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EXPLICIT DEFINITION OF AN EXACT MEASURE FOR THE SEMIFLOW OF A FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1. INTRODUCTION

This paper deals with the semiflow S on $C[0, 1]$ generated by the first order partial differential equation

(1) $$u_t + c(x)\, u_x = f(x, u)\,, \quad 0 \leqq x \leqq 1\,, \quad 0 \leqq t < \infty\,,$$

(2) $$u(0, x) = v$$

and its conjugate — the left shift semigroup T on $C[0, \infty)$. The semiflow S has been introduced in [3] as a model of a selfsustaining cell population. As shown in [1, 2], S is conjugate to T, the latter being a very useful tool for the study of S.

In [2] it has been proved that the left shift on $C[0, \infty)$ is exact, i.e. admits a non-trivial probabilistic measure m such that $\lim_{t\to\infty} m(T_t(A)) = 1$ for each $A \subset C[0, \infty)$ with $m(A) > 0$. Using the conjugacy Φ of S and T this measure could be carried over to certain S-invariant subsets of $C[0, 1]$ to prove exactness of the restriction of S to these subsets. In order to do so a preliminary scaling of the real line was needed before Φ could be used. The definition of this scaling, however, was not constructive.

The purpose of this paper is to remove this shortcoming. In Section 2 we prove that a suitable scaling can be defined explicitly using the distribution function of the normalized normal distribution. In addition we show that the resulting measure is natural in that the marginal cylinder of functions the values of which at a fixed point lie in a given interval has the measure equal to the Lebesgue measure of the interval.

To define a measure on $C[0, 1]$ via the conjugacy Φ using the measure on $C[0, \infty)$ constructed in Section 2 a refinement of the stability theorem of [3] is needed giving the attraction rate of the stable stationary solution of (1), (2). This is the subject of Section 3.

We conclude this section by a survey of assumptions and results on (1), (2) which are necessary for understanding our paper.

Assumptions:

A1. The functions c, f are continuously differentiable.

A2. $c(0) = 0$, $c(x) > 0$ for $x > 0$.

A3. There exist $u_- < 0 < u_+$ such that $f(0, u_-) = f(0, 0) = f(0, u_+) = 0$, $u f(0, u) > 0$ for $0 \neq u \in (u_-, u_+)$, $f_u(0, u_-) < 0$, $f_u(0, u_+) < 0$.

A4. $f(x, u) \operatorname{sign} u \leqq k_1|u| + k_2$ for some $k_1, k_2 > 0$, all $x \in [0, 1]$ and all u.

Note that the assumptions A1, A2, A4 coincide with those of [2]. Assumption A3 is a restriction of A3 of [2] to one interval between "stable" zeros of $f(0, u)$.

Under these assumptions we have the following

Proposition 1. (i) *For any* $v \in [0, 1]$, (1), (2) *has a unique solution* u *in* $C([0, \infty) \times [0, 1])$. *The map* $S\colon [0, \infty) \times C[0, 1] \to C[0, 1]$ *defined by* $(S_t v)(x) = u(t, x)$, *where* u *satisfies* (1), (2), *is continuous and* $S_0 = \mathrm{id}$, $S_t \circ S_s = S_{t+s}$ *for all* $s, t \geqq 0$.

(ii) *There exists a unique solution* $w_+(w_-)$ *of the stationary equation*

$$c(x)\, w' = f(x, w)\,, \quad 0 \leqq x \leqq 1 \tag{3}$$

satisfying $w_+(0) = u_+(w_-(0) = u_-)$. *For each* $v \in C[0,1]$ *such that* $v(0) \in (0, u_+]$ *or* $v(0) \in [u_-, 0)$ *one has* $\lim_{t\to\infty} (S_t v)(x) = u_+$ *or* u_-, *respectively*.

(iii) *The map* $\Phi\colon C[0, 1] \to C[0, \infty)$ *defined by* $\Phi(v)(t) = (S_t v)(1)$ *is continuous one-to-one and satisfies*

$$\Phi \circ S_t = T_t \circ \Phi \tag{4}$$

for $t \geqq 0$, *where* T *is the left shift semigroup on* $C[0, \infty)$ *defined by* $(T_t g)(s) = g(t + s)$ *for* $t, s \geqq 0$, $g \in C[0, \infty)$. *It can be extended to a map* $\tilde{\Phi}\colon C(0, 1] \to C[0, \infty)$ *which has a continuous inverse and the following property*: *There is a continuous strictly decreasing function* φ *from* $[0, \infty)$ *onto* $(0, 1]$ *such that* $\tilde{\Phi}(v_1)(t) > \tilde{\Phi}(v_2)(t)$ *if and only if* $v_1(\varphi(t)) > v_2(\varphi(t))$.

(iv) *The set* $W = \{v \in C[0, 1]\colon v(0) = 0,\ w_-(x) < v(x) < w_+(x)\}$ *is invariant under* S.

For the proofs cf. [3, Sections 1, 2] and [2, Section 3].

2. A NATURAL EXACT MEASURE ON $C[0, \infty)$ WITH BOUNDED SUPPORT

Proposition 2. *Let* $X(t)$ *be the Gaussian stationary process with continuous trajectories and triangular autocovariance function*

$$\operatorname{cov}(X(t), X(s)) = \max\{1 - |t - s|, 0\} \tag{5}$$

(cf. [2]). *Then for every* $t_0 \in [0, \infty)$ *and* $\tau \in [0, 1]$ *the process*

$$Y(\tau) = (X(t_0 + \tau) - X(t_0))/\sqrt{2} \tag{6}$$

coincides with the standard Wiener process on $C[0, 1]$.

Proof. The process $Y(\tau)$ is Gaussian, continuous and has the autocovariance function

(7) $$\operatorname{cov}(Y(\tau), Y(\varrho)) = \min(\tau, \varrho)$$

for $\tau, \varrho \in [0, 1]$.

Corollary 1. *Let m be the measure on $C[0, \infty)$ associated with the process X and let F be the distribution function of the onedimensional normalized normal distribution (usually denoted by Φ). Then, for any $t \in [0, \infty)$,*

(8) $$m(\{g \in C[0, \infty) | \sup \{g(t + \tau) - g(t) \mid \tau \in [0, 1]\} > a\} < 2[1 - F(a/\sqrt{2})] .$$

For the proof cf. [4, p. 227].

Corollary 2. *For any $t \in [0, \infty)$ and $0 < a \in \mathbb{R}$ we have*

(9) $$m\{g \in C[0, \infty) \mid \sup \{g(t + \tau) \mid \tau \in [0, 1]\} > a\} < 3[1 - F(a/\sqrt{6})] .$$

Proof. We have

(10) $$\begin{aligned} M_{t,a} = \{g \in C[0, \infty) \mid \sup g(t + \tau) \mid \tau \in [0, 1]\} > a\} \subset \\ \subset \{g \in C[0, \infty) \mid g(t) > a/\sqrt{6}\} \cup \\ \cup \{g \in C[0, \infty) \mid \sup \{g(t + \tau) - g(t)\} > a(1 - 6^{-1/2})\} . \end{aligned}$$

Hence

$$\begin{aligned} m(M_{t,a}) \leqq [1 - F(a/\sqrt{6}) + 2[1 - F(a(1 - 6^{-1/2})\, 2^{-1/2})] \leqq \\ \leqq 3[1 - F(a/\sqrt{6})] \end{aligned}$$

because of $(1 - 6^{-1/2})\, 2^{-1/2} \geqq 6^{-1/2}$.

Lemma 1. (i) *The inequality*

(11) $$(2\pi)^{-1/2} (a^{-1} - a^{-3})\, e^{-a^2/2} \leqq [1 - F(a)] \leqq a^{-1} e^{-a^2/2} (2\pi)^{-1/2}$$

holds for $a > 0$

(ii) *Let $a \geqq 3$. Then*

(12) $$1 - F(a) \geqq [1 - F(a/\sqrt{6})]^6 .$$

Proof. (i) The inequality (11) can be obtained simply by integrating by parts the formula

$$1 - F(a) = (2\pi)^{-1/2} \int_a^\infty e^{-x^2/2}\, dx .$$

(ii) We have

$$1 - F(a) \geqq (2\pi)^{-1/2}\, a^{-1}(1 - a^{-2})\, e^{-a^2/2} ,$$
$$(2\pi)^{-3}\, 6^3 a^{-6} (e^{-a^2/12})^6 \geqq [1 - F(a/\sqrt{6})]^6$$

because of

$$a^{-2} + 6^3 a^{-5} \leqq 1 \quad \text{for} \quad a \geqq 3\,.$$

Proposition 3. *For* $n \in Z^+ = \{0, 1, \dots\}$ *let* $a_n = F^{-1}(1 - [10^3(1 + n)]^{-7})$. *Then for*

$$(13) \qquad M = \{g \in C[0, \infty) \mid |g(t)| \leqq a_n \text{ for } n \leqq t < n + 1;\, n \in Z^+\}$$

the inequality

$$(14) \qquad m(M) > 0{\cdot}985$$

holds.

Proof. We have $a_n > 3$ for all $n \in Z^+$. Hence,

$$1 - F(a_n\sqrt{6}) \leqq [1 - F(a_n)]^{1/6} = 10^{-3\cdot5}(1 + n)^{-7/6}\,.$$

Moreover,

$$C[0, \infty) - M \subset \bigcup_{n=0}^{\infty} [M_{n,a_n} \cup (-M_{n,a_n})]$$

where $-B = \{-g \mid g \in B\}$ for $B \subset C[0, \infty)$. Therefore,

$$m(M) \geqq 1 - 6\,.\,10^{-3\cdot5} \sum_{n=0}^{\infty} (1 + n)^{-7/6} \geqq 1 - 14\,.\,10^{-3} > 0{\cdot}985\,.$$

Let D be the set of those $g \in C[0, \infty)$ for which $w_-(1) < g(t) < w_+(1)$ for all $0 \leqq t < \infty$. Define $H: D \to C[0, \infty)$ by

$$H(g) = h(g(t)) \quad \text{for} \quad g \in [0, \infty)\,,$$

where

$$h(\xi) = F^{-1}(\sqrt{2}\, d^{-1}(\xi - \eta)) \text{ for } \xi \geqq \eta,\; h(\xi) = -F^{-1}(2d^{-1}(\eta - \xi)) \text{ for } \xi \leqq \eta$$

and

$$d = \tfrac{1}{2}(w_+(1) - w_-(1))\,.$$

Then, $m_0 = m \circ H^{-1}$ is a probabilistic measure on the Borel subsets of D such that $m_0(\{g(t) \mid w_-(1) + d\,10^3(1 + n)^{-7} \leqq g(t) \leqq w_+(1) - d[10^3(1 + n)]^{-7}$ for $t \in [n\; n + 1]\}) > 0{\cdot}985$ and the marginal probabilities of cylinders satisfy

$$m_0(g \mid g(t) \in [a, b]) = d^{-1}(b - a)$$

for each $w_-(1) < a \leqq b < w_+(1)$, i.e. they are proportional to the standard Lebesgue measure on $\mathbb{R}$.

3. EXPONENTIAL ATTRACTION OF THE STABLE EQUILIBRIA OF S

In this section we prove

Proposition 4. $\Phi(W) \supset H^{-1}(C[0, \infty))$.

This proposition expresses the meaning of the results of Section 2 for S, since it allows to define the exact measure μ on W by $\mu = m_0 \circ \Phi$ with m_0 defined in Section 2. In virtue of Proposition 3, in order to prove Proposition 4 it suffices to prove

Proposition 5. *There exists a $\gamma > 0$ such that if*

$$w_-(1) + Ke^{-\gamma t} \leqq g(t) \leqq w_+(1) - Ke^{-\gamma t}$$

for some $K > 0$ then $g \in \Phi(W)$.

For the proof of Proposition 5 we need

Lemma 2. *Let $v(0) \in (0, w_+(1))$. Then for each $\alpha < -f_u(0, u_+)$ there exists a $\beta > 0$ such that*

$$\Phi(v)(t) \geqq w_+(1) - \beta e^{-\alpha t}.$$

A similar statement holds for $v(0) \in (w_-(1), 0)$.

Proof. Denote by $x = \psi(t, \xi)$ the characteristics of (1) passing through the point $t = 0, x = \xi$. That is, ψ satisfies $d\psi/dt = c(\psi)$, $\psi(0, \xi) = \xi$. Note that by Assumption A2, $\psi(t, 0) = 0$, $\psi(t, \xi)$ is strictly increasing for $\xi > 0$. Choose $\xi > 0$ and denote $u_1(t) = u(t, \psi(t, \xi))$, $u_2(t) = w(\psi(t, \xi))$, where u solves (1), (2). Both u_1 and u_2 satisfy the differential equation

$$du/dt = f(\psi(t, \xi), u),$$

so

$$\dot{u}_1(t) - \dot{u}_2(t) = \int_0^1 f_u(\psi(t, \xi), u_2(t) + \vartheta(u_1(t) - u_2(t)))\, d\vartheta (u_1(t) - u_2(t)). \tag{15}$$

There is a $d \in (0, \min\{1, u_+ - v(0)\}$ such that in $Q_d = \{(x, u) \mid 0 \leqq x \leqq d, w_+(x) - d \leqq u \leqq w_+(x)\}$ we have $f_u(x, u) \leqq -\alpha$. Consequently, by (15),

$$d/dt(u_1(t) - u_2(t)) \geqq -\alpha(u_1(t) - u_2(t))$$

whenever $(\psi(t, \xi), u_i(t)) \in Q_d$ for $i = 1, 2$. Thus, we have

$$u_1(t_2) - u_2(t_2) \geqq e^{-\alpha(t_2 - t_1)}(u_1(t_1) - u_2(t_1)) \tag{16}$$

if $(\psi(t, \xi), u_i(t)) \in Q_d$ for $t_1 \leqq t \leqq t_2$, $i = 1, 2$.

Consider the solution $y(t)$ of the equation

$$\dot{y} = f(0, y) \tag{17}$$

satisfying $y(0) = v(0)$. Since $v(0)$ is in the domain of attraction of the equilibrium u_+ of (17) there is a $t_1' > 0$ such that $y(t_1') = u_+ - d$. By the continuous dependence theorem for ordinary differential equations there is a $0 < d_1 \leqq d$ such that if $0 \leqq \xi \leqq d_1$ then for some $t_1(\xi) \leqq t_1' + 1$ we have $\psi(t_1(\xi), \xi) \leqq d$, $u_1(t) = d$.

Let t_2, t_3 be given by $\psi(t_2, \xi) = d$, $\psi(t_3, \xi) = 1$, respectively, and let $L = \sup\{f_u(x, u) \mid 0 \leqq x \leqq 1, \inf w_-(x) \leqq u \leqq \sup w_+(x)\}$. Note that $t_3 - t_2 = \tau$

is given by $\psi(\tau, d) = 1$ and, therefore, is independent of ξ. Thus, from (16) we have

$$\Phi(v)(t_3) - w_+(1) = u(t, 1) - w_+(1) = u_1(t_3) - u_2(t_3) \geqq \\ \geqq e^{L(t_3 - t_2)}(u_1(t_2) - u_2(t_2)) \geqq e^{L\tau} e^{-\alpha(t_2 - t_1' - 1)} d . \tag{18}$$

For $\xi \to 0$ we have $t_3 \to \infty$ (Proposition 1 (iii)), so (18) proves the lemma.

Remark. The proof of Lemma 2 can be used to prove the inequality

$$u(t, x) \geqq w(x) - \beta e^{-\alpha t}$$

for a solution u of (1), (2) with $v(0) > 0$ and $0 \leqq x \leqq 1$. It can be readily checked that β can be chosen independently of x. This refines slightly [3, Theorem 2] in that it gives an estimate of the rate of convergence of the solutions of (1), (2) to the stable equilibrium of the equation.

Proof of Proposition 5. Let φ be as in Proposition 1. Take $\alpha \in (\gamma, -f_u(0, u_+))$. By Lemma 2, for every $\varepsilon \in (0, w_+(1))$ there exists a $\beta > 0$ such that $\Phi(\varepsilon)(t) \geqq w_+(1) - \beta e^{-\alpha t}$ (ε is to be understood as the constant ε-valued function). Therefore, we have $g(t) < \Phi(\varepsilon)(t)$ for $t \geqq T$ sufficiently large. Consequently, $\tilde{\Phi}^{-1}(g)(x) \leqq \varepsilon$ for $x \in \varphi^{-1}(T)$. This proves

$$\limsup_{x \to 0} \tilde{\Phi}^{-1}(g)(x) \leqq 0 . \tag{19}$$

Similarly, from the analogue of Lemma 2 for the case $v(0) \in (u_-, 0)$ we obtain

$$\liminf_{x \to 0} \tilde{\Phi}^{-1}(x) \geqq 0 . \tag{20}$$

From (19), (20) we have $\lim_{x \to 0} \tilde{\Phi}^{-1}(g)(x) = 0$, so $\tilde{\Phi}^{-1}(g) \in W$.

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