INVESTIGATION OF INVARIANT MANIFOLDS OF DYNAMIC SYSTEMS BY MEANS OF QUADRATIC FORMS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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This paper deals with some results concerning the theory of bounded invariant manifolds of dynamic systems which have been recently obtained.

First we will consider the linear system of differential equations

$$\dot{x} = A(t)\,x \tag{1}$$

with continuous and bounded on the whole axis $\mathbb{R} =]-\infty, \infty[$ matrix of coefficients $A(t)$, $\dot{x} = dx/dt$, $x \in \mathbb{R}^r$. We use the following notations: $C^0(\mathbb{R})$ is the space of continuous (vector or matrix) functions $F(t)$ bounded on the whole axis $\mathbb{R}$: $C^1(\mathbb{R})$ is the subspace in $C^0(\mathbb{R})$ of functions $F(t)$ possessing continuous derivatives; $\Omega_\tau^t(A)$ is the fundamental matrix of the system (1) with $\Omega_\tau^\tau(A) = I_n$, I_n being the n-dimensional unit matrix: $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is the scalar product in $\mathbb{R}^n$, $\langle x, x\rangle = \|x\|^2$: A^* is the transposed matrix to A.

Let there exist a quadratic form $V(t, x) = \langle S(t)\,x, x\rangle$, $S = S^* \in C^1(\mathbb{R})$ such that its derivative $\dot{V}$ along the solutions of the system (1) is negative definite,

$$\dot{V}(t, x) = \langle(\dot{S}(t) + S(t)\,A(t) + A^*(t)\,S(t))\,x, x\rangle \leqq -\|x\|^2\,, \tag{2}$$

and $\det S(t) \neq 0$, $t \in \mathbb{R}$. Then the system (1) is exponentially dichotomous on the whole axis $\mathbb{R}$. There is a possibility that the determinant of the matrix $S(t)$ vanishes at some moments $t = t_1, \ldots, t_k$. It is proved that $k \leqq n$, where n is the dimension of the system (1). The condition of non-degeneracy of the matrix $S(t)$ can be substituted by an equivalent condition that there exists an other symmetric matrix $S_1(t) \in C^1(\mathbb{R})$ satisfying the condition

$$\langle(\dot{S}_1(t) - S_1(t)\,A^*(t) - A(t)\,S_1(t))\,x, x\rangle \leqq -\|x\|^2\,. \tag{3}$$

The following statement has been proved.

Theorem 1. *The existence of an n-dimensional symmetric matrix $S_1(t) \in C^1(\mathbb{R})$ satisfying the condition (3) is a necessary and sufficient condition for the system* *of equations* $\dot{x} = A(t)\, x + f(t)$ *to have a unique solution bounded on the whole axis* $\mathbb{R}$ *for every vector function* $f(t) \in C^0(\mathbb{R})$.

Note that the determinant of the matrix $S_1(t)$ at some moments of time can vanish and then the inhomogeneous system will have not one but a family of bounded on $\mathbb{R}$ solutions.

Theorem 2. *Let the matrix* $S_1(t) = S_1^*(t) \in C^1(\mathbb{R})$ *satisfying the condition* (3) *exist and let its determinant vanish at some moments* $t_1, \ldots, t_k$. *Then the system* (1) *is exponentially dichotomous on the semi-axes* $\mathbb{R}_+, \mathbb{R}_-$ *and the dimension of the subspace* $\hat{E}$ *of all solutions bounded on the whole axis* $\mathbb{R}$ *is given by the formula* $\dim \hat{E} = n^-(T_2) - n^-(T_1)$, *where* T_1, T_2 *are fixed moments of time such that* $T_1 < t_i < T_2$, $i = 1, \ldots, k$; $n^-(T)$ *is the number of negative eigenvalues of the matrix* $S(T)$.

In the case of weak regularity of the system (1) on $\mathbb{R}$ the problem of its decomposition by means of Lyapunov Transform as well as the integral representation of solutions bounded on $\mathbb{R}$ were studied. All these results were used for investigating linear extensions of dynamic systems on a torus.

$$\dot{\varphi} = a(\varphi)\,, \quad \dot{x} = A(\varphi)\, x\,. \tag{4'}$$

Such systems of differential equations appear when studying nonlinear multifrequency oscillations. Here $\varphi = (\varphi_1, \ldots, \varphi_m)$, $x = (x_1, \ldots, x_n)$: $a(\varphi)$, $A(\varphi)$ are continuous vector – and matrix – functions, respectively, which are 2π-periodic with respect to each variable φ_j.

$a(\varphi)$ is such that the Cauchy problem $\varphi|_{t=0} = \varphi_0$, $\dot{\varphi} = a(\varphi)$ has a unique solution $\varphi_t(\varphi_0)$ continuously depending on φ_0. We use the following notations: $C^0(T_m)$ is the space of continuous (vector or matrix) functions $F(\varphi)$ which are 2π periodic with respect to each variable φ_j, $j = 1, \ldots, m$, i.e., they are given on the m-dimensional torus T_m, $C'(T_m)$ is the subspace of functions $F(\varphi)$ in $C^0(T_m)$ such that the function $F(\varphi_t(\varphi_0))$ is continuously differentiable with respect to t for all $t \in \mathbb{R}$, $\varphi_0 \in T_m$, $\mathrm{d}F(\varphi_t(\varphi))/\mathrm{d}t|_{t=0} = \dot{F}(\varphi) \in C^0(T_m)$, $\Omega_\tau^t(\varphi_0)$ is a fundamental matrix of the system $\dot{x} = A(\varphi_t(\varphi_0))\, x$.

Recall that the invariant torus of the perturbed system of equations $\dot{\varphi} = a(\varphi)$, $\dot{x} = A(\varphi)\, x + C(\varphi)$, $C(\varphi) \in C^0(T_m)$ is defined by the equality $x = u(\varphi)$ if $u(\varphi) \in C'(T_m)$ and the identity $\dot{u}(\varphi) \equiv A(\varphi)\, u(\varphi) + C(\varphi)$ is valid. Let us introduce one of the main results.

Theorem 3. *Let there exist an n-dimensional symmetric matrix* $S_1(t) \in C'(T_m)$ *satisfying the condition*

$$\langle (\dot{S}_1(\varphi) - S_1(\varphi)\, A^*(\varphi) - A(\varphi)\, S_1(\varphi))\, x, x \rangle \leqq -\|x\|^2\,. \tag{5}$$

Then for every vector-function $C(\varphi) \in C^0(T_m)$ *the system of equations* $\dot{\varphi} = a(\varphi)$, $\dot{x} = A(\varphi)\, x + C(\varphi)$ *has at least one invariant torus* $x = u(\varphi)$. *Moreover, if* $\det S(\varphi)$

vanishes at some point φ_0, *then the system* (4) *has nontrivial invariant tori and each of them can be represented in the form*

$$x = u(\varphi) = \int_{-\infty}^{\infty} H(\varphi)\,(\Omega_0^\tau(\varphi))^* f(\varphi_\tau(\varphi))\,d\tau\,, \tag{6}$$

where $f(\varphi)$ *is any function in* $C^0(T_m)$. *Here* $H(\varphi)$ *is a symmetric matrix function in* $C'(T_m)$ *satisfying the identity* $H(\varphi_\tau(\varphi)) \equiv \Omega_0^\tau(\varphi)\,H(\varphi)\,(\Omega_0^\tau(\varphi))^*$ *and the estimate* $\|\Omega_0^t(\varphi)\,H(\varphi)\,(\Omega_0^\tau(\varphi))\| \leqq K \exp\{-\gamma \mid t - \tau|\}$.

Let us take into account the fact that the equality (6) determines a certain operator $\mathfrak{M}$ acting on functions $f(\varphi) \in C^0(T_m)$. In this case there exists a matrix function $H(\varphi)$ such that the operator $\mathfrak{M}$ is projecting: $\mathfrak{M}^2 = \mathfrak{M}$.

As an example let us consider the system of three equations

$$\begin{aligned} \varphi_1 &= 1 + h_1 \sin \varphi_1 + h_2 \sin m\varphi_2\,, \\ \varphi_2 &= \sqrt{2} + h_3 \cos \varphi_1 + h_4 \sin n\varphi_2\,, \\ \dot{x} &= (h_5 \cos \varphi_1 + h_6 \sin 2\varphi_2)\,x + c(\varphi_1, \varphi_2)\,. \end{aligned}$$

The problem consists in finding the values of parameters h_i $i = 1, \ldots, 6$ for which this system has an invariant torus $x = u(\varphi_1, \varphi_2) \in C'(T_2)$ for each function $c(\varphi_1, \varphi_2) \in C^0(T_2)$. Choosing the scalar function $\cos \varphi_1$ as $S_1(\varphi)$ we obtain the following sufficient condition:

$$h_1 h_5 > 0\,, \quad \min\{|h_1|, 2|h_5|\} \geqq 1 + |h_2| + 2|h_6|\,.$$

Possibilities of integral representations of invariant tori of perturbed systems were studied. If turned out that under the conditions of Theorem 3 there exists an n-dimensional matrix $C(\varphi) \in C'(T_m)$ such that the function

$$G_0(\tau, \varphi) = \begin{cases} \Omega_\tau^0(\varphi)\,C(\varphi_\tau(\varphi))\,, & \leqq 0\,, \\ \Omega_\tau^0(\varphi)\,(C(\varphi_\tau(\varphi)) - I_n), & \tau > 0 \end{cases} \tag{7}$$

satisfies the estimate

$$\|G_0(\tau, \varphi)\| \leqq K \exp\{-\gamma|\tau|\}\,, \quad K, \gamma - \text{const} > 0\,, \quad \tau \in R\,. \tag{8}$$

This is sufficient for representing the invariant torus of the system $\dot{\varphi} = a(\varphi)$, $\dot{x} = A(\varphi)\,x + C(\varphi)$ by the equality

$$x = \int_{-\infty}^{\infty} G_0(\tau, \varphi)\,C(\varphi_\tau(\varphi))\,d\tau\,.$$

The function (7) satisfying the estimate (8) is usually called the Green function of the problem of invariant tori for the system (4).

Sometimes we need only that instead of estimate (8) the function (7) satisfy a weaker condition

(9) $$\int_{-\infty}^{\infty} \|G_0(\tau, \varphi)\| \, d\tau \leqq K = \text{const} < \infty .$$

The problem is: can the estimate (8) be obtained from (9) even for an other function $G_0(\tau, \varphi)$? Note that the assumption of existence of the matrix $C(\varphi) \in C^0(T_m)$, guaranteeing uniform convergence and boundedness of the integral

$$\int_{-\infty}^{\infty} \|G_0(\tau, \varphi)\|^2 \, d\tau ,$$

implies existence of the matrix

$$S_1(\varphi) = 2\left(\int_0^{\infty} G_0(\tau, \varphi)\, G_0^*(\tau, \varphi)\, d\tau - \int_{-\infty}^{0} G_0(\tau, \varphi)\, G_0^*(\tau, \varphi)\, d\tau\right),$$

which satisfies the condition (5) of Theorem 3. Therefore there exists a matrix $C(\varphi) \in C'(T_m)$ which, generally speaking, differs from the previous one, such that the estimate (8) is fulfilled. Note that the constants K, γ in the estimate (8) can be expressed in terms of matrices $S_1(\varphi)$, $A(\varphi)$.

It follows from (5) that small perturbations of the matrix $A(\varphi)$ do not substantially affect the existence of the Green function. If $S_1(\varphi) \in C^1(T_m)$, then the same conclusions would hold for the vector-function $a(\varphi)$, since in this case $\dot{S}_1(\varphi) = (\partial S(\varphi)/\partial\varphi)\, a(\varphi)$. In this connection the problem appears of an approximation of functions $F(\varphi) \in \in C'(T_m)$ by functions $F(\varphi) \in C^1(T_m)$ so that simultaneously its derivative $\dot{F}(\varphi)$ is approximated: $\lim_{n\to\infty} (\|F(\varphi) - F_n(\varphi)\| + \|\dot{F}(\varphi) - \dot{F}_n(\varphi)\|) = 0$. The affirmative solution of this problem is known provided $a(\varphi) \in C'(T_m)$ and $F(\varphi) \in C_{\text{Lip}}(T_m)$. Recently the possibility of such an approximation has been proved provided $\lim_{\sigma\to+0} \sigma^{-1}\mu(a\,;\sigma)\,.$ $.\,\mu(F\,;\sigma) = 0$ where $\mu(a\,;\sigma)$, $\mu(F\,;\sigma)$ – are moduli of continuity of the functions $a(\varphi)$, $F(\varphi)$.

If in addition to the condition (5) we require the existence of an n-dimensional matrix $S(\varphi) = S^*(\varphi) \in C'(T_m)$ satisfying the estimate.

(10) $$\langle(\dot{S}(\varphi) + S(\varphi)\, A(\varphi) + A^*(\varphi)\, S(\varphi))\, x, x\rangle \leqq -\|x\|^2 ,$$

then the matrices $S_1(\varphi)$, $S(\varphi)$ are non-degenerate and the exponential dichotomy of the system $\dot{x} = A(\varphi_t(\varphi_0))\, x$ on $\mathbb{R}$ is uniform with respect to φ_0. In this case the Green function (7) is unique and the matrix $C(\varphi) \in C'(T_m)$ is a projecting matrix, $C^2(\varphi) \equiv C(\varphi)$, satisfying the identity

(11) $$C(\varphi_\tau(\varphi)) \equiv \Omega_0^\tau(\varphi)\, C(\varphi)\, \Omega_\tau^0(\varphi) .$$

In this connection we have the problem of existence of an analogue of the identity (11) for the matrix function $C(\varphi)$ in the case when the Green function (7) is not unique.

Theorem 4. *Let the condition* (5) *be valid with a matrix* $S_1(\varphi) = S_1^*(\varphi) \in C'(T_m)$ *degenerate at some points* φ. *Then there exist unique n-dimensional matrices* $C(\varphi)$, $\hat{C}(\varphi) \in C'(T_m)$, $\hat{C}^*(\varphi) \equiv \hat{C}(\varphi)$ *satisfying the identities*

$$C(\varphi_\tau(\varphi)) \equiv \Omega_0^\tau(\varphi)\, C(\varphi)\, \Omega_\tau^0(\varphi) + \Omega_0^\tau(\varphi)\, \hat{C}(\varphi) \int_\tau^0 (\Omega_0^\sigma(\varphi))^*\, \Omega_\tau^\sigma(\varphi)\, d\sigma\,,$$

$$\hat{C}(\varphi_\tau(\varphi)) \equiv \Omega_0^\tau(\varphi)\, \hat{C}(\varphi)\, (\Omega_0^\tau(\varphi))^*\,, \quad \tau \in \mathbb{R}\,,$$

and estimates

$$\|\Omega_0^t(\varphi)\, C(\varphi)\| \leqq K \exp\{-\gamma t\}\,, \qquad t > 0\,;$$

$$\|\Omega_0^t(\varphi)\, (C(\varphi) - I_n)\| \leqq K \exp\{\gamma t\}\,, \quad t < 0\,;$$

$$\|\Omega_0^t(\varphi)\, \hat{C}(\varphi)\| \leqq K \exp\{-\gamma |t|\}\,, \qquad t \in \mathbb{R}\,,$$

with positive constants K, γ *independent of* t *and* φ. *Moreover,* rank $\hat{C}(\varphi_0) =$ $=$ dim $\hat{E}(\varphi_0)$, *where* $\hat{E}(\varphi_0)$ *is the space of bounded on* $\mathbb{R}$ *solutions of the system* $\dot{x} = A(\varphi_t(\varphi_0))\, x$.

Other problems concern decompositions of the system (4). Supposing that the linear system $\dot{x} = A(\varphi_t(\varphi_0))\, x$ is exponentially dichotomous on $\mathbb{R}$ uniformly in φ_0 we ensure separability of two sets of solutions of this system. It is known that when each φ_0 is fixed there exists a Lyapunov change of variables $x = T_{\varphi_0}(t)\, y$ which transforms the system $\dot{x} = A(\varphi_t(\varphi_0))\, x$ to the corresponding decomposed form $\dot{y} = A^+(t; \varphi_0)\, y_1$, $\dot{y}_2 = A^-(t; \varphi_0)\, y_2$. The problem arises whether it is possible to choose the matrix $T_{\varphi_0}(t)$ in the form $T(\varphi_t(\varphi_0))$ where $T(\varphi) \in C'(T_m)$, i.e., whether there exists a matrix $T(\varphi) \in C'(T_m)$ such that

$$(12) \qquad T^{-1}(\varphi)\, A(\varphi)\, T(\varphi) - T^{-1}(\varphi)\, \dot{T}(\varphi) = \operatorname{diag}\{A^+(\varphi), A^-(\varphi)\}\,,$$

where the matrices A^+, A^- correspond to the ε-dichotomy of the system $\dot{x} =$ $= A(\varphi_t(\varphi_0))\, x$. This problem has a negative answer. In spite of this fact it has been proved that when supposing that the matrix $S(\varphi)$ satisfying the condition (10) can be represented in the decomposed form

$$(13) \qquad S(\varphi) = Q^*(\varphi) \operatorname{diag}\{S_1(\varphi), -S_2(\varphi)\}\, Q(\varphi)\,,$$

where $Q(\varphi) \in C'(T_m)$, $\langle S_i(\varphi)\, \eta_i, \eta_i \rangle \geqq \beta \|\eta_i\|^2$, then there exists a matrix $T(\varphi) \in$ $\in C'(T_m)$ ensuring the decomposition (12). On the other hand, it has been proved that a non-degenerate matrix $T(\varphi) \in C'(T_m)$ reducing the projecting matrix $C(\varphi)$ to the Jordan form $T^{-1}(\varphi)\, C(\varphi)\, T(\varphi) = \operatorname{diag}\{I_r, 0\}$ ensures the decomposition (12). Hence we have the problem of the interconnection of the projecting matrix $C(\varphi) \in C'(T_m)$ with the non-degenerate symmetric matrix $S(\varphi) \in C'(T_m)$. The study of this problem has led to the conclusion that each non-degenerate symmetric matrix $S(\varphi) \in C'(T_m)$ satisfying the condition (10) is connected with the projecting matrix $C(\varphi)$ up to a constant factor by the inequality

$$(14) \qquad \langle (S(\varphi)\, C(\varphi) + C^*(\varphi)\, S(\varphi) - S(\varphi))\, x, x \rangle \geqq \|x\|^2\,.$$

It turned out that the additional supposition (13) concerning the matrix $S(\varphi)$ implies solvability of the system of algebraic equations $C(\varphi)\,x = 0$, $C(\varphi)\,x = x$, i.e., the possibility of reducing the matrix $C(\varphi)$ to the Jordan form. Note that the inequality (14) can be considered as an independent one, not connected with the system (4). Besides, for every projecting matrix $C^2(\varphi) \equiv C(\varphi) \in C^0(T_m)/C'(T_m)/$ there exists a set of matrices $S(\varphi) \in C^0(T_m)/C'(T_m)/$ satisfying the condition (14), in particular, $S(\varphi) = 2(C(\varphi) + C^*(\varphi) - I_n)$. If we suppose that it is possible to reduce the projecting matrix $C(\varphi) \in C^0(T_m)$ to the Jordan form, then each matrix $S(\varphi)$ satisfying (14) is reduced to the diagonal form.

The problem of a possibility of a smooth decomposition of the system (4) into more than two subsystems was studied via quadratic forms. In this direction, the following statements have been proved:

Theorem 5. *Let there exist two n-dimensional non-degenerate matrices $S(\varphi)$, $\bar{S}(\varphi) \in C'(T_m)$ such that the matrix $S(\varphi)$ satisfies the conditions* (10), (13) *where S_1 is an r-dimensional matrix, and $\bar{S}(\varphi)$ satisfies the inequality*

$$\langle(\dot{\bar{S}}(\varphi) + \bar{S}(\varphi)\,A(\varphi) + A^*(\varphi)\,\bar{S}(\varphi) + 2\,\lambda(\varphi)\,\bar{S}(\varphi))\,x, x\rangle \geqq \varepsilon\|x\|^2\,,$$

$$\varepsilon = \text{const} > 0\,,$$

with a certain positive scalar function $\lambda(\varphi) \in C^0(T_m)$ and admits a representation $\bar{S}(\varphi) = \bar{Q}^*(\varphi)\,\mathrm{diag}\,\{\bar{S}_1(\varphi), -\bar{S}_2(\varphi)\}\,Q(\varphi)$, $\bar{Q}(\varphi) \in C'(T_m)$ with positive definite blocks $\bar{S}_i(\varphi)$, $i = 1, 2$, $\bar{S}_1$ being an $\bar{r}$-dimensional matrix, $\bar{r} < r$. Then the inequality $r - \bar{r} < n - m$ where m is the number of variables φ ensures the existence of a non-degenerate matrix $L(\varphi) \in C'(T_m)$ such that

$$L^{-1}(\varphi)\,A(\varphi)\,L(\varphi) - L^{-1}(\varphi)\,\dot{L}(\varphi) = \mathrm{diag}\,\{B_1(\varphi), B_2(\varphi), B_3(\varphi)\}\,, \tag{15}$$

where the matrices B_1, B_2, B_3 have the types $\bar{r} \times \bar{r}$, $(r - \bar{r}) \times (r - \bar{r})$, $(n - r) \times (n - r)$, respectively.

Theorem 6. *Let all the conditions of Theorem 5 except the inequality $r - \bar{r} < n - m$ be fulfilled and let the matrices $S(\varphi)$, $\bar{S}(\varphi)$ have the block-diagonal form $S(\varphi) = \mathrm{diag}\,\{S_1(\varphi), -S_2(\varphi)\}$, $\bar{S}(\varphi) = \mathrm{diag}\,\{\bar{S}_1(\varphi), -\bar{S}_2(\varphi)\}$ where $S_1(\varphi)$, $\bar{S}_1(\varphi)$ are r-dimensional, $S_1(\varphi)$, $S_2(\varphi)$, $\bar{S}_2(\varphi)$ — are positive definite and the matrix $\bar{S}_1(\varphi)$ has $\bar{r}$ positive eigenvalues and $r - \bar{r}$ negative ones.*

Then a non-degenerate matrix $L(\varphi) \in C'(T_m)$ ensuring the decomposition (15) *exists if and only if there exists an r-dimensional matrix $Q(\varphi) \in C'(T_m)$ satisfying the equality $Q^*(\varphi)\,\bar{S}_1(\varphi)\,Q(\varphi) = \mathrm{diag}\,\{I_r, I_{r-\bar{r}}\}$.*

Let us present one of the main results concerning the system of differential equations $\dot{\psi} = a(\psi)$, $\dot{x} = A(\psi)\,x$ with continuous and bounded in the whole space $\mathbb{R}^m$ functions $a(\psi)$, $A(\psi)$. Analogously as before, the uniqueness of solution $\psi_t(\psi_0)$ of the Cauchy problem $\psi|_{t=0} = \psi_0$, $\dot{\psi} = a(\psi)$ is assumed.

Theorem 7. *For each $\psi_0 \in \mathbb{R}^m$ fixed let there exist an n-dimensional symmetric matrix function $S_{\psi_0} \in C^1(R)$ satisfying the conditions*

1) $\langle \dot{S}_{\psi_0}(t) - S_{\psi_0}(t) A^*(\psi_t(\psi_0)) - A(\psi_t(\psi_0)) S_{\psi_0}(t)) x, x \rangle \leqq -\|x\|^2$,

2) $\|S_{\psi_0}(t)\| \leqq \text{const} < \infty$ for all $\psi_0 \in \mathbb{R}^m$, $t \in \mathbb{R}$.

Then: (i) *If* $\det S_{\psi_0}(t) \neq 0$ *for all* $t \in \mathbb{R}$, $\psi_0 \in \mathbb{R}^m$, *then for each vector-function* $f(\psi) \in C^0(\mathbb{R}^m)$ *continuous and bounded on* $\mathbb{R}^m$ *the system of equations*

$$\dot{\psi} = a(\psi), \quad \dot{x} = A(\psi) x + f(\psi) \tag{16}$$

has a unique invariant manifold $x = u(\psi) = \int_{-\infty}^{\infty} G_0(\tau, \psi) f(\psi_\tau(\psi))\, d\tau$.

(ii) *If there exist* $t_0 \in \mathbb{R}$, $\psi_0 \in \mathbb{R}^m$ *such that* $\det S_{\psi_0}(t_0) = 0$ *then the system of equations* (16) *has a family of bounded invariant manifolds and they are represented by the formula*

$$x = u(\psi) = \int_{-\infty}^{\infty} H(\psi) (\Omega_0^\tau(\psi))^* g(\psi_\tau(\psi))\, d\tau + \int_{-\infty}^{\infty} G_0(\tau, \psi) f(\psi_\tau(\psi))\, d\tau ,$$

where $g(\psi)$ *is an arbitrary function in the space* $C^0(\mathbb{R}^m)$, $G_0(\tau, \psi)$ *is the Green function. In this case for every n-dimensional symmetric definite matrix* $B(\psi) \in C^0(\mathbb{R}^m)$ *there exist unique n-dimensional matrices* $C(\psi)$, $H(\psi) \in C^0(\mathbb{R}^m)$, $H^* = H$, *satisfying the identities and estimates*

$$\Omega_\tau^0(\psi)\, C(\psi_\tau(\psi))\, \Omega_0^\tau(\psi) \equiv C(\psi) + H(\psi) \int_\tau^0 (\Omega_0^\sigma(\psi))^* B(\psi_\sigma(\psi))\, \Omega_0^\sigma(\psi)\, d\sigma ,$$

$$H(\psi_\tau(\psi)) \equiv \Omega_0^\tau(\psi) H(\psi) (\Omega_0^\tau(\psi))^* ;$$

$$\|\Omega_0^t(\psi) C(\psi)\| \leqq K \exp\{-\gamma t\} , \quad t \geqq 0 ;$$

$$\|\Omega_0^t(\psi) (C(\psi) - I_n)\| \leqq K \exp\{\gamma t\} , \quad t < 0 ;$$

$$\|\Omega_0^t(\psi) H(\psi)\| \leqq K \exp\{-\gamma |t|\} , \quad t \in \mathbb{R} , \quad K, \gamma = \text{const} > 0 .$$

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