Tomasz Natkaniec

On points of qualitative semicontinuity

Časopis pro pěstování matematiky, Vol. 110 (1985), No. 4, 337–342

Persistent URL: `http://dml.cz/dmlcz/118238`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

Vydává Matematický ústav ČSAV, Praha

SVAZEK 110 * PRAHA 29. 11. 1985 * ČÍSLO 4

ON POINTS OF QUALITATIVE SEMICONTINUITY

TOMASZ NATKANIEC, Bydgoszcz

(Received October 5, 1983)

Let $\mathcal{J}$ be the σ-ideal of sets of the first category on the real line. For a real function $f: R \to R$ let us define the qualitative upper limit at the point x

$$q\text{-}\lim_{t \to x} \sup f(t) = \inf \{y \in R: \{t \in R: f(t) < y\} \text{ is residual at } x\} .$$

Similarly let us define the qualitative lower limit of f at x

$$q\text{-}\lim_{t \to x} \inf f(t) = \sup \{y \in R: \{t \in R: f(t) > y\} \text{ is residual at } x\} .$$

We use the notation introduced in [1]:

$$Q(f) = \{r \in R: q\text{-}\lim_{t \to r} \sup f(t) = f(r) = q\text{-}\lim_{t \to r} \inf f(t)\} ,$$

$$S_q(f) = \{r \in R: q\text{-}\lim_{t \to r} \sup f(t) \leqq f(r)\} ,$$

$$T_q(f) = \{r \in R: q\text{-}\lim_{t \to r} \sup f(t) < f(r)\} ,$$

$$S_q^1(f) = \{r \in R: q\text{-}\lim_{t \to r} \inf f(t) \geqq f(r)\} ,$$

$$T_q^1(f) = \{r \in R: q\text{-}\lim_{t \to r} \inf f(t) > f(r)\} .$$

The following facts are proved in [1].

Fact 0. *There exist sets B and C such that B is a G_δ set, $C \in \mathcal{J}$ and $Q(f) = B - C$.*

Fact 1. *The sets $T_q(f)$ and $T_q^1(f)$ are of the first category.*

Fact 2. *The sets $S_q(f) - Q(f)$ and $S_q^1(f) - Q(f)$ do not contain sets of the second category having the Baire property.*

Z. Grande in [1] showed Theorem 3 and stated the following Problem 1.

Theorem 3. *Let $A, B, C \subseteq R$ satisfy*

(i) *$C \in \mathscr{I}$, $B \subseteq A$, $C \subseteq A - B$ and the set $A - B$ do not contain sets of second category having the Baire property*

and

(ii) *$B = D - C$, where D is a G_δ set. Then there exists a function $g: R \to R$ such that $Q(g) = B$, $S_q(g) = A$ and $T_q(g) = C$.*

Problem 1. Accepting the assumption (i) of Theorem 3 let us suppose furthermore that $B = D - D_1$, where D is a G_δ set and $D_1 \in \mathscr{I}$.

Is there a function $g: R \to R$ such that $Q(g) = B, S_q(g) = A$ and $T_q(f) = C$?

The answer to this question is negative. It follows from the following fact.

Fact 3. Let $D_q(f) = \{r \in R: q\text{-}\liminf_{t \to r} f(t) = q\text{-}\limsup_{t \to r} f(t)\}$. Then $D_q(f)$ is a G_δ set for every function $f: R \to R$.

Proof. It is easy to show that for every $a \in R$ the sets $A = \{x \in R: q\text{-}\liminf_{t \to x} f(t) \leqq a\}$ and $B = \{x \in R: q\text{-}\limsup_{t \to x} f(t) \geqq a\}$ are closed. Indeed, if $q\text{-}\liminf_{t \to x} f(t) > a$ then there exist: $\varepsilon > 0$ and a neighbourhood U of x such that $U \cap \{y \in R: f(y) < a + \varepsilon\} \in \mathscr{I}$. So for every $y \in U$ we we have $q\text{-}\liminf_{t \to x} f(t) \geqq a + \varepsilon$ and $x \notin \mathrm{Cl} A$. Then for all rational numbers $p, q \in Q$ the sets $A(f, p, q) = \{x \in R: q \text{-} \liminf_{t \to x} f(t) \leqq p < q \leqq q\text{-}\limsup_{t \to x} f(t)\}$ are closed. Since $R - D_q(f) = \bigcup\{A(f, p, q): p, q \in Q\}$, $D_q(f)$ is a G_δ set.

It is clear that $Q(f) \subseteq D_q(f)$ and $D_q(f) - Q(f) \subseteq T_q(f) \cup T_q^1(f)$, hence $Q(f) = D_q(f) - [T_q(f) \cup T_q^1(f)]$.

Assume that $A = R$, $C = \emptyset$, $R - B \in \mathscr{I}$ and B is not a G_δ set. Suppose that there exists a function $f: R \to R$ such that $Q(f) = B, S_q(f) = R$ and $T_q(f) = C$. Then $T_q^1(f) = \emptyset$. If D is a G_δ set and $B \subseteq D$ then $D - B$ is non empty and $D - B \nsubseteq T_q(f) \cup T_q^1(f)$. This is impossible since the Fact 3 holds.

In the next part we assume that every set $A \subseteq R$ of cardinality less than continuum is of the first category. Notice that if CH (Continuum Hypothesis) or MA (Martin's Axiom) are assumed then this condition holds. [3]

The following theorem is generalization of Theorem 3 [1].

Theorem. (MA) *For every sets $A, A_1, B, C, C_1 \subseteq R$ the following conditions are equivalent*:

(i) $A \cap A_1 = B$,
$C \cup C_1 \in \mathscr{I}$,
$C \subseteq A - B$, $C_1 \subseteq A_1 - B$,

the sets $A - B$ and $A_1 - B$ do not contain sets of the second category having the Baire property,
there exists a G_δ set D such that $B = D - (C \cap C_1)$,

(ii) there exists a function $f\colon R \to R$ such that $A = S_q(f)$, $A_1 = S_q^1(f)$, $B = Q(f)$, $C = T_q(f)$ and $C_1 = T_q^1(f)$.

Proof. The implication (ii) $\Rightarrow$ (i) follows from the facts 0 – 3.

(i) $\Rightarrow$ (ii). Let $E = \mathrm{Cl}\, B$. Since $E - D \subseteq E - B$, we have $E - D \in \mathscr{I}$. Notice that $E - D$ is a F_σ set and $E - D = \bigcup_{n \in N} F_n$, where F_n are closed, nowhere dense and $F_i \cap F_j = \emptyset$ for $i \neq j$ [4].

Let $(a_n)_{n \in N}$ be a sequence of positive real numbers such that $\sum_{n \in N} a_n = 1$.

For every $n \in N$ we define the function $h_n\colon R \to \langle -a_n, a_n \rangle$,

$$h_n(x) = \begin{cases} a_n \sin \dfrac{1}{\operatorname{dist}(x, F_n)} & \text{for } x \notin F_n, \\ 0 & \text{for } x \in F_n. \end{cases}$$

For $n \in N$ the function h_n is continuous on the set $R - F_n$ and for $x \in F_n$, $q\text{-}\limsup_{t \to x} h_n(t) = a_n \geqq h_n(x) \geqq -a_n = q\text{-}\liminf_{t \to x} h_n(t)$.

In the first step we define a function $h\colon R \to R$ such that $Q(h) = S_q^1(h) = S_q(h) = R - (E - D) = R - \bigcup_{n \in N} F_n$ and $T_q(h) = T_q^1(h) = \emptyset$. Let $h(x) = \sum_{n \in N} h_n(x)$. This function satisfies the above conditions.

Indeed:

a) Assume that $x \notin \bigcup_{n \in N} F_n$. Since h is a sum of a uniformly convergent series, h is continuous at the point x.

b) If $x \in F_n$ then

$$q\text{-}\limsup_{t \to x} h(t) = a_n + \sum_{m \neq n} a_m \sin \frac{1}{\operatorname{dist}(x, F_m)} =$$

$$= h(x) + a_n > h(x) > h(x) - a_n = -a_n + \sum_{m \neq n} a_m \sin \frac{1}{\operatorname{dist}(x, F_m)} = q\text{-}\liminf_{t \to x} h(t).$$

Hence $x \notin S_q(h) \cup S_q^1(h)$.

Assume that $E = R$. Then the following function $f\colon R \to R$ satisfies the conditions of the theorem

$$f(x) = \begin{cases} 2 & \text{for } x \in C, \\ -2 & \text{for } x \in C_1, \\ q\text{-}\limsup_{t \to x} h(t) & \text{for } x \in A - C, \\ q\text{-}\liminf_{t \to x} h(t) & \text{for } x \in A_1 - C_1, \\ h(x) & \text{elsewhere}. \end{cases}$$

Since $\{x \in R: f(x) \neq h(x)\} \in \mathcal{I}$, so for every $x \in R$, $q\text{-}\limsup_{t \to x} f(t) = q\text{-}\limsup_{t \to x} h(t)$ and $q\text{-}\liminf_{t \to x} f(t) = q\text{-}\liminf_{t \to x} h(t)$. Hence $B \subseteq Q(f)$, $C \subseteq T_q(f)$ and $C_1 \subseteq T_q^1(f)$.

If $x \in A - (B \cup C)$ then $x \in E - D$. Hence $f(x) = q\text{-}\limsup_{t \to x} h(t) = q\text{-}\limsup_{t \to x} f(t) > q\text{-}\liminf_{t \to x} h(t) = q\text{-}\liminf_{t \to x} f(t)$ and $A - (B \cup C) \subseteq S_q(f) - [Q(f) \cup T_q(f)]$. Similarly, $A_1 - (B \cup C_1) \subseteq S_q^1(f) - [Q(f) \cup T_q^1(f)]$ and $R - (A \cup A_1) \subseteq R - [S_q(f) \cup S_q^1(f)]$. Consequently, $Q(f) = B$, $S_q(f) = A$, $S_q^1(f) = A_I$, $T_q(f) = C$ and $T_q^1(f) = C_1$.

Now assume that $R - E \neq \emptyset$. We prove the following lemma.

Lemma. *If A is an open, non empty subset of R and $B \subseteq A$ then there exists a partition $(K_n)_{n \in N}$ of A such that sets K_n are of the second category at every point $x \in A$ and if B is of the second category at x then $K_n \cap B$ $(n = 1, 2, \ldots)$ is of the second category at x.*

Proof of lemma. The construction of the sets K_n is very similar to the construction of Bernstein's set [2].

Let:

(r_ξ) be an enumeration of the set A,

$(I_n)_{n \in N}$ be a countable basis of A,

$(H_{n,\eta})_{\eta < 2^{\omega_0}}$ be an enumeration of the family of the residual and G_δ subsets of I_n,

$$H_{n,\eta}^1 = \begin{cases} H_{n,\eta} & \text{if } H_{n,\eta} \cap B \in \mathcal{I}, \\ H_{n,\eta} \cap B & \text{if } H_{n,\eta} \cap B \notin \mathcal{I}, \end{cases}$$

$(H_\xi)_{\xi < 2^{\omega_0}}$ be an enumeration of the family $\{H_{n,\eta}^1 : n \in N,\ \eta < 2^{\omega_0}\}$.

Since MA holds, so for every ξ the cardinality of H_ξ is continuum. We shall construct inductively a sequence $(x_{\xi,n})$ of the type $2^{\omega_0} \times \omega_0$:

$$x_{\eta,0} = \min_\xi \{r_\xi : r_\xi \in H_\eta - \{x_{\gamma,k} : \gamma < \eta\}\},$$
$$x_{\eta,n} = \min_\xi \{r_\xi : r_\xi \in H_\eta - \{x_{\gamma,k} : (\gamma < \eta \vee (\gamma = \eta \,\&\, k < n))\}\}.$$

Let us define sets K_n as follows:

$$K_n = \begin{cases} \{x_{\eta,n} : \eta < 2^{\omega_0}\} & \text{for } n > 0, \\ A - \bigcup\limits_{m > 0} K_m & \text{for } n = 0. \end{cases}$$

It is easy to show that sets K_n $(n = 1, 2, \ldots)$ are of the second category at every point $x \in A$. Suppose that the set B is of the second category at x and there exists a number $n \in N$ such that $K_n \cap B$ is of the first category at x. Then there exist I_n and $H_{n,\eta}$ such that $H_{n,\eta} \subseteq A - K_n \cap B$. This is impossible since the set $H_{n,\eta} \cap B \cap K_n$ is non empty.

In this step we shall construct a function $g\colon R \to R$ such that $Q(g) = B$, $T_q(g) = C$ $T_q^1(g) = C_1$, $S_q(g) = B \cup C$ and $S_q^1(g) = C_1 \cup B$. Let $(b_n)_{n\in N}$ be an enumeration of the set of all rational numbers from the interval $(-1, 1)$.

Let $(K_n)_{n\in N}$ be a partition of $R - E$ such that for every $x \in R - E$ the sets K_n $(n = 1, 2, \ldots)$ are of the second category at x and if the set $R - (A \cup A_1)$ is of the second category at x then $K_n - (A \cup A_1)$ is of the second category at x.

The function g is defined as follows:

$$g(x) = \begin{cases} 2 & \text{for} \quad x \in C\,, \\ -2 & \text{for} \quad x \in C_1\,, \\ h(x) & \text{for} \quad x \in E - (C \cup C_1)\,, \\ h(x) + \dfrac{\operatorname{dist}(x, E)}{1 + \operatorname{dist}(x, E)} \cdot b_n & \text{for} \quad x \in K_n - (C \cup C_1)\,. \end{cases}$$

a) It is clear that $C \subseteq T_q(g)$ and $C_1 \subseteq T_q^1(g)$.

b) If $x \in E - B$ then $g(x) = h(x)$ and $g \mid R - (C \cup C_1)$ is continuous at x. Since $C \cup C_1 \in \mathcal{I}$, g is qualitive continuous at x.

c) If $x \in (E - D) - (C \cup C_1)$ then $g(x) = h(x)$, $q\text{-}\liminf_{t\to x} g(t) = q\text{-}\liminf_{t\to x} h(t)$ and $q\text{-}\limsup_{t\to x} h(t) = q\text{-}\limsup_{t\to x} g(t)$. Since $x \in R - (S_q(h) \cup S_q{}^1(h))$, $x \in R - (S_q(g) \cup$ $\cup\, S_q^1(g))$.

In the next step we define a function $f\colon R \to R$ such that $Q(f) = B$, $S_q(f) = A$, $S_q^1(f) = A_1$, $T_q(f) = C$ and $T_q^1(f) = C_1$. Let us define the function f as follows:

$$f(x) = \begin{cases} q\text{-}\limsup_{t\to x} g(t) & \text{for} \quad x \in A - (B \cup C)\,, \\ q\text{-}\liminf_{t\to x} g(t) & \text{for} \quad x \in A_1 - (B \cup C_1)\,, \\ g(x) & \text{elsewhere}\,. \end{cases}$$

a) If $x \in C$ then $f(x) = g(x) > q\text{-}\limsup_{t\to x} g(t) \geqq q\text{-}\limsup_{t\to x} f(t)$ and $x \in T_q(f)$. Similarly, $C_1 \subseteq T_q^1(f)$.

b) Notice that for $x \in R - E$ we have

$$q\text{-}\limsup_{t\to x} g(t) = q\text{-}\limsup_{t\to x} h(t) + \frac{\operatorname{dist}(x, E)}{1 + \operatorname{dist}(x, E)} = $$

$$= h(x) + \frac{\operatorname{dist}(x, E)}{1 + \operatorname{dist}(x, E)}$$

and

$$q\text{-}\liminf_{t\to x} g(t) = q\text{-}\liminf_{t\to x} h(t) - \frac{\operatorname{dist}(x, E)}{1 + \operatorname{dist}(x, E)} = $$

$$= h(x) - \frac{\operatorname{dist}(x, E)}{1 + \operatorname{dist}(x, E)}\,.$$

Since for $x \in E$ the set

$$\left\{t \in R: |f(t) - h(t)| < \frac{\text{dist}(t, E)}{1 + \text{dist}(t, E)}\right\}$$

is residual at the point x, we have $q\text{-}\limsup_{t \to x} f(t) = q\text{-}\limsup_{t \to x} h(t)$ and $q\text{-}\liminf_{t \to x} f(t) = q\text{-}\liminf_{t \to x} h(t)$. Hence $C_q(f) \cap [E - (C \cup C_1)] = B$, $S_q(f) \cap [E - (C \cup C_1)] = A - C$ and $S_q^I(f) \cap [E - (C \cup C_1)] = A_1 - C_1$.

c) Assume that $x \in R - (E \cup C \cup C_1)$. The following cases may occur: The set $R - (A \cup A_1)$ is of the second category at x. Then for every $n \in N$ the set $K_n - (A \cup A_1)$ is of the second category at x,

$$q\text{-}\limsup_{t \to x} f(t) = h(x) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)} \quad \text{and} \quad q\text{-}\liminf_{t \to x} f(t) = h(x) - \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}.$$

There exists a neighbourhood $U \subseteq R - E$ of x such that $U - (A \cup A_1) \in \mathscr{I}$. Since the sets $A - B$ and $A_1 - B$ do not contain sets of the second category having the Baire property, hence the sets $A - B$ and $A_1 - B$ are of the second category at x. Then

$$q\text{-}\limsup_{t \to x} f(t) = \lim_{t \to x}\left(h(t) + \frac{\text{dist}(t, E)}{1 + \text{dist}(t, E)}\right) = h(x) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}$$

and

$$q\text{-}\liminf_{t \to x} f(t) = h(x) - \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}.$$

Thus, if $x \in A - C$ then $x \in S_q(f) - [Q(f) \cup T_q(f)]$, if $x \in A_1 - C_1$ then $x \in S_q^1(f) - [Q(f) \cup T_q^1(f)]$ and if $x \notin A \cup A_1$ then $x \notin S_q(f) \cup S_q^1(f)$.

Therefore f satisfies the condition (ii).

References

[1] *Z. Grande:* Quelques remarques sur la semicontinuité supérieure. Fund. Math. CXXV. 1 (1985), 1—13.

[2] *K. Kuratowski:* Topologie I. PWN, Waszawa 1958.

[3] *J. Shoenfield:* Martin's Axiom, Am. Math. Monthly 82 (1975), 610—617.

[4] *W. Sierpiński:* Sur une propriété des ensembles F_σ linéaires. Fund. Math. 14 (1929), 216—220.

Author's address: Department of Mathematics WSP Bydgoszcz, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland.