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STATISTICAL PERIODICITY OF DETERMINISTIC SYSTEMS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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INTRODUCTION

Dynamical systems which exhibit a complicated "chaotic" behavior of trajectories quite often can be successfully studied as a special case of stochastic processes. The idea of this approach is the following. Consider a discrete time dynamical system $\{S^n\}$ where S is a transformation of a σ-finite measure space (X, Σ, μ) into itself. The trajectories of this system are the sequences

$$x, S(x), S^2(x) = S(S(x)), \ldots$$

for $x \in X$. We define an operator P from $L^1(X)$ into itself by setting

$$Pf = \frac{\mathrm{d}\mu_f}{\mathrm{d}\mu}, \quad f \in L^1(X) \tag{0.1}$$

where $\mathrm{d}\mu_f/\mathrm{d}\mu$ denotes the Radon-Nikodym derivative of the countably additive function

$$\mu_f(A) = \int_A f \,\mathrm{d}\mu, \quad A \in \Sigma. \tag{0.2}$$

The operator P has a simple probabilistic interpretation. Namely, if x is a random variable with the probability density function f, then the variable $S(x)$ has the probability density function Pf. Thus the sequence $\{P^n f\}$ describes the evolution of the density f in time and $\{P^n\}$ may be considered as a special case of Markov-Hopf process (see [2] and [3]).

The main advantage of studying P instead of the original transformation S is that P is always a bounded linear operator on $L^1(X)$. Thus in examining the behavior of $\{P^n\}$ as $n \to \infty$ it is possible to apply the powerful tools of linear functional analysis. However, this application is seldom straightforward. In order to understand this difficulty consider a classical example of the β-transformation $S_\beta(x) = \beta x (\mathrm{mod}\ 1)$

of the unit interval [0, 1] into itself ($\beta > 1$, real). In this case P has especially simple form, namely

$$P_\beta f(x) = \frac{1}{\beta}\sum_{k=0}^{m-1} f\left(\frac{x}{\beta} + \frac{k}{\beta}\right) + \frac{1}{\beta} f\left(\frac{x}{\beta} + \frac{m}{\beta}\right) 1_{[0,\beta-m]}(x)$$

where m is the largest integer smaller than β and $1_{[0,\beta-m]}$ denotes the characteristic function of the interval $[0, \beta - m]$. The operator P_β maps $L^1[0, 1]$ into itself but it is neither compact nor weakly compact nor even quasi-compact so that the recent general methods of the spectral theory of positive operators [11] are not applicable. Thus in studying the behavior of $\{P_\beta^n\}$ it was necessary to use the specific properties of the transformation S_β. (See [1, 8, 12].)

The same kind of problem appears in studying more complicated transformations. Thus it seems to be necessary to develop a special technique well adjusted to the Markov operators given by formulas (0.1) and (0.2). The purpose of the present paper is to show this is possible. We shall prove namely that the iterates of the operator P have the following interesting property: if for every f that is normalized ($\|f\|_{L^1} = 1$) and nonnegative, the sequence $\{P^n f\}$ converges to a weakly compact set, then there is a finite dimensional subspace of L^1 to which $\{P^n f\}$ converges for every $f \in L^1$ and $\{P^n f\}$ is asymptotically periodic. For our original system $\{S^n\}$ this means that, under some moderate assumption concerning P, the system is "statistically periodic".

The proof is based on two ideas. First we shall use the technique of orthogonal projections in L^2 due to V. A. Rochlin [10]. Then we shall follow the proof of an asymptotic decomposition theorem valid (with stronger assumptions) for all Markov operators [6].

The paper is organized as follows. In Section 1 we state our main result. In Section 2 we mention some possible applications and relationships to the results of G. Keller [5] and M. Misiurewicz [9]. Sections 3–6 are devoted to the proof.

1. ASYMPTOTIC PROPERTIES OF P

Let (X, Σ, μ) be a σ-finite measure space. A linear operator $P: L^1 \to L^1$ will be called a *Markov operator* if it satisfies the following two conditions

$$\text{(a)} \quad Pf \geqq 0 \quad \text{for} \quad f \geqq 0, \quad f \in L^1 ;$$

$$\text{(b)} \quad \|Pf\| = \|f\| \quad \text{for} \quad f \geqq 0, \quad f \in L^1$$

where $\|\cdot\|$ is an abbreviation for $\|\cdot\|_{L^1}$. From (a) and (b) it is easy to derive that P also satisfies

$$\text{(c)} \quad \|Pf\| \leqq \|f\| \quad \text{for} \quad f \in L^1 .$$

A measurable transformation $S: X \to X$ is called *nonsingular* if $\mu(S^{-1}(A)) = 0$

whenever $\mu(A) = 0$. Given a nonsingular transformation S we define the corresponding operator P by the condition

$$\int_A Pf \, \mathrm{d}\mu = \int_{S^{-1}(A)} f \, \mathrm{d}\mu \quad \text{for} \quad A \in \Sigma \,, \quad f \in L^1 \tag{1.1}$$

which is equivalent to (0.1), (0.2). Due to the nonsingularity of S the operator P is well defined; it is called the *Frobenius-Perron operator* corresponding to S. From condition (1.1) it follows that P satisfies conditions (a) and (b) and therefore represents a special kind of a Markov operator.

We denote by $D = D(X, \Sigma, \mu)$ the set of all normalized nonnegative elements of L^1, i.e.

$$D = \{f \in L^1 \colon f \geqq 0, \ \|f\| = 1\} \,.$$

The elements of D will be called *densities*. A Markov operator P is called strongly (weakly) *constrictive* if there exists a strongly (weakly) compact set $F \subset L^1$ such that

$$\lim_{n \to \infty} d(P^n f, F) = 0 \quad \text{for} \quad f \in D \,. \tag{1.2}$$

In condition (1.2), $d(g, F)$ denotes the distance from g to F, that is, the infimum of $\|g - h\|$ for $h \in F$.

The following theorem summarizes the main properties of weakly constrictive Frobenius-Perron operators.

Theorem 1.1. *Let (X, Σ, μ) be a σ-finite measure space, $S\colon X \to X$ a non-singular transformation and P the corresponding Frobenius-Perron operator. If P is weakly constrictive, then there exists a sequence of densities $g_1, \ldots, g_r \in L^1$ and a sequence of functionals $\lambda_1, \ldots, \lambda_r \in L^{1*}$ such that*

$$\lim_{n \to \infty} \|P^n(f - \sum_{i=1}^{r} \lambda_i(f)\, g_i)\| = 0 \quad \textit{for} \quad f \in L^1 \tag{1.3}$$

and

$$P g_i = g_{\alpha(i)} \tag{1.4}$$

where α is a permutation of integers $1, \ldots, r$.

Conditions (1.3) and (1.4) immediately imply that $P^n f$ may be written in the form

$$P^n f = \sum_{i=1}^{r} \lambda_i(f)\, g_{\alpha^n(i)} + R_n f \tag{1.5}$$

where α^n denotes the n-th iterate of the permutation α and the remainder $R_n f$ converges in norm to zero as $n \to \infty$. The summation term in (1.5) is periodic in n with a period which does not exceed $r!$. Thus from Theorem 1.1 we have immediately the following

Corollary 1.1. *If P is a weakly constrictive Frobenius-Perron operator, then for* *very $f \in L^1$ the sequence $\{P^n f\}$ is asymptotically periodic (i.e. $\{P^n f\}$ is the sum of a periodic sequence and a sequence which converges to zero).*

Furthermore, it follows from (1.5) that

$$d(P^n f, F) \leqq \|R_n f\| \quad \textit{for} \quad f \in D$$

where F is the set of all sums of the form

$$\sum_{i=1}^{r} C_i g_i$$

where the $|C_i|$ do not exceed the norms of λ_i. The set F is evidently compact (and is even finite dimensional), and Theorem 1.1 implies

Corollary 1.2. *If a Frobenius-Perron operator P is weakly constrictive, then P is also strongly constrictive.*

For strongly constrictive Markov operators a result analogous to Theorem 1.1 was recently proved [6]. Thus theoretically we may reverse our argument first proving Corollary 1.2 and then the decomposition result stated in Theorem 1.1. However, because of the lack of strong limits of $\{P^n f\}$ in the case when P is assumed to be weakly constrictive, this reverse argument cannot easily proceed.

2. COMMENTS AND APPLICATIONS

The structure of $P^n f$ described by Corollary 1.1 has been discovered in a few special cases. G. Keller considered piecewise expanding transformations on the unit interval [5] and on the square [4]. He pointed out that by the use of the classical spectral theorem of Ionescu-Tulcea and Marinescu it is possible to obtain a decomposition formula for $P^n f$ that is close to our formula (1.5). In both cases the main idea is to observe that P shrinks the variation of the functions, that is, there is a constant $K \geqq 0$ such that

$$\limsup_{n \to \infty} (\operatorname{Var} P^n f) < K \tag{2.1}$$

for every $f \in D$ of bounded variation. Thus $\{P^n f\}$ converges, for $f \in D$, to the strongly compact set

$$F = \{g \in D\colon \operatorname{Var} g \leqq K\}$$

and the decomposition formula (1.5) follows immediately from Theorem 1.1 as well as from the analogous theorem for strongly constrictive Markov operators proved in [6]. In order to apply the Ionescu-Tulcea-Marinescu theorem it is necessary to use a slightly more sophisticated inequality than (2.1), namely, for some m, K, and $C < 1$,

$$\operatorname{Var} P^m f \leqq C \operatorname{Var} f + K\|f\| \quad \text{for} \quad f \in L^1 . \tag{2.2}$$

Another case for which an asymptotic decomposition like (1.5) was found is a class of piecewise smooth transformations with negative Schwarzian derivative. This class is a natural generalization of the family of mappings

$$S_\gamma(x) = 4\gamma x(1 - x) \quad \text{where} \quad 0 \leqq \gamma \leqq 1$$

of the unit interval $[0, 1]$ into itself. M. Misiurewicz has shown [9] that there exists an uncountable set $\Gamma \subset [0, 1]$ such that for every $\gamma \in \Gamma$ the Frobenius-Perron operator P_γ corresponding to S_γ is (using our terminology) weakly constrictive. He also constructed the basis $g_1, \ldots, g_r$ of the limiting finite dimensional space.

Our Theorem 1.1 suggests that the asymptotic periodicity found by G. Keller and M. Misiurewicz is not an exceptional property of special families of transformations of the unit interval into itself but is rather a general rule.

3. CONSTRUCTION OF A LIMITING SET Q

In this and in the following two sections we will assume that the measure μ is finite and that $P1 = 1$. It follows from (1.1) that the last assumption is equivalent to the fact that the measure μ is invariant, i.e.

$$\mu(S^{-1}(A)) = \mu(A) \quad \text{for} \quad A \in \Sigma .$$

Since μ is finite, every square integrable function on X is also integrable and thus Pf is well defined for every $f \in L^2(X)$. Now introduce the Koopmann operator U on L^2 by setting

$$Uf(x) = f(S(x)) \quad \text{for} \quad x \in X \quad \text{and} \quad f \in L^2 .$$

It is well known that U is an isometry on L^2. In fact the following relationships hold:

$$\langle Uf, Ug \rangle = \langle f, g \rangle \equiv \int_X fg \, \mathrm{d}\mu \quad \text{for} \quad f, g \in L^2 , \tag{3.2}$$

$$\langle f, Ug \rangle = \langle Pf, g \rangle \quad \text{for} \quad f, g \in L^2 \tag{3.3}$$

and in particular

$$\|Pf\|_{L^2} \leqq \|f\|_{L^2} \quad \text{and} \quad \|Uf\|_{L^2} = \|f\|_{L^2} \quad \text{for} \quad f \in L^2 . \tag{3.4}$$

These properties imply

Lemma 3.1. *For every integer n the operator P^nU^n is the identity on L^2 and $\pi_n = U^nP^n$ is the orthogonal projection of L^2 onto $Q_n = U^n(L^2)$.*

Since the subspaces Q_n form a decreasing sequence, Lemma 3.1 it in turn yields

Lemma 3.2. *For every $f \in L^2$ the limit*

$$\pi_\infty f = \lim_{n\to\infty} U^n P^n f$$

exists and π_∞ *is the orthogonal projection of* L^2 *onto*

$$Q = \bigcap_{n=0}^{\infty} Q_n = \bigcap_{n=0}^{\infty} U^n(L^2).$$

The most important property of Q *is that in studying the asymptotic properties of* $\{P^n f\}$ *we may restrict ourselves to* $f \in Q$.

Proposition 3.1. *For every* $f \in L^2$,

$$\lim_{n\to\infty} \|P^n f - P^n(\pi_\infty f)\|_{L^2} = 0.$$

Proof. Write $f_n = \pi_n f$ and $f_\infty = \pi_\infty f$. We have

$$P^n f_n = P^n U^n P^n f = P^n f$$

and consequently

$$\|P^n f - P^n f_\infty\|_{L^2} = \|P^n f_n - P^n f_\infty\|_{L^2} \leqq \|f_n - f_\infty\|_{L^2}.$$

The last term converges to zero by Lemma 3.2.

To conclude this section observe that Q is invariant under P. In fact since PU is the identity we have

$$P(Q_n) = PU^n(L^2) = U^{n-1}(L^2) = Q_{n-1}$$

and finally, since $\{Q_n\}$ is a decreasing sequence,

$$P(Q) \subset \bigcap_{n=0}^{\infty} P(Q_n) = \bigcap_{n=1}^{\infty} P(Q_n) = \bigcap_{n=1}^{\infty} Q_{n-1} = Q.$$

4. PROPERTIES OF THE SET Q

The purpose of this section is to prove that Q is a finite dimensional space. The idea of the proof lies in using some special properties of functions from Q listed in Lemmas 4.1 – 4.7.

Lemma 4.1. *If* f *and* g *belong to* Q *and* g *is bounded, then the product* fg *also belongs to* Q.

Proof. For every fixed integer n the functions f and g are elements of Q_n and can be written in the form

$$f = h \circ S^n, \quad g = k \circ S^n$$

where $h, k \in L^2$ and k is bounded. Thus $fg = (hk) \circ S^n$ with $(hk) \in L^2$ which shows that $fg \in Q_n$. Since n was arbitrary this completes the proof.

Let 1_A denote the characteristic function of the set A.

Lemma 4.2. *If $f \in Q$ then $1_{f^{-1}(\Delta)} \in Q$ for every Borel set $\Delta \subset \mathbb{R}$.*

Proof. For every fixed integer n we may write f in the form $f = h \circ S^n$. Thus

$$1_{f^{-1}(\Delta)} = 1_{h^{-1}(\Delta)} \circ S^n$$

which shows that $1_{f^{-1}(\Delta)} \in Q_n$. Since n was arbitrary this completes the proof.

Given a function $f: X \to \mathbb{R}$ and a set $\Delta \subset \mathbb{R}$ we define $f_\Delta(x) = f(x)$ when $f(x) \in \Delta$ and $f_\Delta(x) = 0$ otherwise. Notice $f_\Delta(x) = f(x)\, 1_{f^{-1}(\Delta)}(x)$.

Lemmas 4.1 and 4.2 immediately imply

Lemma 4.3. *If $f \in Q$ then $f_\Delta \in Q$ for every Borel set $\Delta \subset \mathbb{R}$.*

Lemma 4.4. *Let f be an element in Q, let $\Delta \subset \mathbb{R}$ be a Borel set, and let n be an integer. Then for (almost) every $x \in X$ either $P^n f_\Delta(x) \in \Delta$ or $P^n f_\Delta(x) = 0$.*

Proof. Since $f_\Delta \in Q$ we have $U^n P^n f_\Delta = f_\Delta$ or

$$(P^n f_\Delta) \circ S^n = f_\Delta \,.$$

Thus for (almost) every $x \in X$ either

$$(P^n f_\Delta)(S^n(x)) \in \Delta \quad \text{or} \quad (P^n f_\Delta)(S^n(x)) = 0 \,.$$

Since S^n is measure preserving and μ is finite, almost every element in X is of the form $S^n(x)$ for some $x \in X$. The proof is completed.

Now denote by C the set of all characteristic functions which belong to Q. It is easy to see that the supports of the elements of C form an algebra. In fact, since $U^n P^n 1 = 1$ we have $\pi_\infty 1 = 1$ and $1_X \in C$. Furthermore, from Lemma 4.1 it follows that for every pair $1_A, 1_B \in C$ the product $1_A \cdot 1_B = 1_{A \cap B}$ belongs to C. Finally from the linearity of π_∞ it follows that $1_X - 1_A = 1_{X \setminus A}$ belongs to C whenever $1_A \in C$.

The following two lemmas show some more sophisticated properties of C.

Lemma 4.5. *If $1_A \in C$ then $P^n 1_A \in C$ for every integer n.*

Proof. Since $P^n Q \subset Q$ it is enough to prove that $P^n 1_A$ is a characteristic function. This follows immediately from Lemma 4.4 if we choose $\Delta = \{1\}$.

Lemma 4.6. *If $1_{A_1}, 1_{A_2} \in C$ and sets A_1 and A_2 are essentially disjoint, then the supports of $P^n 1_{A_1}$ and $P^n 1_{A_2}$ are also disjoint for each $n = 1, 2, \ldots$.*

Proof. Fix n and write $1_{B_i} = P^n 1_{A_i}$. Since $1_{A_1} + 1_{A_2} \leqq 1$, we have

$$1_{B_1} + 1_{B_2} = P^n(1_{A_1} + 1_{A_2}) \leqq P1 = 1$$

which implies that B_1 and B_2 are (essentially) disjoint.

Observe that so far we have not used our main assumption that P is weakly constrictive. Using this fact we prove the following

Lemma 4.7. *The set C is finite.*

Proof. Assume that $1_{A_1}, \dots, 1_{A_r}$ are different from zero elements of C and have mutually disjoint supports. Consider the densities

$$f_i = \frac{1}{\mu(A_i)} 1_{A_i} .$$

Since P is weakly constrictive the sequence $\{P^n f_i\}$ converges as $n \to \infty$ to a weakly compact set F. Choose an $\varepsilon \in (0, 1)$. Since F is weakly compact there is a $\delta > 0$ such that

$$\int_B |g| \, \mathrm{d}\mu \leqq 1 - \varepsilon \tag{4.1}$$

for every $g \in F$ and for every set B satisfying $\mu(B) \leqq \delta$. Furthermore, since $\{P^n f_i\}$ converge to F there exists an integer m and functions $g_1, \dots, g_r \in F$ such that

$$\|P^m f_i - g_i\|_{L^1} < \varepsilon . \tag{4.2}$$

We claim that

$$\mu(\operatorname{supp} P^m f_i) > \delta \quad \text{for} \quad i = 1, \dots, r . \tag{4.3}$$

If not, then setting $B_i = \operatorname{supp} P^m f_i$ and using (4.1) we obtain (for some integer i)

$$\|P^m f_i - g_i\|_{L^1} \geqq \int_{B_i} |P^m f_i - g_i| \, \mathrm{d}\mu \geqq \int_{B_i} P^m f_i \, \mathrm{d}\mu - \int_{B_i} g_i \, \mathrm{d}\mu \geqq 1 - (1 - \varepsilon) = \varepsilon$$

which contradicts (4.2). Thus (4.3) is proved. By Lemma 4.6 the sets B_i are mutually disjoint and the number r must be smaller than $\mu(X)/\delta$. Thus the largest number of elements of C with mutually disjoint supports is bounded. Since the supports of elements of C form an algebra, this completes the proof.

From now we shall denote the minimal elements of C by

$$1_{A_1}, \dots, 1_{A_r} . \tag{4.4}$$

By Lemma 4.1 their supports are disjoint. From Lemmas 4.5 and 4.6 it follows that

$$P1_{A_1}, \dots, P1_{A_r} \tag{4.5}$$

are also elements of C with disjoint supports. Thus the sequence (4.5) must be simply a permutation of (4.4) and we have

$$P1_{A_i} = 1_{A_{\alpha(i)}} . \tag{4.6}$$

Proposition 4.1. *The space Q is r-dimensional and (4.4) is a basis for Q.*

Proof. Choose $f \in Q$. From Lemma 4.2 it follows that for every real t the function $1_{f^{-1}(t_i, \infty)}$ is an element of C. Since C is finite, there is at most a finite number of values of t (say $t_1, \dots, t_s$) which give distinct functions $1_{f^{-1}(t_i, \infty)}$. Thus f is constant on every set

$$T_i = f^{-1}(t_i, \infty) \setminus f^{-1}(t_{i+1}, \infty)$$

and so we have $f = \Sigma \lambda_i \mathit{1}_{T_i}$, for appropriate λ_i. Since $\mathit{1}_{T_i}$ belong to C, this completes the proof.

5. ASYMPTOTIC DECOMPOSITION OF P WHEN $P\mathit{1} = \mathit{1}$

From now on, our proof will be similar to the arguments in [6], so we abbreviate the details where possible.

We start from a lemma which simplifies the derivation of formula (1.5).

Lemma 5.1. *Let $P: L^1 \to L^1$ be a Markov operator and let $g_1, \ldots, g_r$ be a sequence of linearly independent elements of L^1 such that $Pg_i = g_{\alpha(i)}$ where α is a permutation of integers $1, \ldots, r$. Assume that for every $f \in L^1$ and every $\varepsilon > 0$ there exists an integer n and a sequence of constants $c_1, \ldots, c_r$ such that*

$$\left\| P^n f - \sum_{i=1}^{r} c_i g_i \right\| \leqq \varepsilon \,. \tag{5.1}$$

Then there exists a sequence of functionals $\lambda_1, \ldots, \lambda_r \in L^{1*}$ such that

$$\lim_{n \to \infty} \left\| P^n \left(f - \sum_{i=1}^{r} \lambda_i(f)\, g_i \right) \right\| = 0 \quad \text{for} \quad f \in L^1 \,. \tag{5.2}$$

The proof of Lemma 5.1 is straightforward. For a given sequence $\varepsilon_n \to 0$ we choose the corresponding sequences c_i^n and define $\lambda_i(f)$ as the limit points of the sequences $c^n_{\alpha^{-n}(i)}$. The details of this construction may be found in [6].

Now we apply Lemma 5.1 to the functions

$$f_i = \frac{1}{\mu(A_i)} \mathit{1}_{A_i} \,. \tag{5.3}$$

Integrating (4.6) and having in mind that P preserves the integral of nonnegative functions we obtain $\mu(A_i) = \mu(A_{\alpha(i)})$. From this and (4.6) it follows immediately that $Pf_i = f_{\alpha(i)}$. Now let f be an arbitrary element of L^1 and let $\varepsilon > 0$. There exists $\bar{f} \in L^2$ such that $\|f - \bar{f}\| \leqq (1/2)\,\varepsilon$ and consequently

$$\|P^n f - P^n \bar{f}\| \leqq \varepsilon/2 \quad \text{for} \quad n = 0, 1, \ldots \,. \tag{5.4}$$

Furthermore, from Proposition 3.1 it follows that $P^n \bar{f} - P^n(\pi_\infty \bar{f})$ converges to zero in the L^2-norm and also in the L^1-norm since $\mu(X) < \infty$. Thus

$$\|P^n \bar{f} - P^n(\pi_\infty \bar{f})\| < \varepsilon/2 \tag{5.5}$$

for sufficiently large n, say $n \geqq m$. Since $P^m(\pi_\infty \bar{f})$ belongs to Q it may be written in the form $\sum_{i=1}^{r} c_i f_i$. From (5.4) and (5.5) it follows that

$$\left\|P^{m}f - \sum_{i=1}^{r} c_i f_i\right\| < \varepsilon\,,$$

which by virtue of Lemma 5.1 proves

Proposition 5.1. *There exists functionals $\lambda_1, \ldots, \lambda_r \in L^{1*}$ such that $P^n f$ may be written in the form*

$$P^n f = \sum_{i=1}^{r} \lambda_i(f) f_{\alpha^n(i)} + R_n f \quad \textit{for} \quad f \in L^1\,, \quad n = 0, 1, 2, \ldots$$

where f_i are given by (5.3) and the remainder R_n satisfies $\|R_n f\| \to 0$.

6. ASYMPTOTIC DECOMPOSITION OF P: GENERAL CASE

Thus far we have assumed that $\mu(X) < \infty$ and $P1 = 1$. In this section we shall show how these assumptions may be dropped.

Let P be a Frobenius-Perron operator corresponding to a nonsingular transformation $S: X \to X$ acting on a σ-finite measure space (X, Σ, μ). We assume that P is weakly constrictive. Since X is σ-finite there exists a density $f_0 \in L^1(X, \Sigma, \mu)$ such that $f_0 > 0$ on X. Furthermore, since P is weakly constrictive, the sequence $\{P^n f_0\}$ is weakly precompact and by the mean ergodic theorem the following strong limit exists:

$$g = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k f_0\,. \tag{6.1}$$

The limiting function satisfies $Pg = g$ and $\|g\| = 1$ and consequently the measure

$$\hat{\mu}(A) = \int_A g \,\mathrm{d}\mu\,, \quad A \in \Sigma$$

is normalized and invariant under S. Now consider S in the space $(X, \Sigma, \hat{\mu})$. The Frobenius-Perron operator $\hat{P}$ corresponding to S in this new space is given by

$$\int_A (\hat{P}h)\, g \,\mathrm{d}\mu = \int_{S^{-1}(A)} hg \,\mathrm{d}\mu \quad \text{for} \quad h \in L^1(X, \Sigma, \hat{\mu})\,, \quad A \in \Sigma\,.$$

Therefore $\hat{P}h = P(gh)/g$ and by induction

$$\hat{P}^n h = \frac{1}{g} P^n(hg) \quad \text{for} \quad h \in L^1(X, \Sigma, \hat{\mu})\,. \tag{6.2}$$

Let $\hat{L}$ denote the space $L^1(X, \Sigma, \hat{\mu})$ and let $|||\cdot|||$ be the norm in $\hat{L}$. We have

$$|||h||| = \int_X |h|\, g \,\mathrm{d}\mu = \|gh\| \quad \text{for} \quad h \in \hat{L}\,. \tag{6.3}$$

By construction $\hat{\mu}(X) = 1$ and $\hat{P}1 = 1$. We also have

Lemma 6.1. *The operator $\hat{P}$ is weakly constrictive.*

Proof. Since P is weakly constrictive, there is a weakly compact set F such that

$$\lim_{n\to\infty} d(P^n f, F) = 0 \quad \text{for} \quad \|f\| = 1\,, \quad f \geqq 0\,.$$

Define $\hat{F} = \{f/g : f \in F\}$. The functions f/g are not defined outside of the set

$$G = \{x : g(x) > 0\}$$

but they are well defined as elements of $\hat{L}$ since $\hat{\mu}(X \setminus G) = 0$. Moreover, the mapping $f \to f/g$ from $L^1(X, \Sigma, \mu)$ into $\hat{L}$ is bounded (and even contractive) since

$$|||f/g||| = \|(f/g)\, g\| = \|f 1_G\| \leqq \|f\|\,.$$

Thus $\hat{F}$ as the image of a weakly compact set by a linear bounded transformation is also weakly compact. Now let h be a density in $\hat{L}$. Then hg is a density in $L^1(X, \Sigma, \mu)$ by (6.3) and $P^n(hg)$ converges to F. Thus there exists a sequence $k_n \in F$ such that

$$\|P^n(hg) - k_n\| \to 0$$

and consequently

$$|||\hat{P}^n h - k_n/g||| \leqq \|P^n(gh) - k_n\| \to 0$$

which completes the proof.

From Lemma 6.1 and Proposition 5.1, $\hat{P}^n h$ may be written in the form

$$\hat{P}^n h = \sum_{i=1}^{r} \lambda_i(h)\, \hat{f}_{\alpha^n(i)} + \hat{R}_n h \quad \text{for} \quad h \in \hat{L}, \tag{6.4}$$

where $\hat{f}_i$ are densities in $\hat{L}$ with disjoint supports and $|||\hat{R}_n h||| \to 0$ as $n \to \infty$. Now multiplying (6.4) by g and using (6.2) we obtain

$$P^n(hg) = \sum_{i=1}^{r} \lambda_i(h)\, g_{\alpha^n(i)} + R_n h \quad \text{for} \quad h \in \hat{L} \tag{6.5}$$

where $g_i = g\hat{f}_i$ and $R_n h = g\hat{R}_n h$.

Notice that g_i are densities in $L^1(X, \Sigma, \mu)$ and $Pg_i = g_{\alpha(i)}$. At first glance it seems that we have reached our goal since hg are elements of $L^1(X, \Sigma, \mu)$. Unfortunately, when $G \neq X$ the family of functions of the form $f = hg$ with $h \in \hat{L}$ is not even a dense subset of $L^1(X, \Sigma, \mu)$. In order to overcome this difficulty we shall use the following lemma concerning the density f_0 (see formula (6.1)), which follows from the additive properties of the L^1-norm. (See also [6].)

Lemma 6.2. *For every $\varepsilon > 0$ there is an integer m such that*

$$\int_{X\setminus G} P^m f_0 \,\mathrm{d}\mu < \varepsilon\,. \tag{6.6}$$

We now complete the proof of Theorem 1.1. Fix an $f \in L^1(X, \Sigma, \mu)$ and an $\varepsilon > 0$.

Since $f_0 > 0$ in X, there is a constant $c \geqq 0$ such that

$$|f| \leqq cf_0 + q$$

where $\|q\| \leqq \varepsilon/4$. Furthermore by Lemma 6.2 there is an integer m such that

$$\int_{X\setminus G} P^m f_0 \leqq \varepsilon/4c\,.$$

Finally, since $g > 0$ in G there exists a constant $c_1 \geqq 0$ such that

$$(cP^m f_0)\, 1_G \leqq c_1 g + q_1$$

where $\|q_1\| \leqq \varepsilon/4$. Taking all this into account we have

$$|P^m f| \leqq P^m|f| \leqq cP^m f_0 + P^m q \leqq$$
$$\leqq c_1 g + q_1 + (cP^m f_0)\, 1_{X\setminus G} + P^m q \leqq c_1 g + q_3$$

where $\|q_3\| \leqq 3\varepsilon/4$. This allows us to write $P^m f$ in the form

$$P^m f = hg + q_4$$

where $|h| \leqq c_1$ and $\|q_4\| \leqq 3\varepsilon/4$. Applying formula (6.5) we obtain

$$P^{m+n} f = \sum_{i=1}^{r} \hat{\lambda}_i(h)\, g_{\alpha^n(i)} + R_n h + P^n q_4\,.$$

For sufficiently large n we have $\|R_n h\| \leqq \varepsilon/4$ and therefore

$$\Big|P^{m+n} f - \sum_{i=1}^{r} \hat{\lambda}_i(h) g_{\alpha^n(i)}\Big| \leqq \varepsilon\,.$$

An application of Lemma 5.1 completes the proof.

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