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A NOTE ON PERVASIVE ALGEBRAS

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By a function algebra (on a compact Hausdorff space X) we mean a closed subalgebra, separating the points of X, of the sup-norm algebra $C(X)$ of all continuous complex-valued functions on X.

A function algebra A is said to be *pervasive* provided it satisfies the following condition:

Whenever F is a nonvoid proper closed subset of X, then A/F, the algebra of all restrictions of the functions in A to the set F, is dense in $C(F)$ (naturally with respect to $|\cdot|_F$, the sup-norm on F).

The notion "pervasiveness" is due to Hoffman and Singer [1] who also were the first to investigate the properties of such algebras.

$C(X)$ is of course a pervasive algebra. More interesting are its proper pervasive subalgebras; the simplest of them is the classical disc algebra, the set of all uniform limits of polynomials on the unit circle in the z-plane, and related algebras.

Pervasiveness is a rather strong property, and it is interesting to seek for a non-trivial additional property which guarantees the pervasive algebra to be equal to the whole $C(X)$. In this sense we have investigated pervasive algebras in [2]. Our aim here is to strengthen the following Theorem A proved therein:

Theorem A. *Let A be a function algebra on X. Suppose that for any closed nonvoid proper subset F of X and for any function f in $C(F)$ there exists a positive constant $k(F, f)$ with the following property:*

Whenever e is a positive number, then there exists a g in A satisfying

$$|f - g|_F \leqq e, \quad |g| \leqq k(F, f).$$

Then A is equal to $C(X)$.

Remark that the assumption of Theorem A comprises the pervasiveness of the algebra A.

In this note we shall require the pervasiveness of A, and the bounded approximation by functions in A solely of a single function on a certain set, and come to the same conclusion. More specifically, we shall prove the following.

Theorem B. *Let A be a pervasive algebra on X. Let F and H be a disjoint couple of closed subsets of X which both have nonvoid interiors. Suppose that there is a constant c with the following property:*

Whenever e is positive, then there is an f in A satisfying

$$(1) \qquad |f|_F < e\,, \quad |f - 1|_H < e\,, \quad |f| \leqq c\,.$$

Then A is equal to $C(X)$.

Proof. Fix an arbitrary g in $C(X)$ and e positive. To prove Theorem B, it suffices to find an h in A satisfying

$$(2) \qquad |g - h| < e\,.$$

It is obvious that $\overline{X - F}$ and $\overline{X - H}$ (where the bar denotes the closure in X) are closed nonvoid proper subsets of X. A being pervasive contains a couple j, k of functions satisfying

$$(3) \qquad |g - j|_{\overline{X-F}} < \frac{e}{4c}\,, \quad |g - k|_{\overline{X-H}} < \frac{e}{4c}\,,$$

where c is the constant from (1). Remark that c is not less than 1.

Without loss of generality we may assume that j and k are not both identically zero (in the opposite case the function $h = 0$ satisfies (2)) and put

$$(4) \qquad \check{e} = \frac{e}{2(|j| + |k|)}\,.$$

Take, with regard to (1), an f in A for which

$$(5) \qquad |f|_F < \check{e}\,, \quad |f - 1|_H < \check{e}\,, \quad |f| \leqq c\,.$$

The function

$$h = fj + (1 - f)\,k\,,$$

satisfies (2). In fact, it is undeniable that

$$F \subset \overline{X - H}\,, \quad H \subset \overline{X - F}\,, \quad X = F \cup H \cup (\overline{X - F} \cap \overline{X - H})\,,$$

and, by (3), (4) and (5)

$$\begin{aligned}
|g - h|_F &= |g - fj - (1 - f)\,k|_F \leqq \\
&\leqq |g - k|_F + |f|_F(|j|_F + |k|_F) \leqq \\
&\leqq |g - k|_{\overline{X-H}} + |f|_F(|j| + |k|) < \frac{e}{4c} + \frac{e}{2} < e\,, \\
|g - h|_H &= |g - j + j - fj - (1 - f)\,k|_H \leqq \\
&\leqq |g - j|_{\overline{X-F}} + |1 - f|_H\,(|j| + |k|) < \frac{e}{4c} + \frac{e}{2} < e,
\end{aligned}$$

and finally

$$|g - h|_{\overline{X-F} \cap \overline{X-H}} = |g - fk - (1 - f)\,k + fk - fj|_{\overline{X-F} \cap \overline{X-H}} \leqq$$

$$\leqq |g - k|_{\overline{X-H}} + |f| \cdot |j - k|_{\overline{X-F} \cap \overline{X-H}} <$$

$$< \frac{e}{4c} + c(|g - j|_{\overline{X-F}} + |g - k|_{\overline{X-H}}) < \frac{e}{4c} + \frac{e}{2} < e .$$

Theorem B is proved.

Remark. Evidently, the condition of pervasiveness for A may be omitted; it suffices to require an approximation of any continuous function on $X - F$ and $X - H$ by functions in A, and a norm-bounded approximation of the function which is equal to 0 on F and to 1 on H on the set $F \cup H$ by functions in A.

Problem. So far we have proved the following:

Whenever A is a proper pervasive algebra (i.e., a pervasive algebra which is a proper subalgebra of $C(X)$) and F, H are arbitrary disjoint closed proper *fat* (i.e., with interior points) subsets of X, then any approximation of the function 0 on F and 1 on H is unbounded in the norm of A.

Now we ask the following question: *Is, in general, the assumption of F and H being fat necessary?*

For the classical disc algebra mentioned above this is not the case:

It is well-known that the disc algebra A is pervasive; it follows, for instance, from the famous Wermer's Maximality Theorem [3]; also it is well-known that any nontrivial analytic measure on C (i.e., a measure m on the unit circle C which annihilates A in the sense that $\int f \, dm = 0$ for any f in A) and the Lebesgue measure on C are mutually absolutely continuous — this is the classical F. and M. Riesz Theorem.

Let F and H be closed disjoint subsets of C having positive Lebesgue measures. Let $\{f_n\}_n$ be a sequence of functions in A which approximates 0 on F and 1 on H. Then $\{f_n\}_n$ is unbounded.

Admit the boundedness of $\{f_n\}$ and fix an arbitrary nontrivial analytic measure m. Then the sequence $\{f_n m\}_n$ is a norm-bounded sequence of analytic measures and has, in the weak-star topology, a limit point, say p. It is evident that

$$p/F = 0 , \quad p/H = m/H ,$$

where y/Y denotes the restriction of the measure y to the set Y. However, p is analytic and F has a positive measure, hence p has to be trivial, and at the same time $p/H \neq 0$, which is a contradiction.

References

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[3] *J. Wermer:* On algebras of continuous functions, Proc. Amer. Math. Soc. 4 (1953), 866—869.

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