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ON LOCAL AND GLOBAL CONTROLLABILITY

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1.

We consider the control process represented by a family of ordinary differential equations

$$(\mathrm{A}, \mathrm{c}) \qquad \mathrm{d}x/\mathrm{d}t = Ax + c$$

where x, the state vector, is a function of time $t \geqq 0$ with values $x(t) \in \mathbb{R}^n$, A is a real $n \times n$ matrix, and c, the control parameter, is a function of t with values $c(t)$ in a subset Γ of $\mathbb{R}^n$.

We shall denote by C_Γ the set of measurable, locally integrable functions of $t \geqq 0$, $c: t \to c(t) \in \Gamma$.

For each $c \in C_\Gamma$ the solution of (A, c) starting from an initial state $v \in \mathbb{R}^n$ at time $t = 0$ is represented, at time t, by

$$(1.1) \qquad x(t, v, c) = \mathrm{e}^{tA} v + \int_0^t \mathrm{e}^{(t-s)A} c(s)\, \mathrm{d}s . \qquad \square$$

In order that c might be considered as a genuine control it must not be constant, so that we shall assume that Γ is not reduced to a single point, or, equivalently, that

$$\operatorname{rel\,int} \operatorname{co} \Gamma \neq \emptyset$$

holds, where $\operatorname{co} \Gamma$ is the convex hull of Γ and $\operatorname{rel\,int} \operatorname{co} \Gamma$ is the interior of $\operatorname{co} \Gamma$ relative to the affine hull of Γ. $\square$

A point $w \in \mathbb{R}^n$ is said to be reachable from the origin O of $\mathbb{R}^n$ at time t if there exist some $c \in C_\Gamma$ such that $x(t, 0, c) = w$.

According to (1.1) the set of points reachable from O at time t is

$$W(t, A, \Gamma) = \left\{ \int_0^t \mathrm{e}^{(t-s)A} c(s)\, \mathrm{d}s : c \in C_\Gamma \right\} .$$

The union of these sets with respect to $t > 0$,

$$W(A, \Gamma) = \bigcup_{t>0} W(t, A, \Gamma) ,$$

is the set of points reachable from O. $\square$

We say that a pair (A, Γ) is O-locally reachable if

$$(\mathrm{R})_{O-\mathrm{loc}} \qquad O \in \operatorname{int} W(A, \Gamma)$$

holds.

We say that (A, Γ) is O-globally reachable if

$$(\mathrm{R})_{O-\mathrm{gl}} \qquad W(A, \Gamma) = \mathbb{R}^n$$

holds.

Obviously

$$(\mathrm{R})_{O-\mathrm{gl}} \Rightarrow (\mathrm{R})_{O-\mathrm{loc}} .$$

In what follows we shall discuss

Problem P. *To find a property* (x) *of the pair* (A, Γ) *such that*

$$(\mathrm{R})_{O-\mathrm{loc}} \ \text{ plus } \ (\mathrm{x}) \Leftrightarrow (\mathrm{R})_{O-\mathrm{gl}} . \qquad \square$$

2.

Before we go further let us recall some known properties of reachable sets (see, for instance, [Q]).

First of all ($\mathrm{Cl}\, S$ = closure of S),

$$W(t, A, \Gamma) = \mathrm{co}\, W(t, A, \Gamma), \quad \forall t, A, \Gamma, \tag{2.1}$$

$$\mathrm{Cl}\, W(t, A, \Gamma) = \mathrm{Cl}\, W(t, A, \mathrm{Cl}\,\mathrm{co}\,\Gamma), \quad \forall t, A, \Gamma, \tag{2.2}$$

$$O \in \operatorname{int} W(A, \Gamma) \Leftrightarrow \exists t > 0 \colon O \in \operatorname{int} W(t, A, \Gamma) \tag{2.3}$$

hold. Consequently,

$$O \in \operatorname{int} W(A, \Gamma) \Leftrightarrow O \in \operatorname{int} W(A, \mathrm{Cl}\,\mathrm{co}\,\Gamma) . \tag{2.4}$$

From (2.2) we have also

$$\mathrm{Cl}\, W(A, \Gamma) = \mathrm{Cl}\, W(A, \mathrm{Cl}\,\mathrm{co}\,\Gamma), \quad \forall A, \Gamma$$

and since

$$\mathrm{Cl}\, W(A, \Gamma) = \mathbb{R}^n \Leftrightarrow W(A, \Gamma) = \mathbb{R}^n, \tag{2.5}$$

we obtain

$$W(A, \Gamma) = \mathbb{R}^n \Leftrightarrow W(A, \mathrm{Cl}\,\mathrm{co}\,\Gamma) = \mathbb{R}^n . \tag{2.6}$$

From (2.4), (2.6) we conclude that, in dealing with **Problem P**, it is not restrictive to assume

$$\Gamma = \mathrm{Cl}\,\mathrm{co}\,\Gamma$$

as we shall do from now on. $\square$

One further remark: $(\mathrm{R})_{O-\mathrm{gl}}$ does not imply that

(h) $$O \in \Gamma \,,$$

hence the same applies to $(\mathrm{R})_{O-\mathrm{loc}}$. Therefore, also the condition (x) we are looking for must be independent of assumption (h). □

3.

Let us now review what is known about Problem P. Notice that the results we are going to review are originally stated in terms of the pair $(-A, -\Gamma)$ rather than (A, Γ). □

The first contribution to Problem P is contained in a wellknown paper of J. P. LaSalle [1] who proved the following.

Let $\Gamma = BU^m$ where U^m is a cube, namely

$$U^m = \{u \in \mathbb{R}^m \colon |u_i| \leqq 1,\ i = 1, 2, \ldots, m\}\,,$$

and B is a real $n \times m$ matrix. Then $(\mathrm{R})_{O-\mathrm{gl}}$ is equivalent to $(\mathrm{R})_{O-\mathrm{loc}}$ plus

(c^{i}) $$z \neq 0\,, \quad z^*A = \lambda z^* \Rightarrow \operatorname{Re} \lambda \geqq 0\,.$$ □

The same result was obtained independently and almost simultaneously by J. Kurzweil and Z. Vorel [2] by means of an entirely different proof.

Other cases where $\Gamma = B\Omega$, Ω a bounded subset of $\mathbb{R}^m$, were considered by E. B. Lee - L. Markus ([3], p. 96), A. M. Formal'skii [4], R. F. Brammer ([5], Th. 3.5), S. H. Saperstone ([6], Cor. 5.2); V. I. Korobov - A. P. Marinic - E. N. Podol'skii ([7], Cor. p. 1978), L. A. Kun - Yu. F. Pronosin [8]. □

All these results were finally extended by L. A. Kun [9] who proved that

(3.2) $$(\mathrm{R})_{O-\mathrm{loc}} \ \text{plus} \ (\mathrm{c}^{\mathrm{i}}) \Rightarrow (\mathrm{R})_{O-\mathrm{gl}}$$

holds with no supplementary assumptions on Γ, and that if

(H_0) Γ is a bounded set

then the converse

(3.3) $$(\mathrm{R})_{O-\mathrm{loc}} \ \text{plus} \ (\mathrm{c}^{\mathrm{i}}) \Leftarrow (\mathrm{R})_{O-\mathrm{gl}}$$

is also true.

In other words,

$$(\mathrm{x}) \Leftrightarrow (\mathrm{c}^{\mathrm{i}})$$

provided that (H_0) holds. □

Assumption (H_0) is quite a reasonable one for applications, but unsatisfactory from a theoretical viewpoint. So we must try to get rid of it.

Taking for instance $\Gamma = \mathbb{R}^n$ it is obvious that $(\mathrm{R})_{O-\mathrm{gl}}$ holds for every A, so that $(\mathrm{c}^{\mathrm{i}})$ is no longer necessary and (3.3) is no longer valid.

In other words $(\mathrm{c}^{\mathrm{i}})$ is stronger than the condition (x) we are after.

4.

A weaker condition than (c^i), reducing to (c^i) when (H_0) holds, is represented by ([7], Theorem 2):

$$(c^{ii}) \qquad z \neq 0\,, \quad z^*A = \lambda z^*\,, \quad \operatorname{Re}\lambda < 0 \Rightarrow \{z^*\gamma\colon \gamma \in \Gamma\} \text{ unbounded}\,,$$

and it is easy to show that

$$(4.1) \qquad (\mathrm{R})_{O-\mathrm{gl}} \Rightarrow (c^{ii})\,.$$

In fact, $w \in W(A, \Gamma)$ if and only if $w = \int_0^t e^{sA} c(t-s)\,ds$ for some $t > 0$ and $c \in C_\Gamma$. Let there exist $z \neq 0$ and $\varrho > 0$ such that $z^*A = \lambda z^*$, $\operatorname{Re}\lambda < 0$, $|z^*\gamma| \leqq \varrho$ for $\gamma \in \Gamma$. Then

$$z^*w = \int_0^t e^{\lambda s} z^* c(t-s)\,ds\,,$$

hence

$$|z^*w| \leqq \int_0^t e^{\operatorname{Re}\lambda s} |z^*c(t-s)|\,ds \leqq -\varrho/\operatorname{Re}\lambda$$

so that $(\mathrm{R})_{O-\mathrm{gl}}$ cannot hold. □

On the other hand,

$$(\mathrm{R})_{O-\mathrm{loc}} \quad \text{plus} \quad (c^{ii}) \Rightarrow (\mathrm{R})_{O-\mathrm{gl}}$$

is *not* true, as is shown, for instance, by

Example 4.1. Let $n = 2$ and

$$A = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \Gamma = \{(\gamma_1, \gamma_2)\colon \gamma_1 \in \mathbb{R},\ \gamma_2 \geqq \gamma_1^2\}\,.$$

It can be shown (see [Q]) that

$$W(A, \Gamma) = \{(w_1, w_2)\colon (w_1 - 1/2)^2 < w_2 + 1/2\}$$

so that $(\mathrm{R})_{O-\mathrm{loc}}$ holds, but $(\mathrm{R})_{O-\mathrm{gl}}$ does not.

On the other hand, it is easily seen that (c^{ii}) holds. □

From the preceding we have

$$(4.2) \qquad (c^i) \Rightarrow (x) \Rightarrow (c^{ii})\,. \qquad □$$

5.

Let us now consider the condition

$$(c^{iii}) \qquad y \neq 0\,, \quad y^*A = \lambda y^*\,, \quad \lambda < 0 \Rightarrow$$
$$\Rightarrow \exists\{y^k\} \text{ in } \Gamma \text{ such that } |y^k| \to +\infty \text{ and } y^*y^k \geqq \delta|y^k| \text{ for some } \delta > 0\,;$$

$$z \neq 0, \quad z^*A = \lambda z^*, \quad \operatorname{Re}\lambda < 0, \quad \operatorname{Im}\lambda \neq 0 \Rightarrow$$

$$\Rightarrow \exists\{y^k\} \text{ in } \Gamma \text{ such that } |y^k| \to +\infty \text{ and } |z^* y^k| \geqq \delta |y^k| \text{ for some } \delta > 0 .$$

Obviously, $(c^{\mathrm{i}}) \Rightarrow (c^{\mathrm{iii}})$.

It was proved in [7] (Theorem 2), under the additional assumption (h), that

$$(\mathrm{R})_{O-\mathrm{loc}} \text{ plus } (c^{\mathrm{iii}}) \Rightarrow (\mathrm{R})_{O-\mathrm{gl}} \tag{5.1}$$

holds. The proof, entirely analytic, makes use of the properties of almost periodic functions. □

6.

Recall that (see T. R. Rockafellar: Convex Analysis, Princeton, 1970) the recession cone of a non empty convex set $S \subset \mathbb{R}^n$ is defined as the set $O^+S = \{x\colon S + x \subset S\}$.

Then let us consider the condition

(c^{iv}) $O^+\Gamma$ is not supported by any y, $y^*A = \lambda y^*$, $\lambda < 0$;

$O^+\Gamma$ is not orthogonal to any z, $z^*A = \lambda z^*$, $\operatorname{Re}\lambda < 0$, $\operatorname{Im}\lambda \neq 0$.

Recently Nguyen Khoa Son [10] proved, under the additional assumption (h), that

$$(\mathrm{R})_{O-\mathrm{loc}} \text{ plus } (c^{\mathrm{iv}}) \Rightarrow (\mathrm{R})_{O-\mathrm{gl}} \tag{6.1}$$

holds. However we can see that

$$(c^{\mathrm{iv}}) \Leftrightarrow (c^{\mathrm{iii}}) . \tag{6.2}$$

Proof. Assume first that (c^{iii}) does not hold. This gives two possibilities, namely

a) there exist y, $y^*A = \lambda y^*$, $\lambda < 0$, such that for any sequence $y^k \in \Gamma$, $\lim |y^k| = +\infty$, we have

$$\limsup \frac{y^* y^k}{|y^k|} \leqq 0 ;$$

b) there exist z, $z^*A = \lambda z^*$, $\operatorname{Re}\lambda < 0$, $\operatorname{Im}\lambda \neq 0$, such that for any sequence $y^k \in \Gamma$, $\lim |y^k| = +\infty$, we have

$$\limsup \frac{|z^* y^k|}{|y^k|} = 0 .$$

Let us fix $y_0 \in \Gamma$. Then for any non-zero $y \in O^+\Gamma$ we have $y^k \triangleq y_0 + ky \in \Gamma$ $(k = 0, 1, \ldots)$, $\lim |y^k| = +\infty$ and

$$\lim \frac{k}{|y^k|} = \lim k(|y_0|^2 + 2ky_0^* y + k^2|y|^2)^{-1/2} = \frac{1}{|y|} .$$

In case a) we have

$$\limsup \frac{y^*\gamma^k}{|\gamma^k|} = \lim \left\{\frac{y^*\gamma_0}{|\gamma^k|} + \frac{k}{|\gamma^k|} y^*\gamma\right\} = \frac{y^*\gamma}{|\gamma|} \leqq 0$$

or $y^*\gamma \leqq 0$, i.e., $O^+\Gamma$ is supported by y.

The proof in case b) is analogous.

So we have $(c^{iv}) \Rightarrow (c^{iii})$. Let us now prove $(c^{iv}) \Leftarrow (c^{iii})$.

Let $\gamma^k \in \Gamma$ be such that $\lim |\gamma^k| = +\infty$. We can assume that $\gamma^k/|\gamma^k| \to \gamma_\infty$, $|\gamma_\infty| = 1$. Let $\gamma \in \Gamma$ and $\vartheta \geqq 0$. Then

$$\gamma + \vartheta\gamma_\infty = \lim \left\{\frac{\vartheta}{|\gamma^k|} \gamma^k + \left(1 - \frac{\vartheta}{|\gamma^k|}\right)\gamma\right\}.$$

Since $\gamma^k, \gamma \in \Gamma$ and $\vartheta/|\gamma^k| \in [0, 1]$ if k is sufficiently large, we have

$$\frac{\vartheta}{|\gamma^k|} \gamma^k + \left(1 - \frac{\vartheta}{|\gamma^k|}\right) \gamma \in \Gamma,$$

hence $\gamma + \vartheta\gamma_\infty \in \Gamma = \mathrm{Cl}\,\mathrm{co}\,\Gamma$, so that $\gamma_\infty \in O^+\Gamma$.

If (c^{iii}) holds, then $y^*\gamma_\infty \geqq \delta > 0$ for every y, $y^*A = \lambda y^*$, $\lambda < 0$, and $|z^*\gamma_\infty| \geqq$ $\geqq \delta > 0$ for every z, $z^*A = \lambda z^*$, $\mathrm{Re}\,\lambda < 0$, $\mathrm{Im}\,\lambda \neq 0$, so that neither $O^+\Gamma$ is supported by y, nor $O^+\Gamma$ is orthogonal to z, i.e., (c^{iv}) holds. □

From (6.2) we conclude that (5.1) and (6.1) are equivalent results. It should be noticed, however, that, unlike the proof of (5.1) in [7], the proof of (6.1) in [10] is entirely geometric and makes use of Schauder fixed point theorem. □

7.

We shall now show that assumption (h) can be omitted to obtain (5.1), i.e., (6.1).

To see this let

$$W(A, \Gamma, x) = W(A, \Gamma + Ax) + x \tag{7.1}$$

denote the set of points which can be reached from a given $x \in \mathbb{R}^n$: in particular, $W(A, \Gamma, 0) = W(A, \Gamma)$.

Let also define the set

$$Q = \{x\colon -Ax \in \mathrm{rel\,int\,co}\Gamma\}.$$

Then it can be shown [11] that $(\mathrm{R})_{O-\mathrm{loc}}$ implies $Q \neq \emptyset$ and

$$x \in \mathrm{int}\, W(A, \Gamma, x), \quad x \in Q, \tag{7.2}$$

hence, by (7.1),

$$O \in \mathrm{int}\, W(A, \Gamma + Ax), \quad x \in Q.$$

Since $x \in Q$ implies $O \in \Gamma + Ax$ we can use condition (c^{iii}) or (c^{iv}), i.e., (5.1) or (6.1), to obtain

$$W(A, \Gamma + Ax) = \mathbb{R}^n, \quad x \in Q.$$

Hence

$$W(A, \Gamma, x) = \mathbb{R}^n\,, \quad x \in Q\,. \tag{7.3}$$

On the other hand, it can also be shown ([11]) that if $x \in \operatorname{int} W(A, \Gamma, x)$, $y \in \operatorname{int} W(A, \Gamma, y)$, then $W(A, \Gamma, x) = W(A, \Gamma, y)$, so that $(\mathrm{R})_{O-\mathrm{loc}}$, due to (7.2), yields

$$W(A, \Gamma) = W(A, \Gamma, x)\,, \quad x \in Q\,,$$

and from (7.3) we have $W(A, \Gamma) = \mathbb{R}^n$, i.e., $(\mathrm{R})_{O-\mathrm{gl}}$. □

8.

The following example shows that condition $(\mathrm{c}^{\mathrm{iii}}) = (\mathrm{c}^{\mathrm{iv}})$ is not necessary for $(\mathrm{R})_{O-\mathrm{gl}}$ to hold.

Example 8.1. Let A be the same as in Example 4.1 but let, instead, Γ be the union of the cone $\Gamma' = \{(\gamma_1, \gamma_2): \gamma_1 \leqq 0,\ \gamma_2 \geqq 0\}$ and the closed region $\Gamma'' = \{(\gamma_1, \gamma_2): \gamma_1 \geqq 0,\ \gamma_2 \geqq \varphi(\gamma_1)\}$, where $\varphi(\gamma) = (|\gamma| + 1)\log(|\gamma| + 1) - |\gamma|$.

Since $O^+\Gamma = \Gamma'$ we see that $(\mathrm{c}^{\mathrm{iv}})$ does not hold.

Obviously $\Gamma = \Gamma' + \Gamma''$, so that $W(t, A, \Gamma) = W(t, A, \Gamma') + W(t, A, \Gamma'')$, $\forall t > 0$.

It is readily seen that $\mathrm{Cl}\, W(t, A, \Gamma') = \Gamma'$: namely, $W(t, A, \Gamma')$ is Γ' without the points $w_1 < 0$, $w_2 = 0$.

Taking $\gamma_1(t - s) = \mathrm{e}^s - 1$, $\gamma_2(t - s) = \varphi(\gamma_1(t - s)) = \mathrm{e}^s s - \mathrm{e}^s + 1$ we obtain the point

$$P_t = (t - 1 + \mathrm{e}^{-t},\ -t + 2 - 2\mathrm{e}^{-t} - t\mathrm{e}^{-t}) \in W(t, A, \Gamma'')$$

so that $\mathrm{Cl}\, W(t, A, \Gamma) \supset \Gamma' + P_t$.

As t goes from 0 to $+\infty$, P_t describes a curve in the region $w_1 > 0$, $w_2 < 0$ going from the origin to the asymptote $w_1 + w_2 = 1$, and it follows that $W(A, \Gamma) = \mathbb{R}^2$. □

Summing up, (4.2) can be replaced by the stronger implication

$$(\mathrm{c}^{\mathrm{iii}}) \Leftrightarrow (\mathrm{c}^{\mathrm{iv}}) \Rightarrow (\mathrm{x}) \Rightarrow (\mathrm{c}^{\mathrm{ii}}) \tag{8.1}$$

and the arrows are not invertible. □

9.

Let us recall that the barrier cone K_S of a convex non empty set $S \subset \mathbb{R}^n$ is the convex cone with vertex at 0 defined by

$$K_S = \{y \in \mathbb{R}^n: \sup_{x \in S} y^* x < +\infty\}\,.$$

K_S is not necessarily closed even if S is closed. Also, $K_S = \mathbb{R}^n$ if S is bounded. Therefore the assumption (H_0) is stronger than the assumption

(H_1) the barrier cone of Γ is closed.

Nguyen Khoa Son [10] noted that if (H_1) holds then

$$(\mathrm{c}^{\mathrm{iv}}) \Leftarrow (\mathrm{R})_{O-\mathrm{gl}}\,.$$

In fact, if $(\mathrm{c}^{\mathrm{iv}})$ does not hold there are two possibilities:

a) there exist $y \neq 0$, $y^*A = \lambda^* y$, $\lambda < 0$, such that $y^*x \leqq 0$, $\forall x \in O^+\Gamma$, which means $y \in (O^+\Gamma)^0$, the polar cone of $O^+\Gamma$. But (cf. T. R. Rockafellar, loc. cit. p. 123) we have $O^+\Gamma = (K_\Gamma)^0$, hence $(O^+\Gamma)^0 = (K_\Gamma)^{00} = \mathrm{Cl}\, K_\Gamma$. Therefore if $\mathrm{Cl}\, K_\Gamma = K_\Gamma$ we have $y \in K_\Gamma$, against condition $(\mathrm{c}^{\mathrm{ii}})$;

b) there exist $z \neq 0$, $z^*A = \lambda z^*$, $\mathrm{Re}\,\lambda < 0$, $\mathrm{Im}\,\lambda \neq 0$, such that $z^*x = 0$, $x \in O^+\Gamma$, so that $\mathrm{Re}\, z$, $\mathrm{Im}\, z \in (O^+\Gamma)^0$, hence $\mathrm{Re}\, z$, $\mathrm{Im}\, z \in K_\Gamma$, against $(\mathrm{c}^{\mathrm{ii}})$ again.

Therefore, in our notation

$$(\mathrm{x}) \Leftrightarrow (\mathrm{c}^{\mathrm{iv}}) \Leftrightarrow (\mathrm{c}^{\mathrm{iii}})$$

provided (H_1) holds, i.e., Problem P is solved under the additional assumption (H_1).

As far as we know, the solution is still unknown when (H_1) does not hold. □

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