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Vlastimil Pták

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EXTREMAL OPERATORS AND OBLIQUE PROJECTIONS

VLASTIMIL PTÁK, Praha

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INTRODUCTION

It is the purpose of this note to point out a geometric explanation of some earlier results concerning the extremal operators $T(\varphi) = S \mid \operatorname{Ker} \varphi(S)$, where S is the backward shift operator on H^2 and φ is a given inner function. The note is a revised version of an earlier paper where we have shown how some of the results proved in [9] and [10] may be deduced from a simple proposition concerning norms of oblique projections. The note was rewritten because it turned out that the geometric proposition from which the results were deduced was known already; the author is indebted to T. Ando and B. Sz. Nagy for having called his attention to the fact that it is contained already in a paper of V. E. Ljance [1]; the journal is not accessible to the author. Under a different form the proposition was also used in the paper [12].

The present note (as well as its predecessor) is the result of an attempt to extract the geometric substance of the results concerning the exttremal operators mentioned above; it is interesting that the geometric idea used in [9] to prove the symmetry of the $C(\varphi, \psi)$ is, in fact, quite close to the proposition on oblique projections due to Ljance; however, it was only quite recently that the author realized fully that the geometric substance of the properties of the operators $\psi(T(\varphi))$ is contained in the proposition about the properties of oblique projections.

The note is divided into two parts. The first section, which is included for the sake of completeness, is devoted to a discussion of oblique projections — the propositions are formulated in a form suitable for application to the $T(\varphi)$. There is no claim originality for this section. The essential results are contained in the paper of Ljance. In the second section the results are applied to the particular case of projections whose ranges are the wandering subspaces of some isometries.

Suppose we are given a Hilbert space H which is decomposed into the direct sum of two closed subspaces M_0 and M_1; denote by R the projection operator of H onto M_0 along M_1. It is possible to show that

$$|R|^2 = (1 - |P_1 P_0|^2)^{-1}$$

where P_0 and P_1 are respectively the orthogonal projections of H onto M_0 and M_1. This equality will be used to explain the geometric substance of the properties of the extremal operators $T(\varphi) = S \mid \operatorname{Ker} \varphi(S)$ where S is the backward shift operator on H^2 and φ is a given inner function [4, 2]; we believe it throws some more light onto some properties of the constants $C(\varphi, \psi)$ but it is not without interest in its own right.

The $C(\varphi, \psi)$ are defined as follows: we take two functions φ, ψ such that $\varphi(A)$ and $\psi(A)$ is meaningful for any contraction A. Then

$$C(\varphi, \psi) = \sup \{|\psi(A)|;\ |A| \leqq 1,\ \varphi(A) = 0\} .$$

The operators $T(\varphi)$ appear first in the author's papers [4, 5]. It turns out that $T(\varphi)$ realizes the maximum in $C(\varphi, \psi)$. The motivation for the study of the $C(\varphi, \psi)$ is explained in [5]. We believe that the result of the present note forms the geometrical substance of some interesting properties of the $C(\varphi, \psi)$ as described in [9] and [10]. In particular, we show how it may be used to obtain a recent result of N. J. Young.

1. PRELIMINARIES

In the whole note H will stand for a Hilbert space. If H_0 is a closed subspace of H and if P stands for the orthogonal projection of H onto H_0 then $P = V(P)^*$ where $V(P)$ is the natural injection operator of H_0 into H.

If the space H is the direct sum of two subspaces A and B then every $x \in H$ may be written in a unique way in the form $x = a + b$ with $a \in A$ and $b \in B$. The operator which assigns to each $x \in H$ the element a will be called the projection of H onto A along B. It follows from the closed graph theorem that R is bounded if and only if both A and B are closed subspaces; R is selfadjoint if and only if A and B are orthogonal to each other. We shall adopt the convention commonly used in the theory of C^* and W^* algebras: if x belongs to the range of the orthogonal projection P we shall write $x \in P$. Thus $x \in P$ is equivalent to $x = Px$.

The following simple proposition characterizes oblique projections onto $\operatorname{Im} Q$ along $\operatorname{Ker} P$ in terms of P and Q.

(1,1) *Let H be a Hilbert space and let P and Q be two orthogonal projections in H. Consider a bounded linear operator R on H. Then these are equivalent*

1° *H is the direct sum of* $\operatorname{Im} Q$ *and* $\operatorname{Ker} P$ *and R is the projection onto* $\operatorname{Im} Q$ *along* $\operatorname{Ker} P$

2° *satisfies the following four relations*

21° $QR = R$
22° $RP = R$
23° $RQ = Q$
24° $PR = P$

3° *H is the direct sum of* $\operatorname{Im} P$ *and* $\operatorname{Ker} Q$ *and R^* is the projection onto* $\operatorname{Im} P$ *along* $\operatorname{Ker} Q$.

Furthermore, suppose H is the sum of Im Q *and* Ker P *and that R satisfies* 21°, 22° *and, in addition, the condition* 23°. *Then R satisfies the condition* 24° *as well (this means, in particular, that the sum* Im Q + Ker P *is a direct one and R is the projection onto* Im Q *along* Ker P).

Similarly, if the intersection of Im Q *and* Ker P *is the zero vector only and if R satisfies conditions* 21°, 22°, 24° *then condition* 23° *is satisfied as well.*

Proof. The verification of the first part of the proposition is immediate.

Taking adjoints in the conditions sub 2° and using the equivalence of 1° and 2° we obtain the equivalence of 2° and 3°.

To prove the second part, assume that H is the sum of Im Q and Ker P and that 21°, 22°, 23° are satisfied. Given $x \in H$, we have $x = q + k$ with $q \in Q$ and $Pk = 0$.

Using $Qq = q$, $R = RP$ and $RQ = Q$ we obtain $PRx = PRq + PRk = PRQq + PRPk = PQq = Pq = P(q + k) = Px$ and this proves 24°.

To conclude the proof, suppose that the intersection of Im Q and Ker P is only the zero vector and that R satisfies conditions 21°, 22°, 24°. Then condition 23° is satisfied as well.

Take any $q \in Q$ and set $x = (1 - R)\, q$. Since $R = QR$ we have $x = q - QRq \in Q$. By 24° we have $P(1 - R) = 0$ so that $Px = P(1 - R)\, q = 0$ thus $x \in \operatorname{Im} Q \cap \operatorname{Ker} P$ and consequently $x = 0$. Thus $(1 - R)\, q = 0$ for every $q \in Q$ so that $RQ = Q$.

Remark. Conditions 21° and 23° alone imply that $R^2 = R$. Indeed, applying 21°, 23° and 21° again, we obtain

$$R^2 = R(QR) = (RQ)\, R = QR = R\,.$$

In a similar manner conditions 22° and 24° yield $R^2 = R$ as well.

(1,2) *Suppose H is the direct sum of* Im Q *and of* Ker P; *denote by P_0 the restriction of P to* Im Q *and by Q_0 the restriction of Q to* Im P.

Then

1° P_0 *is an injective mapping of* Im Q *onto* Im P

2° $|Q(1 - P)\, Q| < 1$

3° Q_0 *is an injective mapping of* Im P *onto* Im Q

4° $|P(1 - Q)\, P| < 1$

Proof. If $q \in Q$ and $Pq = 0$ then $q \in \operatorname{Im} Q \cap \operatorname{Ker} P$ so that $q = 0$. Thus P_0 is injective. Denote by R the projection of H onto Im Q along Ker P. Since $PR = P$ we have, for each $p \in P$

$$p = Pp = P(Rp)$$

and $Rp \in Q$; thus every $p \in P$ is of the form Pq for a suitable $q \in Q$ and P_0 is surjective. This proves condition 1°.

It follows from condition 1° that P_0 is invertible: thus there exists an $\alpha > 0$ such that

$$|Pq|^2 \geqq \alpha |q|^2$$

for every $q \in Q$. Thus

$$(QPQx, x) = (PQx, PQx) \geqq \alpha(Qx, Qx) = \alpha(Qx, x)$$

so that $QPQ \geqq \alpha Q$ and

$$1 - Q(1 - P) Q = 1 - Q + QPQ \geqq 1 - Q + \alpha Q \geqq \alpha$$

Thus $Q(1 - P) Q \leqq 1 - \alpha < 1$.

The second part of the proposition is obtained using the equivalence of 1° and 3° from Proposition (1,1).

Now we are ready to state the main

(1,3) Proposition. *Let H be a Hilbert space and let P and Q be two orthogonal projections in H such that H is the direct sum of* Im Q *and* Ker P. *Denote by R the projection onto* Im Q *along* Ker P. *Then*

$$|R|^2 = \frac{1}{1 - |(1 - Q) P|^2} = \frac{1}{1 - |(1 - P) Q|^2}.$$

Proof. We observe first that R is a bounded operator by the closed graph theorem. According to (1, 2) the operator P_0 has a bounded inverse; if we denote it by Y then Y is an injective mapping of Im P onto Im Q so that

$$P_0 Y = \mathrm{I}_{\mathrm{Im}(P)} \qquad YP_0 = \mathrm{I}_{\mathrm{Im}(Q)}$$

Now let us prove that YP is the projection onto Im Q along Ker P. The relations

$$Q(YP) = YP \quad \text{and} \quad (YP) P = YP$$

are obvious. Furthermore, we have

$$P(YP) = (PY) P = P$$

$$(YP) Q = YPQ = YP_0Q = Q\,;$$

now it suffices to use Proposition (1,1) to prove that $YP = R$.

The next step consists in proving the equality

$$|R| = |Y|\,.$$

Since $R = YP$ the inequality $|R| \leqq |Y|$ is obvious. Now it suffices to observe that $Rp = Yp$ for $p \in P$ so that Y is a restriction of R (to the range of P) whence $|Y| \leqq |R|$.

Since $R = YP = QR$ we have $R^* = Y^*Q$ and $RR^* = YY^*Q$ and $R^*R = Y^*YP$.

Let us prove now the identity

$$YY^* = (1 - Q(1 - P) Q)^{-1} \mid \mathrm{Im}\, Q\,.$$

Since $|Q(1 - P) Q| < 1$ the inverse $(1 - Q(1 - P) Q)^{-1}$ exists. We have $(1 - Q(1 - P) Q) = 1 - Q + QPQ = 1 - Q + (PQ)^* PQ = 1 - Q + (P_0Q)^* P_0Q = 1 - Q + QP_0^*P_0Q$. Since

$$QP_0^*P_0Q \,.\, QYY^*Q = Q$$

it follows that

$$(1 - Q(1 - P)\,Q)^{-1} = 1 - Q + QYY^*Q$$

whence

$$QYY^*Q = Q(1 - Q(1 - P)\,Q)^{-1}\,Q$$

$$YY^* = (1 - Q(1 - P)\,Q)^{-1}\,|\,\mathrm{Im}\,Q$$

Since

$$Q(1 - P)\,Q + (1 - Q(1 - P)\,Q) = 1$$

the identity operator is represented as a sum of two nonnegative operators, the second one being invertible. If x ranges over all vectors of norm 1, we obtain

$$\sup\,(Q(1 - P)\,Qx, x) + \inf\,((1 - Q(1 - P)\,Q)\,x, x) = 1\,,$$

in other words,

$$|Q(1 - P)\,Q| + |(1 - Q(1 - P)\,Q)^{-1}|^{-1} = 1\,.$$

Since $(1 - Q(1 - P)\,Q)^{-1} = 1 - Q + YY^*Q$, we have

$$|(1 - Q(1 - P)\,Q)^{-1}| = \max\,(|1 - Q|, |YY^*|) = |YY^*|\,.$$

Since $|YY^*| = |Y|^2 = |R|^2$, we have

$$|(1 - P)\,Q|^2 = |Q(1 - P)\,Q| = 1 - |R|^{-2}\,.$$

The proof is complete.

It follows from the preceding considerations that $RR^* = (1 - Q(1 - P)\,Q)^{-1}\,Q$; we have thus an expression directly in terms of P and Q.

It is, sometimes, more convenient to work with operators defined on the whole space. It is not difficult to prove the following expression for R

$$R = (1 - (1 - P)\,Q)^{-1}\,P\,.$$

The inverse of $(1 - (1 - P)\,Q)$ exists since $|(1 - P)\,Q|^2 = |Q(1 - P)\,Q| < 1$ by (1,2). One way of proving this formula for R is to verify the relations from (1,1). We shall use the properties of Y. To show that

$$(1 - (1 - P)\,Q)^{-1}\,P = YP$$

it suffices to prove the equality

$$(1 - (1 - P)\,Q)\,YP = P\,.$$

This equality is an easy consequence of the equalities $QY = Y$ and $PYP = P$.

The following result is fairly well known but it is interesting to note that it may be considered as a particular case of Proposition (1,3). For many readers the result will be no surprise.

(1,4) Corollary. *For every real t*

$$\sin^2 t + \cos^2 t = 1$$

2. APPLICATION TO EXTREMAL OPERATORS

Let us turn now to projections of a particular type: their ranges will be the wandering subspaces of some isometries. We believe that Proposition (1,3) when applied to these projections provides further insight into the properties of the extremal operators

$$T(\varphi) = S \mid \operatorname{Ker} \varphi(S)$$

where S is the backward shift operator on H^2 and φ is a given inner function. These operators appear first in the author's paper [5] where the motivation for their study is explained. For a systematic account see the survey [9] or the more recent one [11]. In connection with the study of these operators N. J. Young formulated the following conjecture: given two Blaschke products φ, ψ of the same length, then

$$|\psi(T(\varphi))| = |\varphi(T(\psi))| .$$

Let us remark here that V. V. Peller recently proved that the conjecture is true in a much more general situation [11]. However, the proof given in [9] is based on a geometric idea, which already implicitly contains, in a rudimentary form, some elements of the proof of Proposition (1,3). The present note is the result of an attempt to extract the geometric substance of the behaviour of the operators $\psi(T(\varphi))$.

For $\psi(z) = z^n$ the author proved that the maximum of $|\psi(T(\varphi))|$ provided φ ranges over all Blaschke products of lentgh n with roots in the disc $|z| \leqq r$ is attained if $\varphi(z) = ((z - r)/(1 - rz))^n$. It seems natural to expect a similar result even for functions other than $z \to z^n$. The conjecture that $|\psi(T(\varphi))|$ is plurisubharmonic as a function of the roots of φ was formulated first by P. Vrbová. In the particular case where ψ is a Blaschke product of the same length as φ a proof was recently given by N. J. Young. Although not explicitly formulated expressis verbis the idea of considering the restriction of one projection to the range of another one plays, similarly as in [9], a decisive role in the proof. In our opinion Proposition (1,3) brings out the geometrical substance of these considerations in their pure form.

To illustrate this, we intend to show that Proposition (1,3) contains the result of N. J. Young [10] mentioned above.

Suppose V and W are two coisometries so that $VV^* = 1$ and $WW^* = 1$. Also, $\operatorname{Ker} V = H \ominus V^*H = \operatorname{Range}(1 - V^*V)$ and similar relations hold for W. Let $P = 1 - V^*V$ and $Q = 1 - W^*W$ so that P and Q are respectively the orthogonal projections onto $\operatorname{Ker} V$ and $\operatorname{Ker} W$. Suppose further that H is the direct sum of $\operatorname{Ker} W$ and $(\operatorname{Ker} V)^\perp$.

The norm of the restriction of W to $\operatorname{Ker} V$ may be expressed as follows

$$|W|\operatorname{Ker} V|^2 = |WP|^2 = |PW^*WP| = |P(1-Q)P| = |(1-Q)P|^2 .$$

It follows from the identity proved in Theorem (1,3) that

$$|W|\operatorname{Ker} V|^2 = 1 - |R|^{-2}$$

where R is the projection onto Ker W along $(\operatorname{Ker} V)^{\perp}$.

It is not difficult to give a detailed description of the projection operator.

The operator R has to satisfy the relations $QR = R$ and $RP = R$, in other words

(1) $$(1 - Q)R = 0$$

(2) $$R(1 - P) = 0$$

Praemultiplying (1) by W and using the relation $WW^* = 1$ we find that (1) is equivalent to

(1′) $$WR = 0 .$$

Similarly, postmultiplying (2) by V^* and using the relation $VV^* = 1$ we find that (2) is equivalent to

(2′) $$RV^* = 0 .$$

From now on we shall assume that the operators V and W are of the following particular form. We consider two polynomials of degree n

$$p(z) = (z - \alpha_1) \dots (z - \alpha_n)$$
$$q(z) = (z - \beta_1) \dots (z - \beta_n)$$

with all roots inside the unit disc and the corresponding polynomials

$$p_0(z) = (1 - \alpha_1^* z) \dots (1 - \alpha_n^* z)$$
$$q_0(z) = (1 - \beta_1^* z) \dots (1 - \beta_n^* z)$$

and set $\varphi(z) = p(z)\, p_0(z)^{-1}$, $\psi(z) = q(z)\, q_0(z)^{-1}$.

We shall consider the two coisometries $\varphi(S)$ and $\psi(S)$ where S is the backward shift operator on H^2. Now we are ready to show that the result mentioned above is an immediate consequence of (1,3).

(2,1) *Let φ and ψ be two Blaschke products of length n and let S be the backward shift operator on H^2. Let M be the restriction of $\psi(S)$ to* Ker $\varphi(S)$. *Then*

$$|M|^2 = 1 - |1 - p(S)^*\, q_0(S)^{*-1}\, p_0(S)^{-1}\, q(S)|^{-2} .$$

Proof. To simplify the formulae, we shall abbreviate $f(S)$ to f if f is a polynomial. Set $V = \varphi(S)$, $W = \psi(S)$. Since W is given as the product of two commuting operators, $W = qq_0^{-1}$ and since q_0 is invertible relation (1′) is equivalent to

(1″) $$qR = 0$$

and, in analogous manner, (2′) is equivalent to

(2″) $$Rp^* = 0 .$$

We have established, in [9, page 370] an interesting relation

$$qp^* = p_0 q_0^* .$$

The operator $qp^* = p_0 q_0^*$ is invertible; denote by Z its inverse. Set

$$A = q_0^* Z q = p_0^{-1} q$$

$$B = p^* Z p_0 = p^* q_0^{*-1}$$

and observe that

$$AB = q_0^* Z q p^* Z p_0 = q_0^* Z p_0 = 1$$

so that BA is a projection.

Furthermore

$$qBA = qp^* q_0^{*-1}(q_0^* Z q) = q$$

$$BAp^* = (p^* Z p_0)\, p_0^{-1} q p^* = p^* ;$$

hence $1 - BA$ satisfies

$$q(1 - BA) = 0 \quad \text{and} \quad (1 - BA)\, p^* = 0$$

so that $R = 1 - BA = 1 - p^* Z q$ satisfies $QR = R$ and $RP = R$.

The relation $RQ = Q$ or the equivalent relation $p^* Z q Q = 0$ follows immediately from the fact that Q is the projection onto Ker $W =$ Ker q.

The proof of $PR = P$ is similar.

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Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).