ *are nondecreasing,*

(ii) $NW^{-p_1\beta(p_1)/q}\,P^{-p_1(1-\beta(p_1))/r}$ *is nondecreasing where*

$$p_1 := \max\{q, r\},$$

and

(iii) *There is a number* c *such that* $p'c + 1 > 0$, $1 - q'c > 0$ *and* $W/N^{p'c+1}$, $P/N^{1-q'c}$ *are nondecreasing where*

$$p' = q/p_0\,\beta(p_0), \quad q' = r/p_0(1 - \beta(p_0)), \quad \text{and} \quad p_0 := n\left(\frac{n-j}{q} + \frac{j}{r}\right)^{-1}.$$

Then there is a number K *so that* (1) *holds for all* $y \in \mathscr{D}$.

Proof. We take the case $q \leqq r$; the $r < q$ case is similar. Then $q \leqq p_0 \leqq p \leqq \leqq p_1 = r$. From (i)–(ii) we have by Corollary 1 and Theorem 1 that for $y \in \mathscr{D}$,

$$\int_I N|y^{(j)}|^r \leqq K_2\left(\int_I W|y|^q\right)^{r\beta(r)/q}\left(\int_I P|y^{(n)}|^r\right)^{1-\beta(r)}. \tag{26}$$

Note that from (i) above we have that $\int_I P|y^{(n)}|^r = 0$ implies that $\int_I N|y^{(j)}|^p = \int_I W|y|^q = 0$.

Define $\tilde{N} = 1$, $\tilde{W} = W/N^{p'c+1}$, and $\tilde{P} = P/N^{1-q'c}$. Then with $f(t) \equiv 1$ and $p = p_0$, we apply Theorem 2 to obtain for $y \in \mathscr{D}$ and $a < c < \infty$,

$$\int_c^\infty |y^{(j)}|^{p_0} \leqq \tilde{K}_3\left(\int_c^\infty \tilde{W}|y|^q\right)^{p_0\beta(p_0)/q}\left(\int_c^\infty P y|^{(n)}|^r\right)^{p_0(1-\beta(p_0))/r} \tag{27}$$

where

$$K_3 = 2^{p_0}\max\{M^{p_0}\,\tilde{W}(c)^{-p_0/q}, \tilde{P}(c)^{-p_0/r}\} \leqq$$
$$\leqq 2^{p_0}\max\{M^{p_0}\,\tilde{W}(a)^{-p_0/q}, \tilde{P}(a)^{-p_0/r}\} := \tilde{K}_3.$$

A calculation shows $1/p' + 1/q' = 1$; hence Lemma 3 applies with $s(t) = t^c$ and $\mu = N$. Since $u(t) = t^{p'c+1}/(p'c + 1)$ and $v(t) = t^{1-q'c}/(1 - q'c)$ this gives for $y \in \mathscr{D}$,

$$\int_I N|y^{(j)}|^{p_0} \leqq K_4\left(\int_I W|y|^q\right)^{p_0\beta(p_0)/q}\left(\int_I P|y^{(n)}|^r\right)^{p_0(1-\beta(p_0))/r} \tag{28}$$

where $K_4 = \tilde{K}_3/(p'c + 1)^{p_0\beta(p_0)/q}(1 - q'c)^{p_0(1-\beta(p_0))/r}$. We set $K_5 = \max\{K_2, K_4\}$ and apply Lyapunov's (interpolation) inequality [12, p. 459] to $\int_I N|y^{(j)}|^p$ and then use (26) and (28). This gives

$$\int_I N|y^{(j)}|^p \leqq \left(\int_I N|y^{(j)}|^{p_0}\right)^{(r-p)/(r-p_0)}\left(\int_I N|y^{(j)}|^r\right)^{(p-p_0)/(r-p_0)} \leqq$$
$$\leqq \left[K\left(\int_I W|y|^q\right)^{p_0\beta(p_0)/q}\left(\int_I P|y^{(n)}|^r\right)^{p_0(1-\beta(p_0))/r}\right]^{(r-p)/(r-p_0)}.$$

$$\cdot\left[K\left(\int_I W|y|^q\right)^{r\beta(r)/q}\left(\int_I P|y^{(n)}|^r\right)^{(1-\beta(r))}\right]^{(p-p_0)/(r-p_0)} = K\left(\int_I W|y|^q\right)^{\Delta_1}\left(\int_I P|y^{(n)}|^r\right)^{\Delta_2}$$

where

$$\Delta_1 = \frac{p_0\,\beta(p_0)\,(r-p)}{q(r-p_0)} + \frac{r\,\beta(r)\,(p-p_0)}{q(r-p_0)}$$

with a similar expression for Δ_2. A lengthy calculation shows that

$$\Delta_1 = p\,\beta(p)/q\,,\quad \Delta_2 = p(1-\beta(p))/r\,;$$

thus the proof is complete.

As an application of Theorem 3, we apply it to the case

(29) $$N = \Gamma^{\phi}\,,\quad W = \Gamma^{\gamma}\,,\quad P = \Gamma^{\alpha}$$

where Γ is a positive, non-decreasing function on I. Condition (i) of Theorem 3 holds if

(30) $$0 \leqq \alpha\,,\phi,\gamma\,,\quad \gamma/q \leqq \alpha/r\,,$$

and since $p\,\beta(p)$ and $p(1-\beta(p))$ are nondecreasing with respect to p, (ii) holds if

$$\phi - [p_0\,\beta(p_0)\,\gamma/q + p_0(1-\beta(p_0))\,\alpha/r] \leqq 0$$

which is equivalent to

(31) $$\phi/p_0 \leqq \Delta := \beta(p_0)\,\gamma/q + (1-\beta(p_0))\,\alpha/r\,.$$

Condition (iii) of Theorem 3 requires $p'c + 1 > 0$, $1 - q'c > 0$ and

(32) $$\phi \leqq \gamma/(p'c+1)\,,\quad \phi \leqq \alpha/(1-q'c)$$

We choose c by making the right sides of (32) equal; this results in

$$c = (\gamma - \alpha)/(\alpha p' + \gamma q')$$

and consequently

$$p'c + 1 = \frac{p'\gamma + \gamma q'}{\alpha p' + \gamma q'} = \frac{\gamma(1/q' + 1/p')}{\alpha/q' + \gamma/p'} = \frac{\gamma}{p_0\Delta} > 0$$

after substitution. Thus with this choice of c, $p'c + 1 > 0$, $1 - q'c > 0$, and (32) is equivalent to (31). We summarize these calculations as

Corollary 2. *Suppose* (3), (4), (5), (29), (30), *and* (31) *hold. Then there is a* K *so that* (1) *holds for all* $y \in \mathscr{D}$.

Note that (30)–(31) hold for $\phi = \alpha = \gamma = 1$ if $r \leqq q$ so that (1) is

$$\int_I \Gamma|y^{(j)}|^p \leqq K\left[\int_I \Gamma|y|^q\right]^{p\beta(p)/q}\left[\int_I \Gamma|y^{(n)}|^r\right]^{p(1-\beta(p))/r}$$

which is a generalization of the Gabushin inequality in the $r \leqq q$ case.

For $\Gamma(t) = t$ in (29) and $I = [a, \infty)$, $a > 0$, we now derive a necessary condition for (1). Suppose (1) holds for all $y \in \mathscr{D}$. Let $\psi \in C_0^\infty$ be such that $\psi(0) = 1$, $\psi^{(k)}(0) = 0$ for $k \geqq 1$, and $\psi^{(k)}(1) = 0$ for $k \geqq 0$. Define

$$y_T(t) = \begin{cases} t^\delta, & a \leqq t \leqq T \\ t^\delta \psi\left(\dfrac{t-T}{T}\right), & T < t \leqq 2T \\ 0, & 2T < t \end{cases}$$

where δ is chosen so that each of $\delta - n$, $\phi + (\delta - j)\, p$, $\gamma + q\delta$, and $\alpha + (\delta - n)\, r$ are positive. Then $y_T \in \mathscr{D}$ and calculations show there are positive constants m and M, independent of T, such that

$$\int_a^\infty N|y_T^{(j)}|^p \geqq m\left(T^{\phi+(\delta-j)p+1} - a^{\phi+(\delta-j)p+1}\right), \tag{33}$$

$$\int_a^\infty W|y_T|^q \leqq MT^{\gamma+q\delta+1},$$

$$\int_a^\infty P|y_T^{(n)}|^r \leqq MT^{\alpha+r(\delta-n)+1}.$$

From (1) and (33) we conclude, since T is arbitrary, that

$$\phi + (\delta - j)\, p + 1 \leqq (\gamma + q\delta + 1)\, p\, \beta(p)/q + (\alpha + r(\delta - n) + 1)\, p(1 - \beta(p))/r. \tag{34}$$

After simplification, (34) becomes

$$\phi/p \leqq \gamma\, \beta(p)/q + \alpha(1 - \beta(p))/r. \tag{35}$$

When (30) holds and $p \geqq \max\{q, r\}$, Theorem 2 implies that (35) is a sufficient condition for (1). For $p = p_0$, (35) is equivalent to (31). We conjecture that when (29)–(30) hold, (35) is also a sufficient condition for (1) in the range $p_0 < p < \max\{q, r\}$. At present however we are only able to establish sufficiency for the somewhat stronger hypothesis (31).

Clearly if (1) holds for $N(t) = t^{\phi_0}$ on $[a, \infty)$, $a > 0$, it also holds for $N(t) = t^\phi$ with $\phi \leqq \phi_0$ since t^{ϕ_0}/t^ϕ is bounded below. An open question when (29) holds is the determination of what negative values of γ and α will imply (1) under the Gabushin condition (4).

References

[1] *R. A. Adams:* Sobolev Spaces. Academic Press, New York, 1975.

[2] *R. C. Brown, D. B. Hinton:* Sufficient conditions for weighted inequalities of sum form. J. Math. Anal. Appl., to appear.

[3] *V. N. Gabushin:* Inequalities for the norms of a function and its derivatives in metric L_p. Mat. Zametki *1* (1967), 291—298. (English version in Mathematical Notes vol. *1*, 194—198).
[4] *E. Gagliardo:* Proprieta di alcune classi di funzioni in piu variabili. Ricerche di Mathematica di Napoli 7 (1958), 102—137.
[5] *E. Gagliardo:* Ulteriori di alcume classi funzioni in piu variabili. Ricerche di Mathematica di Napoli 8 (1959), 23—51.
[6] *J. A. Goldstein, M. K. Kwong, A. Zettl:* Weighted Landau Inequalities. J. Math. Anal. Appl. 95 (1983), 20—28.
[7] *D. Henry:* How to remember the Sobolev inequalities, in Differential Equations, (proceedings of the 1st Latin American School of Differential Equations, Sao Paulo, Brazil, 1981). Lecture Notes in Mathematics 957, Springer, Berlin, 1982, 97—109.
[8] *M. K. Kwong, A. Zettl:* Ramifications of Landau's inequality. Proc. Royal Soc. Edinburgh 86A(1980, 175—212.
[9] *M. K. Kwong, A. Zettl:* Weighted norm inequalities of sum form involving derivatives. Proc. Royal Soc. Edinburgh 88A(1981), 121—134.
[10] *M. K. Kwong, A. Zettl:* Norm Inequalities of product form in weighted L^p spaces. Proc. Royal. Soc. Edinburgh 89A(1981), 293—307.
[11] *A. Kufner, O. John, S. Fučík:* Function Spaces. Noordhoff International Publishers, Leyden, 1977.
[12] *A. Marshall, I. Olkin:* Inequalities: Theory of Majorization and Its Applications. Academic Press, New York, 1979.
[13] *C. Miranda:* Su alcuni teoremi di inclusione. Annales Polonici Mathematici *16* (1965), 305—315.
[14] *D. S. Mitrinovic:* Analytic Inequalities. Springer, Berlin, 1970.
[15] *L. Nirenberg:* On elliptic partial differential equations. Annali della Scuola Norm. Sup. Pisa, Ser. III *13* (1958), 115—162.
[16] *H. Triebel:* Theory of function spaces. Monographs in Mathematics, vol. *78*, Birkhauser, Basel, 1983.

Authors' addresses: R. C. Brown, Department of Mathematics, University of Alabama, University, Alabama 35486, D. B. Hinton, Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1300, U.S.A.