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TOTAL STABILITY PROPERTY IN LIMITING EQUATIONS FOR A FUNCTIONAL DIFFERENTIAL EQUATION WITH INFINITE DELAY

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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The study of relationships between the behaviors of solutions of a given differential equation and those of its limiting equations is important and fruitful for discussing stability properties in nonautonomous systems. Many efforts devoted to this topics can be found in many references. For the references, see [1], [3], [7], [8], [10]. For ordinary differential equations, many authors have shown that it is possible to deduce some stability properties in a given equation from analogous properties in its limiting equations (cf. [1], [3]). Recently D'Anna [3] has shown that the total stability of a bounded solution can be deduced from the total stability in a certain limiting equation which is obtained by employing the Bohr topology. This result is false when the limiting equations are obtained by using the compact open topology as is seen in [2]. The equation considered in [2] is

$$ax'' + bx' + cx = x \sin \sqrt{t}\,, \quad a > 0\,, \quad b > 0\,, \quad 0 < c < 1\,,$$

and for every $\mu \in [-1, 1]$, $ax'' + bx' + cx = \mu x$ is a limiting equation under the compact open topology (cf. [10]). For $-1 \leqq \mu < c$, the null solution of the limiting equation is totally stable, but the null solution of the given equation is not so.

In this article, we shall extend D'Anna's results to functional differential equations with infinite delay, where the arguments in ordinary differential equations can not be applied since the phase spaces are not locally compact. Let $|x|$ be any norm of x in $\mathbb{R}^n$, and let B be a real linear vector space of functions mapping $(-\infty, 0]$ into $\mathbb{R}^n$ with a semi-norm $|\cdot|_B$. If x is a function defined on $(-\infty, a)$, then for each $t \in (-\infty, a)$ we define the function x_t by the relation $x_t(s) = x(t + s)$, $-\infty < s \leqq 0$. The space B is assumed to have the following properties:

(I) If $x(t)$ is defined on $(-\infty, a)$ and is continuous on $[\sigma, a)$, $\sigma < a$, and if $x_\sigma \in B$, then for each $t \in [\sigma, a)$,

(i) $x_t \in B$ and x_t is continuous in t with respect to $|\cdot|_B$,

(ii) there exist a $K > 0$ and a nonnegative continuous function $M(\beta)$ such that $M(\beta) \to 0$ as $\beta \to \infty$ and

$$|x_t|_B \leqq K \sup_{\sigma \leqq s \leqq t} |x(s)| + M(t - \sigma)\, |x_\sigma|_B\,,$$

(iii) there exists a constant $N > 0$ such that $|x(t)| \leqq N|x_t|_B$.

(II) The space of equivalent classes $B/|\cdot|_B$ is a separable Banach space.

Typical examples of such phase spaces are the space C_γ of ϕ such that $e^{\gamma s}\phi(s)$ is bounded and uniformly continuous on $(-\infty, 0]$ with the norm $|\phi| = \sup\{e^{\gamma s}|\phi(s)|; s \leqq 0\}$, where $\gamma > 0$ is a constant, and the space M_γ of measurable functions ϕ with a finite norm $|\phi| = |\phi(0)| + \int_{-\infty}^0 e^{\gamma s}|\phi(s)|\, ds$, $\gamma > 0$ (cf. [4], [7]).

Let S be a compact subset in B, and let $\alpha > 0$, $\beta > 0$ be constants. Denote by $X(S, \alpha, \beta)$ the set $\{x_t;\ t \geqq 0$, where $x(\cdot)$ is a function such that $x_0 \in S$, $|x(s)| \leqq \alpha$ for $s \in [0, \infty)$ and $|x(s_1) - x(s_2)| \leqq \beta|s_1 - s_2|$ for $0 \leqq s_1, s_2 < \infty\}$. Then it is known that the closure $\bar{X}(S, \alpha, \beta)$ of $X(S, \alpha, \beta)$ is compact in B (Corollary 3.2 in [4]).

We denote by $C(I \times B, \mathbb{R}^n)$ the set of continuous functions defined on $I \times B$ with values in $\mathbb{R}^n$, where $I = [0, \infty)$. A sequence $\{f_k\}$ in $C(I \times B, \mathbb{R}^n)$ is said to converge to g Bohr-uniformly on $I \times B$ if f_k converges to g uniformly on $I \times S$ for any compact set S in B as $k \to \infty$. A function $f(t, \phi) \in C(I \times B, \mathbb{R}^n)$ is said to be positively precompact if for any sequence $\{t_k\}$ in I such that $t_k \to \infty$ as $k \to \infty$, the sequence $\{f(t + t_k, \phi)\}$ contains a Bohr-uniformly convergent subsequence. Then, if $f(t, \phi) \in C(I \times B, \mathbb{R}^n)$ and $f(t, \phi)$ is positively precompact, $f(t, \phi)$ is asymptotically almost periodic in t uniformly for $\phi \in B$ (cf. [11]). Also, if $f \in C(I \times B, \mathbb{R}^n)$ is asymptotically almost periodic in t uniformly for $\phi \in B$, $f(t, \phi)$ is positively precompact. We denote by $\Omega(f)$ the set of all limit functions g such that $f(t + t_k, \phi)$ converges to g Bohr-uniformly for some sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$.

Now we shall consider the functional differential equation

$$\dot{x}(t) = f(t, x_t)\,, \tag{1}$$

where $f(t, \phi) \in C(I \times B, \mathbb{R}^n)$. We assume that

(i) $f(t, \phi)$ is positively precompact,

(ii) for any $H > 0$, there is an $L(H) > 0$ such that $|f(t, \phi)| \leqq L(H)$ for all $t \geqq 0$ and $\phi \in B$ such that $|\phi|_B \leqq H$,

(iii) equation (1) has a bounded solution $u(t)$ defined on I such that $|u_t|_B \leqq c$ for all $t \in I$.

A system

$$\dot{x}(t) = g(t, x_t) \tag{2}$$

is called a limiting equation of (1) when $g \in \Omega(f)$. Under the above assumptions, it is clear that the closure of the set $\{u_t;\ t \geqq 0\}$ is contained in the compact set $\bar{X}(\{u_0\}, Nc, L(c))$ and that $g(t, \phi)$ is almost periodic in t uniformly for $\phi \in B$ if $g \in \Omega(f)$. We shall write $(v, g) \in \Omega(u, f)$ when there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $f(t + t_k, \phi) \to g(t, \phi) \in \Omega(f)$ Bohr-uniformly and $u(t + t_k) \to v(t)$ uniformly on any compact set in I. In this case, there exists a subsequence $\{\tau_k\}$ of $\{t_k\}$ such that $u_{\tau_k} \to w \in \bar{X}(\{u_0\}, Nc, L(c))$ and $u(t + \tau_k) \to v(t)$ uniformly on any compact interval in I. Since $|u(\tau_k) - w(0)| \leqq N|u_{\tau_k} - w|_B$, $w(0) = v(0)$.

Thus, if we let $v_0 = w$, then $v_t \in B$ for all $t \geqq 0$ and we have

$$(3) \qquad |u_{t+\tau_k} - v_t|_B \leqq K \sup_{0 \leqq s \leqq t} |u(s + \tau_k) - v(s)| + M(t) |u_{\tau_k} - v_0|_B .$$

Thus we can see that $|u_{t+\tau_k} - v_t|_B \to 0$ uniformly on any compact interval in I as $k \to \infty$. This implies that $v(t)$ is a solution of (2).

Definition. The bounded solution $u(t)$ of (1) is said to be *totally stable*, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $s \geqq 0$, $|u_s - \psi|_B < \delta(\varepsilon)$ and $h(t, \phi)$ is a continuous function which satisfies $|h(t, \phi)| < \delta(\varepsilon)$ for $t \in [s, \infty)$ and ϕ such that $|u_t - \phi|_B \leqq \varepsilon$ for $t \geqq s$, then

$$|u_t - x_t(s, \psi, f + h)|_B < \varepsilon \quad \text{for} \quad t \geqq s ,$$

where $x(s, \psi, f + h)$ is a solution through (s, ψ) of

$$(4) \qquad \dot{x}(t) = f(t, x_t) + h(t, x_t) .$$

Then we have the following equivalent definition.

Lemma 1. *The solution $u(t)$ of (1) is totally stable, if and only if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $s \geqq 0$, $|u_s - \psi|_B < \delta(\varepsilon)$ and $k(t)$ is a continuous function which satisfies $|k(t)| < \delta(\varepsilon)$ on $[s, \infty)$, then $|u_t - x_t(s, \psi, f + k)|_B < \varepsilon$ for $t \geqq s$, where $x(s, \psi, f + k)$ is a solution through (s, ψ) of*

$$(5) \qquad \dot{x}(t) = f(t, x_t) + k(t) .$$

Proof. The necessity is clear. Now consider a solution x through (s, ψ) of (4), where $s \geqq 0$, $|u_s - \psi|_B < \delta(\varepsilon)$ and $h(t, \phi)$ is a continuous function such that $|h(t, \phi)| < \delta(\varepsilon)$ for $t \in [s, \infty)$ and ϕ such that $|u_t - \phi|_B \leqq \varepsilon$ for $t \geqq s$. Suppose that $|u_\tau - x_\tau|_B = \varepsilon$ at some $\tau > s$ and $|u_t - x_t|_B < \varepsilon$ on $s \leqq t < \tau$. Then x is a solution of (4) defined on $[s, \tau]$ and $|h(t, x_t)| < \delta(\varepsilon)$ on $s \leqq t \leqq \tau$. Thus there exists a continuous function $k(t)$ on $[s, \infty)$ such that $|k(t)| < \delta(\varepsilon)$ for all $t \geqq s$ and $k(t) = h(t, x_t)$ on $s \leqq t \leqq \tau$. Then x is also a solution defined on $[s, \tau]$ of (5), and $|u_s - \psi|_B < \delta(\varepsilon)$ and $|k(t)| < \delta(\varepsilon)$ for all $t \geqq s$. Therefore $|u_\tau - x_\tau|_B < \varepsilon$, which contradicts $|u_\tau - x_\tau|_B = \varepsilon$. Thus we have $|u_t - x_t|_B < \varepsilon$ for $t \geqq s$. This shows that $u(t)$ is totally stable.

Remark 1. It is known that $|u_t - x_t|_B < \varepsilon$ in the definition can be replaced by $|u(t) - x(t)| < \varepsilon$ (cf. Theorem 6.1 in [4]).

To prove our theorems, we use the following lemma which can be easily proved (cf. [5], [11]).

Lemma 2. *If the solution $v(t)$ of (2), where $(v, g) \in \Omega(u, f)$, is totally stable, then it is asymptotically almost periodic in t.*

Theorem 1. *If system* (1) *admits a limiting equation* (2) *whose solution* $v(t)$ *such that* $(v, g) \in \Omega(u, f)$ *is totally stable, then* $u(t)$ *is asymptotically almost periodic in* t.

Proof. Since $(v, g) \in \Omega(u, f)$, there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $u(t + t_k) \to v(t)$ uniformly on any compact interval in I and $f(t + t_k, \phi) \to \to g(t, \phi)$ uniformly for $(t, \phi) \in I \times X_0$ as $k \to \infty$, where $X_0 = \bar{X}(\{u_0\}, Nc, L(c))$. For any $\varepsilon > 0$, there is a $k_0(\varepsilon) > 0$ such that if $k \geqq k_0(\varepsilon)$, taking a subsequence if necessary, we have

$$|u_{t_k} - v_0|_B < \delta(\varepsilon/2) \quad \text{and} \quad |f(t + t_k, \phi) - g(t, \phi)| < \delta(\varepsilon/2) \quad \text{on} \quad I \times X_0 ,$$

where $\delta(\cdot)$ is the number for the total stability of $v(t)$. For $k \geqq k_0(\varepsilon)$, $u(t + t_k)$ is a solution of

$$\dot{x}(t) = g(t, x_t) + f(t + t_k, u_{t+t_k}) - g(t, u_{t+t_k}) ,$$

$|u_{t_k} - v_0|_B < \delta(\varepsilon/2)$ and $|f(t + t_k, u_{t+t_k}) - g(t, u_{t+t_k})| < \delta(\varepsilon/2)$ for $t \geqq 0$. Since $v(t)$ is totally stable, we have

$$|u(t + t_k) - v(t)| < \varepsilon/2 \quad \text{for all} \quad t \geqq 0 \tag{6}$$

if $k \geqq k_0(\varepsilon)$.

Now let $\{h'_k\}$ be any sequence such that $h'_k \to \infty$ as $k \to \infty$. Choose a subsequence $\{h_k\}$ of $\{h'_k\}$ such that $2t_k < h_k$ and set $h_k = t_k + s_k$. Then $s_k \to \infty$ as $k \to \infty$, because $s_k > t_k$. Since $v(t)$ is asymptotically almost periodic by Lemma 2, there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ and a function $w(t)$ such that

$$v(t + s_{k_j}) \to w(t) \quad \text{uniformly on } I \text{ as} \quad j \to \infty ,$$

and hence there is a $j_0(\varepsilon) > 0$ such that if $j \geqq j_0(\varepsilon)$, then

$$|v(t + s_{k_j}) - w(t)| < \varepsilon/2 \quad \text{for all} \quad t \geqq 0 . \tag{7}$$

It follows from (6) and (7) that if j is greater than some positive integer $j_1(\varepsilon)$, then $|u(t + h_{k_j}) - w(t)| < \varepsilon$ for all $t \geqq 0$, since we have

$$|u(t + h_{k_j}) - w(t)| \leqq |u(t + t_{k_j} + s_{k_j}) - v(t + s_{k_j})| + |v(t + s_{k_j}) - w(t)| .$$

This shows that $u(t)$ is asymptotically almost periodic in t.

The following lemma holds for a more general case where $C(I \times B, \mathbb{R}^n)$ is a space with the compact open topology. In this case, the convergence of a sequence $\{f_k\}$ in $C(I \times B, \mathbb{R}^n)$ means that f_k converges uniformly on any compact set in $I \times B$ as $k \to \infty$. Moreover, $\Omega(f)$ is the set of all limit functions g such that $f(t + t_k, \phi)$ converges to g uniformly on any compact set in $I \times B$ for some sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$.

Lemma 3. *Let* $C(I \times B, \mathbb{R}^n)$ *be the space of continuous functions defined on* $I \times B$ *with values in* $\mathbb{R}^n$, *with the compact open topology. Assume that* f *in system* (1) *is in* $C(I \times B, \mathbb{R}^n)$. *If the bounded solution* $u(t)$ *of* (1) *is totally stable, then for any* $(v, g) \in \in \Omega(u, f)$, v *is totally stable with a common pair* $(\varepsilon, \delta^*(\varepsilon))$.

Proof. Let y be a solution of

$$\dot{x}(t) = g(t, x_t) + h(t)$$

through (s, y_s), where $s \geqq 0$, $(v, g) \in \Omega(u, f)$ and $h(t)$ is a continuous function on $[s, \infty)$, and assume that

$$|v_s - y_s|_B < \delta(\varepsilon/2)/2 \quad \text{and} \quad |h(t)| < \delta(\varepsilon/2)/2 \quad \text{on} \quad [s, \infty),$$

where $\delta(\cdot)$ is the number for the total stability of $u(t)$. Suppose that for some $\tau > 0$,

$$|v_{s+\tau} - y_{s+\tau}|_B = \varepsilon \quad \text{and} \quad |v_t - y_t|_B < \varepsilon \quad \text{for} \quad s \leqq t < s + \tau .$$

Since $(v, g) \in \Omega(u, f)$, there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $u(t + t_k) \to v(t)$ uniformly on any compact interval in I and $f(t + t_k, \phi) \to g(t, \phi)$ uniformly on any compact set in $I \times B$ as $k \to \infty$. Therefore there is a $k_0 = k_0(\varepsilon, y) > 0$ such that

$$|u_{s+\sigma} - v_s|_B < \delta(\varepsilon/2)/2 \quad \text{and} \quad |f(t + \sigma, \phi) - g(t, \phi)| < \delta(\varepsilon/2)/2 \quad \text{on}$$
$$[s, s + \tau] \times \{X_0 \cup X_y\},$$

where $\sigma = t_{k_0}$, $X_0 = \overline{X}(\{u_0\}, Nc, L(c))$ and $X_y = \overline{X}(\{y_s\}, N(c + \varepsilon), L(c + \varepsilon) + \varepsilon)$. Thus there are continuous functions $p(t)$ and $q(t)$ defined on $[s, \infty)$ such that $|p(t)| < \delta(\varepsilon/2)$, $|q(t)| < \delta(\varepsilon/2)$,

$$p(t) = g(t, v_t) - f(t + \sigma, v_t) \quad \text{on} \quad [s, s + \tau]$$

and

$$q(t) = g(t, y_t) - f(t + \sigma, y_t) + h(t) \quad \text{on} \quad [s, s + \tau] .$$

Then $v(t)$ is a solution of

$$\dot{x}(t) = f(t + \sigma, x_t) + p(t)$$

on $[s, s + \tau]$, and $y(t)$ is a solution of

$$\dot{x}(t) = f(t + \sigma, x_t) + q(t)$$

on $[s, s + \tau]$. On the other hand, it is clear that $u(t + \sigma)$ is a solution of

$$\dot{x}(t) = f(t + \sigma, x_t)$$

and $u(t + \sigma)$ is totally stable with the same $\delta(\cdot)$ as for $u(t)$. Since $|u_{s+\sigma} - v_s|_B < \delta(\varepsilon/2)$ and $|p(t)| < \delta(\varepsilon/2)$, the total stability of $u(t + \sigma)$ implies

(8) $$|u_{t+\sigma} - v_t|_B < \varepsilon/2 \quad \text{for} \quad s \leqq t \leqq s + \tau .$$

Moreover, $|q(t)| < \delta(\varepsilon/2)$ and $|u_{s+\sigma} - y_s|_B \leqq |u_{s+\sigma} - v_s|_B + |v_s - y_s|_B < \delta(\varepsilon/2)$, and hence we have

(9) $$|u_{t+\sigma} - y_t|_B < \varepsilon/2 \quad \text{for} \quad s \leqq t \leqq s + \tau .$$

Thus it follows from (8) and (9) that $|v_{s+\tau} - y_{s+\tau}|_B < \varepsilon$, which contradicts

$|v_{s+\tau} - y_{s+\tau}|_B = \varepsilon$. This shows that $v(t)$ is totally stable with $(\varepsilon, \delta^*(\varepsilon))$, where $\delta^*(\varepsilon) = \delta(\varepsilon/2)/2$.

Lemma 4. *Under the assumptions of Theorem* 1, *for any* $(w, p) \in \Omega(u, f)$, $w(t)$ *is totally stable with a common pair* $(\varepsilon, \delta^*(\varepsilon))$.

Proof. By Theorem 1 and Lemma 2, $u(t)$ and $v(t)$ are asymptotically almost periodic in t, and hence $\Omega(u, f) = \Omega(v, g)$ since $f(t, \phi)$ also is asymptotically almost periodic. Thus, for any $(w, p) \in \Omega(u, f) = \Omega(v, g)$, $w(t)$ is totally stable with a common pair $(\varepsilon, \delta^*(\varepsilon))$ which follows by applying Lemma 3 to (v, g) since $v(t)$ is totally stable.

Now we are ready to prove the following theorem which is a generalization of a result obtained by D'Anna [3].

Theorem 2. *Assume that the bounded solution* $u(t)$ *of* (1) *is the unique solution through* $(0, u_0)$. *If system* (1) *admits a limiting equation* (2) *whose solution* $v(t)$ *such that* $(v, g) \in \Omega(u, f)$ *is totally stable, then* $u(t)$ *is totally stable.*

Proof. First of all, we shall show that $u(t)$ is eventually totally stable, that is, for any $\varepsilon > 0$ there exist $\alpha(\varepsilon) \geqq 0$ and $\delta(\varepsilon) > 0$ such that if $s \geqq \alpha(\varepsilon)$, $|u_s - \psi|_B < \delta(\varepsilon)$ and $h(t)$ is a continuous function which satisfies $|h(t)| < \delta(\varepsilon)$ on $[s, \infty)$, then

$$|u_t - x_t(s, \psi, f + h)|_B < \varepsilon \quad \text{for} \quad t \geqq s\,,$$

where $x(s, \psi, f + h)$ is a solution through (s, ψ) of $\dot{x}(t) = f(t, x_t) + h(t)$.

Suppose that $u(t)$ is not eventually totally stable. Then there exist an $\varepsilon > 0$ and sequences $\{t_k\}, \{r_k\}, \{h^k(t)\}, \{x^k(t)\}$ such that $t_k > k$, $r_k > t_k$, $|x^k_{t_k} - u_{t_k}|_B < 1/k$, $|h^k(t)| < 1/k$ on $[t_k, \infty)$, $|x^k_{r_k} - u_{r_k}|_B = \varepsilon$ and $|x^k_t - u_t|_B < \varepsilon$ on $[t_k, r_k)$, where $h^k(t)$ is a continuous function and $x^k(t)$ is a solution through $(t_k, x^k_{t_k})$ of

$$\dot{x}(t) = f(t, x_t) + h^k(t)\,.$$

Then there is an s_k, $t_k < s_k < r_k$, such that

$$|x^k_{s_k} - u_{s_k}|_B = \delta^*(\varepsilon/2)/2 \quad \text{and} \quad |x^k_t - u_t|_B < \delta^*(\varepsilon/2)/2 \quad \text{on} \quad [t_k, s_k)\,,$$

where $\delta^*(\cdot)$ is the number given in Lemma 4. Taking a subsequence if necessary, we can assume that $u(t + s_k) \to w(t)$ uniformly on I and $f(t + s_k, \phi) \to p(t, \phi)$ uniformly on $I \times X_0$ as $k \to \infty$, where $X_0 = \overline{X}(\{u_0\} \cup \mathrm{Cl}\,\{x^k_{t_k}\}, N(c + \varepsilon), L(c + \varepsilon) + 1)$, since $s_k \to \infty$ as $k \to \infty$ and $u(t)$ is asymptotically almost periodic by Theorem 1. Moreover, we can assume that $w_t \in B$ for all $t \geqq 0$ and $|u_{t+s_k} - w_t|_B \to 0$ uniformly on I as $k \to \infty$, because $u(t + s_k) \to w(t)$ uniformly on I and $\sup_{\beta \geqq 0} M(\beta) < \infty$ in (3). Thus $(w, p) \in \Omega(u, f)$. Then there exists a $k_0(\varepsilon) > 0$ such that if $k \geqq k_0(\varepsilon)$, then

$$|u_{t+s_k} - w_t|_B < \delta^*(\varepsilon/2)/2 \quad \text{for all} \quad t \geqq 0$$

and

$$|f(t + s_k, \phi) + h^k(t + s_k) - p(t, \phi)| < \delta^*(\varepsilon/2) \quad \text{on} \quad I \times X_0\,.$$

On the other hand, $x^k(t + s_k)$ is a solution defined on $[0, r_k - s_k]$ of

$$\dot{x}(t) = p(t, x_t) + f(t + s_k, x^k_{t+s_k}) + h^k(t + s_k) - p(t, x^k_{t+s_k})$$

and

$$|x^k_{s_k} - w_0|_B \leqq |x^k_{s_k} - u_{s_k}|_B + |u_{s_k} - w_0|_B < \delta^*(\varepsilon/2) \quad \text{if} \quad k \geqq k_0(\varepsilon).$$

Therefore we have

$$|x^k_{r_k} - w_{r_k - s_k}|_B < \varepsilon/2 ,$$

because $w(t)$ is a totally stable solution of $\dot{x}(t) = p(t, x_t)$ with $\delta^*(\cdot)$ by Lemma 4. However, we have

$$|x^k_{r_k} - u_{r_k}|_B \leqq |x^k_{r_k} - w_{r_k - s_k}|_B + |w_{r_k - s_k} - u_{r_k}|_B < \varepsilon/2 + \delta^*(\varepsilon/2)/2 < \varepsilon ,$$

which contradicts $|x^k_{r_k} - u_{r_k}|_B = \varepsilon$.

This shows that $u(t)$ is eventually totally stable. Since $u(t)$ is unique for the initial condition, the continuous dependence on initial conditions implies the total stability of $u(t)$.

Under the regularity condition on system (1), we have a result for uniform asymptotic stability. We say that system (1) is regular if the solutions of every limiting equation of (1) are unique for the initial value problem.

The following lemma can be found in [6], [9].

Lemma 5. *If the bounded solution $u(t)$ of (1) is unique for the initial condition and if any w such that $(w, p) \in \Omega(u, f)$ is uniformly asymptotically stable with a common $(\delta_0, \delta(\cdot), T(\cdot))$, then $u(t)$ is uniformly asymptotically stable and is also totally stable.*

Theorem 3. *Assume that the bounded solution $u(t)$ of (1) is the unique solution through $(0, u_0)$. If system (1) is regular and admits a limiting equation (2) whose solution $v(t)$ such that $(v, g) \in \Omega(u, f)$ is uniformly asymptotically stable, then $u(t)$ is uniformly asymptotically stable.*

Proof. Since system (1) is regular and $v(t)$ is uniformly asymptotically stable, any $w(t)$ such that $(w, p) \in \Omega(v, g)$ is uniformly asymptotically stable with a common $(\delta_0, \delta(\cdot), T(\cdot))$ by Proposition 1 in [6]. Therefore, by applying Lemma 5 to (v, g), $v(t)$ is totally stable, and hence $u(t)$ is asymptotically almost peroidic in t by Theorem 1. Since $\Omega(u, f) = \Omega(v, g)$, we have the conclusion by applying Lemma 5 again.

Remark 2. As was shown in [9], if system (1) is regular or periodic and if the bounded solution $u(t)$ is uniformly asymptotically stable, then $u(t)$ is asymptotically stable under M perturbations, that is, for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $s \geqq 0$, $|u_s - \psi|_B < \delta(\varepsilon)$ and $\sup_{t \geqq 0} \int_t^{t+1} |h(s)| \, ds < \delta(\varepsilon)$, then $|u_t - x_t(s, \psi, f + h)|_B < \varepsilon$ for all $t \geqq s$, and moreover, there exists a $\delta_0 > 0$ and for any $\eta > 0$ there exist

$\gamma(\eta) > 0$ and $T(\eta) \geqq 0$ such that $|u_t - x_t(s, \psi, f + h)|_B < \eta$ for all $t \geqq s + T(\eta)$ whenever $|u_s - \psi|_B < \delta_0$ at some $s \geqq 0$ and $\sup_{t \geqq 0} \int_t^{t+1} |h(s)| ds < \gamma(\eta)$, where $x(s, \psi, f + h)$ is a solution through (s, ψ) of

$$\dot{x}(t) = f(t, x_t) + h(t) .$$

Therefore Theorem 3 is a generalization of Theorem 3.2 in [3].

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