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A GENERALIZATION OF TICHONOV THEOREM

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1. INTRODUCTION

In a great number of biological, ecological and sociological systems the variables considered change with very different rates. Modelling these systems by ODE's leads, in the simplest case, to the following system

$$\frac{dx}{dt} = f(x, y), \tag{1.1}$$

$$\varepsilon \frac{dy}{dt} = g(x, y),$$

$\varepsilon \ll 1$ being a small parameter and $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Such systems were first studied by Tichonov [5], [6]. He supposed that a C^1-solution $y = \varphi(x)$ of the algebraic equation $g(x, y) = 0$ is such that for each $x \in \Omega \subset \mathbb{R}^m$, $\varphi(x)$ represents the one point attractor* of the equation

$$\frac{dy}{dt} = g(x, y), \tag{1.2}$$

where x is considered as a parameter. Then, provided some technical assumptions are fulfilled, the solutions of the system (1.1) converge, for $\varepsilon \to 0$, to the solutions of the system

$$\frac{dx}{dt} = f(x, y)$$

$$g(x, y) = 0 .$$

If the solution $y = \varphi(x)$ of the equation $g(x, y) = 0$ is not a one-point attractor for some x, Tichonov Theorem can not be used. This may be the case when solutions of the equation (1.2) are attracted by more complicated attractors. The simplest

*) In other words, the solution $y = \varphi(x)$ defined on the domain $\Omega \subset \mathbb{R}^m$ of the equation $g(x, y) = 0$ is such that for each x the point $y = \varphi(x)$ is an asymptotically stable point of the equation $dy/dt = g(x, y)$.

model of such a situation is the system with three independent variables x, y_1, y_2, x being the slow variable and y_1, y_2 the fast oscillating variables, where the frequency of oscillations depends on ε. The smaller ε, the higher the frequency of oscillations. In this highly nonlinear situation, it is difficult to calculate the time dependence of x, even by methods of numerical integration.

For this and even for more general cases a theorem will be given which for the slow and fast components of solutions gives an arbitrarily close approximation provided $\varepsilon > 0$ is small enough. This makes it possible to solve numerically the system (1.1) with a complicated attractor for the fast variable in a very efficient way.

The paper is organised as follows. Notation and some definitions are recalled in Section 2. In the concluding part of this Section the Main Theorem of this paper generalizing the Tichonov Theorem is stated. In Section 3 preliminary results necessary for proving Main Theorem are developed. The proof of Main Theorem is given in the last Section.

II. NOTATIONS AND DEFINITIONS

If $X \times Y$ is a Cartesian product of sets X, Y, then the corresponding projections will be denoted by $\pi_1: X \times Y \to X$, $\pi_2: X \times Y \to Y$. Supposing $S \subset X \times Y$, $x \in X$ and $y \in Y$, we will denote $S_x = \{y \in Y \mid (x, y) \in S\}$, $S^y = \{x \in X \mid (x, y) \in S\}$. The Lebesgue measure will be denoted by m throughout this paper and with respect to the Lebesgue measure we will use the term "almost all" in the generally accepted sense. We are not going to mention explicitly the dimension of the Euclidean space on which the Lebesgue measure is considered.

Let M be a compact manifold with a Riemannian metric and let $\Phi: R \times M \to M$ be a flow on M. The flow Φ is said to be *topologically transitive* on a closed invariant set Λ if there exists a dense trajectory inside Λ.

A fixed point $x \in M$ of the flow Φ (i.e. $\Phi(t, x) = x$ for every $t \in \mathbb{R}$) is called a *hyperbolic fixed point* if the tangent bundle at $x \in M$ can be represented as the Whitney sum of two $T\Phi(t, -)$-invariant subbundles – $T_x M = E_x^s + E_x^u$, and if there exist constants $c > 0$, $\lambda > 0$ such that

a) if $v \in E_x^s$, then $\|T\Phi(t, -)(v)\| \leqq ce^{-\lambda t}\|v\|$ for all $t > 0$,

b) if $v \in E_x^u$, then $\|T\Phi(t, -)(v)\| \leqq ce^{\lambda t}\|v\|$ for all $t < 0$.

Similarly, let Λ be an invariant set for the flow Φ containing no fixed points and such that

i) the restriction of the tangent bundle to Λ can be represented as the continuous Whitney sum of three $T\Phi(t, -)$-invariant subbundles

$$T_\Lambda M = E + E^s + E^u,$$

ii) dim $E_x = 1$ and E_x is the tangent space to the trajectory through x,

iii) there are constants $c, \lambda > 0$ such that

a) if $v \in E_x^s$, then $\|T\Phi(t, -)(v)\| \leqq ce^{-\lambda t}\|v\|$ for all $t > 0$,

b) if $v \in E_x^u$, then $\|T\Phi(t, -)(v)\| \leqq ce^{\lambda t}\|v\|$ for all $t < 0$.

Then we say that Λ is a *hyperbolic set.*

A subset Λ of M is an *attractor* of Φ if

i) Φ is topologically transitive on Λ,

ii) there exists a closed neighborhood U of Λ such that

$$\Phi(t, U) \subset U \quad \text{for} \quad t > 0 \quad \text{and} \quad \bigcap_{t>0} \Phi(t, U) = \Lambda .$$

We shall say that *the attractor Λ of Φ on M satisfies Axiom A* if Λ is either a hyperbolic point or a hyperbolic set which is densely filled with periodic orbits. Let us define the *basin of attraction* of Λ as the set of points whose ω-limits are contained in Λ. (The ω-limit of x means $\bigcap_t \mathrm{cl}(\bigcup_{s>t} \Phi(s, x))$.)

Let ν be a probability measure on M. Then we define the time average of ν as the measure

$$\mu = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(t, \nu)\, \mathrm{d}t^{*)} ;$$

we shall call it *the Bowen-Ruelle measure.*

The following theorem of Bowen and Ruelle and its Corollary form a basis for the proof of Main Theorem.

Theorem (*Bowen, Ruelle* [1]). *If Λ is an Axiom* A *C^2-attractor with a basin B, then for any continuous probability measure ν with support in B the Bowen-Ruelle measure μ exists, has support Λ, and is invariant, ergodic, and independent of ν.*

Corollary. *Suppose that $f\colon B \to R$ is a continuous function and define the time average $\bar{f}$ of f by*

$$\bar{f}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\Phi(t,x))\, \mathrm{d}t .$$

Then there exists $B^0 \subset B$ such that $m(B - B^0) = 0$ and for $x \in B^0$,

$$\bar{f}(x) = \int_\Lambda f\, \mathrm{d}\mu .$$

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*) This expression means the vague limit of measures

$$\mu_T = \frac{1}{T}\int_0^T \Phi(t, \nu)\, \mathrm{d}t, \quad \mu_T(B) = \frac{1}{T}\int_0^T \nu(\Phi(-t, B))\, \mathrm{d}t$$

for any ν-measurable set B.

For more information on dynamical systems see Smale [4], Bowen, Ruelle [1] and references quoted there.

Now, let us formulate the main result generalizing Tichonov Theorem

Main Theorem. *Assume that* $U \subset \mathbb{R}^m \times \mathbb{R}^n$ *is an open set,* $f\colon U \to \mathbb{R}^m$ *is a* C^1-*function,* $g\colon U \to \mathbb{R}^n$ *is a* C^2-*function with a compact support* $S_g \subset \mathbb{R}^m \times \mathbb{R}^n$. *Let us consider the system*

$$(1.1) \qquad \frac{dx}{dt} = f(x, y),$$

$$\varepsilon \frac{dy}{dt} = g(x, y),$$

denote by $\Phi(t, x, y)$ *the flow on* U_x *induced by the equation*

$$(1.2) \qquad \frac{dy}{dt} = g(x, y)$$

for each $x \in \pi_1(U)$. *Suppose that*

i) $B \subset U$ *is a region such that for each* $x \in \pi_1(B)$ *the set* B_x *is the basin of an A-axiom attractor* $A(x) \subset B_x$ *of the flow* $\Phi(t, x, y)$ *on* U_x;

ii) μ_x *are the Bowen-Ruelle measure on* $A(x)$ *for each* $x \in \pi_1(B)$ *and the mapping*

$$\bar{f}(x) = \int_{B_x} f(x, y)\, d\mu_x(y)$$

is defined on $\pi_1(B)$ *and is Lipschitz on this domain*;

iii) Φ *is uniformly Lipschitz in* y *on the set* $S = \{t \in \mathbb{R} \mid t \geqq 0\} \times B$, $\partial\Phi/\partial x$, $\partial\Phi/\partial y$ *are uniformly Lipschitz in* x *and* y *on* S, $\|\partial\Phi/\partial x\|$ *is bounded on* S *and* $\|\partial\Phi/\partial y\| > \alpha > 0$ *on* S. *Consider the system*

$$(2.1) \qquad \frac{dx}{dt} = \bar{f}(x),$$

$$\varepsilon \frac{dy}{dt} = g(x, y) + \varepsilon \frac{\partial \Phi}{\partial x} (\bar{f}(x) - f(x, y)),$$

defined on B *and denote, for* $L > 0$, *by* $B_L \subset B$ *the set of the initial conditions* (x_0, y_0) *for which the solutions of* (2.1) *are defined on* $[0, L]$ *and are contained in* B *together with some of their* $\varrho_{(x_0,z_0)}$-*neighborhoods**).

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*) The solution $(x(t; x_0, y_0), y(t, x_0, y_0))$ defined on $[0, L]$ is contained in B together with its $\varrho_{(x_0,y_0)}$-neighborhood if the set $\bigcup_{t\in[0,L]}\{(x, y) \mid \|x - x(t; x_0, y_0)\| + \|y - y(t; x_0, y_0)\| < \varrho_{(x_0,y_0)}\}$ is included in B.

Then for each $\delta > 0$, $L > 0$ and $x_0 \in \pi_1(B_L)$ there exists a set $B^0_{L,x_0} \subset (B_L)_{x_0}$ such that $m((B_L)_{x_0} - B^0_{L,x_0}) = 0$, and for each $y_0 \in B^0_{L,x_0}$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, L]$ the inequalities

$$(2.2) \qquad \|\bar{x}(t) - x_\varepsilon(t)\| < \delta\,,$$
$$\|\bar{y}(t) - y_\varepsilon(t)\| < \delta$$

hold, where $(x_\varepsilon(t), y_\varepsilon(t))$, $(\bar{x}(t), \bar{y}(t))$, are the solutions of (1.1), (2.1), respectively, both satisfying the same initial condition (x_0, y_0).

III. PRELIMINARIES TO THE PROOF OF THE MAIN THEOREM

Let us consider the system

$$(3.1) \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = f(x)\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t, x, y)$$

where $(x, y) \in U \subset \mathbb{R}^m \times \mathbb{R}^n$, the function f defined on $\pi_1(U)$ belongs to the class $C^1(\pi_1(U))$ and the function g defined on $\{t \in \mathbb{R} \mid t \geqq 0\} \times U$ is continuous and of the class C^1 in the variables x and y.

Let $x(t; x_0, y_0) = x(t; x_0)$, $y(t; x_0, y_0)$ be the solution of (3.1) such that $x(0; x_0, y_0) = x_0$, $y(0; x_0, y_0) = y_0$. We will suppose that all solutions are defined for $t \geqq 0$.

Let $N_x \subset U_x$ be given for every $x \in \pi_1(U)$. Since $x(t; x_0, y_0) \in \pi_1(U)$ we can define $I(x_0, y_0) = \{t \in I \mid y(t; x_0, y_0) \notin N_{x(t;x_0,y_0)}\}$ for every interval $I = [0, T]$, $T > 0$. Then the following lemma holds.

Lemma. *If $m(N_x) = 0$ for every $x \in \pi_1(U)$ then the set $Q = \{y_0 \in U_{x_0} \mid I(x_0, y_0)$ is not dense in $I = [0, T]\}$ fulfils $m(Q) = 0$ for every $x_0 \in \pi_1(U)$ and $T > 0$.*

Proof. Let $T > 0$ and $x_0 \in \pi_1(U)$ be given and denote $Q_n = \{y \in U_{x_0} \mid$ there is $t_0 \in [0, T]$ such that $y(t, x_0, y) \in N_{x(t,x_0,y)}$ for every $t \in (t_0 - 1/n,\ t_0 + 1/n) \cap I\}$.

It is easy to see that $Q = \bigcup_{n \in N} Q_n$. Let us show that $m(Q_n) = 0$ for every $n \in N$. Let us suppose on the contrary that $m(Q_n) > 0$ for some $n \in N$ and take the intervals $J_k = [k/3n, (k + 1)/3n]$. Define $Q_{n,k} = \{y \in Q_n \mid$ there exists $t_0 \in J_k$ such that for $y(t; x_0, y) \in N_{x(t,x_0,y)}$ every $t \in (t_0 - 1/n,\ t_0 + 1/n) \cap I\}$; then there must exist k for which $m(Q_{n,k}) > 0$. To complete the proof, take the map $F(t, x_0)\colon U_{x_0} \to U_{x(t,x_0)}$ defined by $F(t, x_0)\,(y) = y(t;\ x_0,\ y)$. This is a one-to-one map and thus $m(F(t, x_0)\,(Q_{n,k})) > 0$. Hence a contradiction is obtained since at the same time $F(t, x_0)\,(Q_{n,k}) \subset N_{x(t,x_0)}$ holds for each $t \in J_k$. ∎

This lemma will be used in the proof of the following theorem which treats the case of partial averaging provided the time averages exist for almost all initial conditions. The theorem slightly generalizes the well-known fundamental theorem on averaging, cf. [2], [3].

Theorem. *Assume that $U \subset \mathbb{R}^m \times \mathbb{R}^n$ is an open set and that X, Y are functions defined on $\{t \in R \mid t \geqq 0\} \times U$ with values in $\mathbb{R}^m$, $\mathbb{R}^n$, respectively. Suppose the following properties are satisfied:*

i) *$X(t, x, y)$, $Y(t, x, y)$ are bounded, continuous in the variable t and satisfy the Lipschitz condition in x and y uniformly with a constant M;*

ii) *for every $x \in \pi_1(U)$ and for almost all $y \in U_x$ the limit*

$$X_0(x, y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x, y) \, dt$$

exists and is independent of y, i.e. $X_0(x, y) = X_0(x)$;

iii) *the function $X_0(x)$ satisfies the Lipschitz condition with constant N.*

Denote by $(x(t), y(t))$ the solutions of the system ($\varepsilon > 0$)

$$\text{(3.2)} \qquad \frac{dx}{dt} = \varepsilon X(t, x, y) \,, \qquad \frac{dy}{dt} = \varepsilon Y(t, x, y)$$

defined for $t \geqq 0$.

For $L > 0$ denote by $S_L \subset U$ the set of initial conditions (x_0, y_0) such that the solution $(\bar{x}(t), \bar{y}(t))$ with $(\bar{x}(0), \bar{y}(0)) = (x_0, y_0)$ of the system

$$\text{(3.3)} \qquad \frac{dx}{dt} = \varepsilon X_0(x) \qquad \frac{dy}{dt} = \varepsilon Y(t, x, y)$$

is defined on $[0, L\varepsilon^{-1}]$ and contained in U together with some of its $\varrho_{(x_0,y_0)}$-neighborhoods.

Then for every $L > 0$, $\delta > 0$ and $x_0 \in \pi_1(S_L)$ there exists a set $S^0_{L,x_0} \subset (S_L)_{x_0}$ such that $m((S_L)_{x_0} - S^0_{L,x_0}) = 0$ and for each $y_0 \in S^0_{L,x_0}$ there exists $\varepsilon_0 > 0$ such that for $t \in [0, L\varepsilon^{-1}]$, $0 < \varepsilon < \varepsilon_0$ the inequalities

$$\|x(t) - \bar{x}(t)\| < \delta \,, \qquad \|y(t) - \bar{y}(t)\| < \delta$$

hold where $(x(t), y(t))$, $(\bar{x}(t), \bar{y}(t))$ *are solutions of* (3.2), (3.3), *respectively and* $(x(0), y(0)) = (\bar{x}(0), \bar{y}(0)) = (x_0, y_0)$.

Proof. Let $\delta > 0$, $L > 0$ and $x_0 \in \pi_1(S)$ be given. Let us denote by $N_x \subset U_x$ the exceptional sets of zero Lebesgue measure for which the time averages

$$X_0(x, y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t, x, y)\,\mathrm{d}t$$

do not exist. Let us set $I = [0, L)$; using the preceding lemma we can conclude that the measure of the set $Q = \{y \in U_{x_0} \mid I(x_0, y) \text{ is not a dense subset in } I\}$ is zero. We define $S^0_{L,x_0} = (S_L)_{x_0} - (N_{x_0} \cup Q \cup \{y_0 \in S_{x_0} \mid \bar{y}(L; x_0, y_0) \in N_{\bar{x}(L,x_0)}\})$; obviously we have $m((S_L)_{x_0} - S^0_{L,x_0}) = 0$.

By the definition of S^0_{L,x_0}, for every sufficiently small $\kappa > 0$ we can choose a subset $A = \{0 = t_0 < t_1 < \ldots < t_m = L\} \subset [0, L]$ such that $\kappa/2 < |t_{i+1} - t_i| < \kappa$, for $i = 0, \ldots, m-1$ and such that $\bar{y}(t_i\varepsilon^{-1}) \notin N_{\bar{x}(t_i\varepsilon^{-1})}$; let us denote $\bar{x}(t_i\varepsilon^{-1}) = \bar{x}_i$ and $\bar{y}(t_i\varepsilon^{-1}) = \bar{y}_i$.

First, we estimate

$$\left\|\varepsilon\int_0^t (x(\tau; \bar{x}(\tau), \bar{y}(\tau)) - X_0(\bar{x}(\tau)))\,\mathrm{d}\tau\right\| \tag{3.4}$$

for $t \in [0, L\varepsilon^{-1}]$. For the sake of simplicity we denote

$$\varphi(t, \bar{x}(t), \bar{y}(t)) = X(t, \bar{x}(t), \bar{y}(t)) - X_0(\bar{x}(t)) .$$

We may write

$$\begin{aligned}
&\left\|\varepsilon\int_0^t \varphi(\tau, \bar{x}(\tau), \bar{y}(\tau))\,\mathrm{d}\tau\right\| = \\
&= \left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} [\varphi(\tau, \bar{x}(\tau), \bar{y}(\tau)) - \varphi(\tau, \bar{x}_i, \bar{y}_i)]\,\mathrm{d}\tau\, + \right.\\
&\qquad \left. + \varepsilon\sum_{i=0}^{m-1}\int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_i, \bar{y}_i)\,\mathrm{d}\tau\right\| \leqq \\
&\leqq \left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} [\varphi(\tau, \bar{x}(\tau), \bar{y}(\tau)) - \varphi(\tau, \bar{x}_i, \bar{y}_i)]\,\mathrm{d}\tau\right\| + \\
&\qquad + \left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_i, \bar{y}_i)\,\mathrm{d}\tau\right\| .
\end{aligned}$$

To estimate the first term, we suppose X, Y are bounded by C_X, C_Y respectively. Then

$$\left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} [X(\tau, \bar{x}(\tau), \bar{y}(\tau)) - X_0(\bar{x}(\tau)) - X(\tau, \bar{x}_i, \bar{y}_i) + X_0(\bar{x}_i)]\,\mathrm{d}\tau\right\| \leqq$$

$$\leqq \varepsilon \sum_{i=0}^{m-1} \int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} [M(\|\bar{x}(\tau) - \bar{x}_i\| + \|\bar{y}(\tau) - \bar{y}_i\|) + N\|\bar{x}(\tau) - \bar{x}_i\|] \,\mathrm{d}\tau =$$

$$= \varepsilon \sum_{i=0}^{m-1} \int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} [(M + N)\,\|\bar{x}(\tau) - \bar{x}_i\| + M\|\bar{y}(\tau) - \bar{y}_i\|] \,\mathrm{d}\tau \leqq$$

$$\leqq \varepsilon \sum_{i=0}^{m-1} \int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} \left[(M + N)\left\| \int_{t_i\varepsilon^{-1}}^{\tau} X_0(\bar{x}(s)) \,\mathrm{d}s\right] + M \left\|\varepsilon \int_{t_i\varepsilon^{-1}}^{\tau} Y(s, \bar{x}(s), \bar{y}(s) \,\mathrm{d}s\right\|\right] \mathrm{d}\tau \leqq$$

$$\leqq \varepsilon \sum_{i=0}^{m-1} \int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} [(M + N)\,\varepsilon C_X|\tau - t_i\varepsilon^{-1}| + M\varepsilon C_Y|\tau - t_i\varepsilon^{-1}|] \,\mathrm{d}\tau =$$

$$= \varepsilon^2[(M + N)\,C_X + MC_Y] \sum_{i=0}^{m-1} \frac{(t_{i+1}\varepsilon^{-1} - t_i\varepsilon^{-1})^2}{2} \leqq$$

$$\leqq \varepsilon^2[(M + N)\,C_X + MC_Y]\,\varepsilon^{-2}(2L\kappa^{-1})\,(\kappa^2 \,.\, 2^{-1}) =$$

$$= [(M + N)\,C_X + MC_Y]\,L\kappa\,.$$

To estimate the second term we write

$$\varepsilon \left\|\sum_{i=0}^{m-1} \int_{t_i\varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_i, \bar{y}_i) \,\mathrm{d}\tau\right\| =$$

$$= \varepsilon \left\| \int_0^{t_0\varepsilon^{-1}} \varphi(\tau, \bar{x}_0, \bar{y}_0) \,\mathrm{d}\tau + \sum_{k=1}^{m-1} \left[\int_0^{t_{k+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_k, \bar{y}_k) \,\mathrm{d}\tau - \int_0^{t_k\varepsilon^{-1}} \varphi(\tau, \bar{x}_k, \bar{y}_k) \,\mathrm{d}\tau\right]\right\| \leqq$$

$$\leqq \varepsilon \left\| \int_0^{t_1\varepsilon^{-1}} \varphi(\tau, \bar{x}_0, \bar{y}_0) \,\mathrm{d}\tau\right\| + \varepsilon \sum_{k=1}^{m-1} \left\| \int_0^{t_{k+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_k, \bar{y}_k) \,\mathrm{d}\tau\right\| +$$

$$+ \varepsilon \sum_{k=1}^{m-1} \left\| \int_0^{t_k\varepsilon^{-1}} \varphi(\tau, \bar{x}_k, \bar{y}_k) \,\mathrm{d}\tau\right\| .$$

Let us denote

$$\Phi(t, x, y) = \left\|\frac{1}{t} \int_0^t [X(\tau, x, y) - X_0(x)] \,\mathrm{d}\tau\right\| .$$

By the assumption ii) we have

$$\lim_{\varepsilon \to 0+} \Phi(t\varepsilon^{-1}, x, y) = 0$$

or t, x, and y fixed. Thus we obtain the following estimates:

$$\varepsilon \left\| \int_0^{t_1\varepsilon^{-1}} \varphi(\tau, \bar{x}_0, \bar{y}_0) \,\mathrm{d}\tau\right\| = t_1\Phi(t_1\varepsilon^{-1}, \bar{x}_0, \bar{y}_0) \leqq L\Phi(t_1\varepsilon^{-1}, \bar{x}_0, \bar{y}_0)\,,$$

$$\varepsilon \left\| \int_0^{t_{k+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_k, \bar{y}_k) \,\mathrm{d}\tau\right\| = t_{k+1}\Phi(t_{k+1}\varepsilon^{-1}, \bar{x}_k, \bar{y}_k) \leqq L\Phi(t_{k+1}\varepsilon^{-1}, \bar{x}_k, \bar{y}_k)\,,$$

$$\varepsilon \left\| \int_0^{t_k \varepsilon^{-1}} \varphi(\tau, \bar{x}_k, \bar{y}_k) \, d\tau \right\| = t_k \Phi(t_k \varepsilon^{-1}, \bar{x}_k, \bar{y}_k) \leqq L \Phi(t_k \varepsilon^{-1}, \bar{x}_k, \bar{y}_k) .$$

Hence we have

$$\varepsilon \left\| \sum_{i=0}^{m-1} \int_{t_i \varepsilon^{-1}}^{t_{i+1}\varepsilon^{-1}} \varphi(\tau, \bar{x}_i, \bar{y}_i) \, d\tau \right\| \leqq L[\Phi(t_1 \varepsilon^{-1}, \bar{x}_0, \bar{y}_0) +$$

$$+ \sum_{k=1}^{m-1} \Phi(t_{k+1} \varepsilon^{-1}, \bar{x}_k, \bar{y}_k) + \sum_{k=1}^{m-1} \Phi(t_k \varepsilon^{-1}, \bar{x}_k, \bar{y}_k)] = H(\varepsilon, \kappa) .$$

Thus we have obtained the estimate

$$\left\| \varepsilon \int_0^t X(\tau, \bar{x}(\tau), \bar{y}(\tau)) - X_0(\bar{x}(\tau)) \, d\tau \right\| \leqq$$

$$\leqq [(M + N) C_X + M C_Y] L\kappa + H(\varepsilon, \kappa) = a(\varepsilon, \kappa) .$$

The term $a = a(\varepsilon, \kappa)$ converges to zero for $\varepsilon \to 0_+$ and $\kappa \to 0_+$.

We set

$$x(t) = \bar{x}(t) + a\, u(t) ,$$

$$y(t) = \bar{y}(t) + a\, v(t)$$

and obtain the following equations for $u(t)$ and $v(t)$:

$$u(t) = \frac{\varepsilon}{a} \int_0^t [X(t, \bar{x} + au, \ \bar{y} + av) - X(t, \bar{x}, \bar{y}) + X(t, \bar{x}, \bar{y}) - X_0(\bar{x})] \, dt$$

$$v(t) = \frac{\varepsilon}{a} \int_0^t [Y(t, \bar{x} + au, \bar{y} + av) - Y(t, \bar{x}, \bar{y})] \, dt .$$

This yields

$$\|u(t)\| = \frac{\varepsilon}{a} \int_0^t Ma(\|u(\tau)\| + \|v(\tau)\|) \, d\tau + 1 ,$$

$$\|v(t)\| = \frac{\varepsilon}{a} \int_0^t Ma(\|u(\tau)\| + \|v(\tau)\|) \, d\tau$$

and thus

$$\|u(t)\| + \|v(t)\| \leqq 2\varepsilon M \int_0^t (\|u(\tau)\| + \|v(\tau)\|) \, d\tau + 1 .$$

Applying the Gronwall lemma we obtain

$$\|u(t)\| + \|v(t)\| \leqq e^{2\varepsilon M t} < e^{2ML}$$

and this implies

(3.5) $$|x(t) - \bar{x}(t)| \leqq a e^{2ML}$$

$$|y(t) - \bar{y}(t)| \leqq a e^{2ML}.$$

Now let us choose $y_0 \in S^0_{L,x_0}$ and suppose first that the solution $x(t) = x(t; x_0, y_0)$, $y(t) = y(t; x_0, y_0)$ of the equation (3.2) satisfies $(x(t), y(t)) \in U$ for $t \in [0, L\varepsilon^{-1}]$. Choosing ε and κ so that $a \leqq \delta e^{-2ML}$ we obtain by (3.5) the required inequalities

$$\|x(t) - \bar{x}(t)\| < \delta$$
$$\|y(t) - \bar{y}(t)\| < \delta .$$

To complete the proof we have to show that $(x(t), y(t)) \in U$ for $t \in [0, L\varepsilon^{-1}]$ provided $(x(t), y(t))$ is a solution of (3.2). Let us suppose this is not the case and denote

$$\hat{t} = \sup\{t > 0 \mid (x(\tau), y(\tau)) \in U \text{ for } \tau \in [0, t]\} .$$

We can choose $a(\varepsilon, \kappa)$ so that by (3.5)

$$\|x(t) - \bar{x}(t)\| < \varrho/2$$
$$\|y(t) - \bar{y}(t)\| < \varrho/2$$

for $t \in [0, \hat{t}]$; we recall that the solution $(\bar{x}(t), \bar{y}(t))$ lies in U together with its ϱ-neighborhood. Supposing $\hat{t} < L\varepsilon^{-1}$ and using the continuity of the solution of the differential equation we find $\eta > 0$ such that for $t \in [\hat{t}, \hat{t} + \eta]$

$$\varrho/2 < \|x(t) - \bar{x}(t)\| < \varrho ,$$
$$\varrho/2 < \|y(t) - \bar{y}(t)\| < \varrho .$$

Hence the solution $(x(t), y(t))$ belongs to U for $t \in [0, \hat{t} + \eta]$ and consequently $\hat{t} = L\varepsilon^{-1}$. ∎

IV. PROOF OF THE MAIN THEOREM

Now we are ready to prove Main Theorem. Let $L > 0$, $\delta > 0$ and $x_0 \in \pi_1(B_L)$ be given. The transformation $t = \varepsilon\tau$ of the time scale changes the system (1.1) into

$$\frac{dx}{d\tau} = \varepsilon f(x, y) \tag{4.1}$$
$$\frac{dy}{d\tau} = g(x, y)$$

and the system (2.1) into

$$\frac{dx}{d\tau} = \varepsilon \bar{f}(x) \tag{4.2}$$
$$\frac{dy}{d\tau} = g(x, y) + \varepsilon \frac{\partial \Phi}{\partial x} (\bar{f}(x) - f(x, y)) .$$

Defining the diffeomorphism $F: \mathbb{R} \times U \to \mathbb{R} \times U$ by $F(\tau, x, z) = (\tau, x, \Phi(\tau, x, z))$, we transform the system (4.1) into

$$\text{(4.3)} \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \varepsilon f(x, \Phi(\tau, x, z)) \equiv \varepsilon X(\tau, x, z)$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = -\varepsilon \left(\frac{\partial \Phi}{\partial z}\right)^{-1} \left(\frac{\partial \Phi}{\partial x}\right) f(x, \Phi(\tau, x, z)) \equiv -\varepsilon Y(\tau, x, z)$$

and the system (4.2) into

$$\text{(4.4)} \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \varepsilon \bar{f}(x),$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = -\varepsilon Y(\tau, x, z).$$

Let us restrict our attention to the behavior of the systems (4.3) and (4.4) on the set $\{\tau \in \mathbb{R} \mid \tau \geqq 0\} \times B$. Using the Bowen-Ruelle theorem and its Corollary we observe that for each $x \in \pi_1(B)$ the limit

$$X(x, z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(\tau, x, z)\, \mathrm{d}\tau$$

exists for almost all $z \in B_x$ and, moreover, that $X(x, z) = \bar{f}(x)$.

At this moment, we use the Theorem from the preceding section for approximating solutions of the system (4.3) restricted to $\{\tau \in \mathbb{R} \mid \tau > 0\} \times B$ by solutions of the system (4.4). In other words, for $L > 0$, $\hat{\delta} > 0$ and $x_0 \in \pi_1(B_L)$ we can find a set $\hat{B}^0_{L,x_0} \subset (B_L)_{x_0}$ such that $m((B_L)_{x_0} - \hat{B}^0_{L,x_0}) = 0$ and such that for each $z_0 \in \hat{B}^0_{L,x_0}$ there exists $\hat{\varepsilon}_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in [0, L\varepsilon^{-1}]$ the inequalities

$$\|\bar{x}(\tau) - x(\tau)\| < \hat{\delta},$$

$$\|\bar{z}(\tau) - z(\tau)\| < \hat{\delta}$$

hold; $(x(\tau), z(\tau)) = (x(\tau; x_0, z_0), z(\tau; x_0, z_0)$, and $(\bar{x}(\tau), \bar{z}(\tau)) = (\bar{x}(\tau; x_0, z_0), z(\tau; x_0, z_0))$, are the solution of (4.3), and of (4.4), respectively.

Let us suppose that Φ is Lipschitz in x and y with a constant M and define $B^0_{L,x_0} = \{y_0 \in (B_L)_{x_0} \mid y_0 = \Phi(0, x_0 z_0), z_0 \in \hat{B}^0_{L,x_0}\}$; we immediately see that $B^0_{L,x_0} = \hat{B}^0_{L,x_0}$. Thus provided $y_0 \in B^0_{L,x_0}$,

$$\|x(\tau) - \bar{x}(\tau)\| < \hat{\delta},$$

$$\|y(\tau) - \bar{y}(\tau)\| = \|\Phi(\tau, x(\tau), z(\tau)) - \Phi(\tau, \bar{x}(\tau), \bar{z}(\tau))\| \leqq$$

$$\leqq M(\|x(\tau) - \bar{x}(\tau)\| + \|z(\tau) - \bar{z}(\tau)\| \leqq 2M\hat{\delta}$$

for $\tau \in [0, L\varepsilon^{-1}]$.

Choose $\hat{\delta} < \delta/2M$ and set $B^0_{L,x_0} = \hat{B}^0_{L,x_0}$. Then we can for each $y_0 \in B^0_{L,x_0}$ find ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, L]$,

$$\|x_\varepsilon(t) - \bar{x}(t)\| < \delta,$$

$$\|y_\varepsilon(t) - \bar{y}(t)\| < \delta. \qquad \blacksquare$$

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