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AN EXISTENCE THEOREM FOR SEMILINEAR FUNCTIONAL PARABOLIC EQUATIONS

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1. INTRODUCTION

The semilinear evolution problem

$$(E) \qquad \dot{u}(t) + A\,u(t) = f(t, u)\,, \quad u(0) = u_0\,,$$

has a rather long history. It is well known that if $-A$ generates a C_0-semigroup (e^{-At}) and a function f is C^1 then (E) possesses a strong solution, i.e. a continuous solution on some interval $[0, T)$ and continuously differentiable on $(0, T)$. If more is supposed about an operator A, namely that A is a sectorial operator, then there exists a strong solution to (E) under weaker assumptions on f. Moreover, f can also depend on the gradient of u with respect to the space variables. Sufficient conditions on f can be expressed in terms of fractional powers of A. For example, the following theorem is proved in [11], p. 54.

Theorem. *Let A be a sectorial operator in a Banach space X, $0 \leqq \alpha < 1$, and let f map an open subset G of $R^+ \times \mathscr{D}(A^\alpha)$ into X and be locally Hölder continuous in t and locally Lipschitzian in x. Then for any $[0, u_0] \in G$ there exists a unique strong solution to (E) on some interval $(0, T)$.*

If we are not interested in the uniqueness of a solution we can prove the existence of a so called mild solution supposing the compactness of e^{-At}, $t > 0$, and only the continuity of $f: G \to X$. If f is locally Hölder continuous then the regularity of the mild solution follows. See e.g. [18] for $\alpha = 0$ and [19] for more general cases. Similar theorems were proved also for semilinear functional evolution equations, see e.g. [21], [24]. Such equations frequently occur in various biological applications.

A very simple example of $f(u) = 2Au$ shows that it is not sufficient to suppose $\alpha = 1$ in the above theorem. Nonetheless, nonlinearities depending on Au are very important e.g. for equations describing materials with memories (see Section 4 and references given there). Because of memories these equations become functional and existence and uniqueness theorems have recently been proved in this case also for $\alpha = 1$. See [9], [10], [22], [23].

In the present paper the existence theorem for the initial problem to the equation

$$\dot{u}(t) + A\,u(t) = f(t, u_t) + \int_{t_0}^{t} g(t, s, u(s))\,\mathrm{d}s + h(t)$$

is proved. Here u_t denotes the shift of a function u, i.e. $u_t(s) = u(t + s)$. The existence of a mild solution is proved with help of the fixed point theorem for a sum of compact and contractive operators, and then the regularity result is derived. This approach requires weaker conditions on smoothness of the right hand side, but does not guarantee uniqueness of the solution.

As an application, the nonlinear equation for the heat conduction in materials with memory is treated.

2. PRELIMINARIES

Throughout this paper we assume

(H 1) *A is a sectorial operator in a Banach space X,* i.e. A is a closed densely defined operator such that for some ϑ in the interval $(0, \pi/2)$, a real a and some $M \geqq 1$ the estimate of the resolvent operator

$$\|(I - A)^{-1}\| \leqq \frac{M}{|\lambda - a|}$$

holds for any λ in the sector $\{\lambda;\ \vartheta \leqq \arg(\lambda - a) \leqq \pi\}$. If A is sectorial then $-A$ generates an analytic C_0-semigroup which will be denoted by e^{-At}, $t \geqq 0$. We shall frequently use further basic facts about sectorial operators (for more details see e.g. [7], [11]), which are collected in the following two lemmas. First note that there is always a real number b such that the spectrum of $A_1 \equiv A + bI$ satisfies the condition

$$\inf\{\operatorname{Re}\lambda,\ \lambda \in \sigma(A_1)\} > 0\,. \tag{2,1}$$

For such sectorial operators all real powers A_1^α are defined and X^α will stand for their domains. Spaces X^α are Banach spaces if they are endowed with the norms $\|x\|_\alpha = \|A_1^\alpha x\|_X$. Note that the spaces X^α do not depend on the particular choice of the shift b. The first lemma is an easy modification of Theorem 1.4.3 in [11].

Lemma 1. *Let A be a sectorial operator. Then for any positive T we have*

(i) $(\forall\alpha \in [0, \infty)\ \exists K_1(\alpha)\ \forall x \in X\ \forall t)\ 0 < t \leqq T \Rightarrow \|\mathrm{e}^{-At}x\|_\alpha \leqq K_1(\alpha)\, t^{-\alpha}\|x\|$;

(ii) $(\forall\alpha \in (0, 1]\ \exists K_2(\alpha)\ \forall x \in X^\alpha\ \forall t)\ \ 0 < t \leqq T \Rightarrow \|(\mathrm{e}^{-At} - I)\, x\| \leqq K_2(\alpha)\, t^\alpha\|x\|_\alpha$.

In order to establish solvability of semilinear equations we shall need the following assumption which is often satisfied in practice.

(H 2) *There is λ such that $(\lambda I - A)^{-1}$ is a compact operator on X.*

The second lemma can be found in [11], pp. 27, 29.

Lemma 2. *Let A be a sectorial operator and let b be such that (2,1) holds. Then the following statements are equivalent.*

(i) *A satisfies* (H 2).

(ii) *e^{-At} is a compact operator on X for any $t > 0$.*

(iii) *A_1^α is a compact operator on X for any $\alpha < 0$.*

(iv) *For any $0 \leqq \beta < \alpha$ the natural embedding $X^\alpha \to X^\beta$ is compact.*

Now we start to study some special integral operators. The following two lemmas can also be viewed as sufficient conditions for solvability of the linear equation

$$\dot{x} + Ax = f . \tag{2,2}$$

It is well known that the mere continuity into X of a map $f: [t_0, T] \to X$ is not sufficient for the existence of a solution of (2,2).

Lemma 3. *Let* (H 1) *be satisfied and let f be a continuous map of $[t_0, T)$ into X^β for some $\beta > 0$. Then a function F given by*

$$F(t) = \int_{t_0}^{t} e^{-A(t-s)} f(s) \, ds , \quad t \in [t_0, T) , \tag{2,3}$$

has the following properties:

(i) *F is a continuous map of $[t_0, T)$ into X^1.*

(ii) *For any $t \in (t_0, T)$ the strong derivative (in the norm of X) $\dot{F}(t)$ exists and $\dot{F}(t) + A\,F(t) = f(t)$.*

Proof. (i) Because of commutativity of both a closed operator and an integral (see e.g. [6], Th. 3.6.20), it is sufficient to prove the integrability of the function $s \to \|A_1 e^{-A(t-s)} f(s)\|$ on $[t_0, t]$. Denote $\sup\{\|f(s)\|_\beta, s \in [t_0, t]\}$ by $M(t)$. Using Lemma 1, we get

$$\int_{t_0}^{t} \|A_1 e^{-A(t-s)} f(s)\| \, ds = \int_{t_0}^{t} \|e^{-A(t-s)} A_1^\beta f(s)\|_{1-\beta} \, ds =$$

$$= \beta^{-1} K_1(1 - \beta) \, M(t) \, (t - t_0)^\beta .$$

Moreover, for $t_0 \leqq t_1 < t_2 \leqq T_1 < T$,

$$\|F(t_2) - F(t_1)\|_1 \leqq \int_{t_0}^{t_1} \|(e^{-A(t_2-t_1)} - I)\, A_1^{1-\beta} e^{-A(t_1-s)} A_1^{\beta} f(s)\| \,ds +$$

$$+ \int_{t_1}^{t_2} \|e^{-A(t_2-s)} A_1^{\beta} f(s)\|_{1-\beta} \,ds \leqq$$

$$\leqq 2K_2(\beta/2)\,(t_2 - t_1)^{\beta/2}\, K_1(1 - \beta/2)\, M(T_1)\, \beta^{-1}(t_1 - t_0)^{\beta/2} +$$

$$+ \beta^{-1} K_1(1 - \beta)\, M(T_1)\,(t_2 - t_1)^{\beta},$$

which completes the proof of (i).

(ii) Fix a $t \in (t_0, T)$. For $\delta > 0$ we have

$$\frac{F(t + \delta) - F(t)}{\delta} = \delta^{-1} \int_{t_0}^{t} (e^{-A\delta} - I)\, e^{-A(t-s)} f(s) \,ds +$$

$$+ \delta^{-1} \int_{t}^{t+\delta} e^{-A(t+\delta-s)} f(s) \,ds = \frac{e^{-A\delta} - I}{\delta} F(t) +$$

$$+ \delta^{-1} \int_{t}^{t+\delta} e^{-A(t+\delta-s)} [f(s) - f(t)] \,ds +$$

$$+ \delta^{-1} \int_{t}^{t+\delta} (e^{-A(t+\delta-s)} - I) f(t) \,ds + f(t).$$

By the first part of this lemma. $F(t) \in X^1 = \mathscr{D}(A)$ and therefore

$$\lim_{\delta \to 0_+} \frac{e^{-A\delta} - I}{\delta} F(t) = -A\, F(t).$$

The X-norm of the first integral on the right hand side of the above equality can be made arbitrarily small by the continuity of the function f at the point t, and by the boundedness of a C_0-semigroup. Also the X-norm of the second integral tends to zero as $\|(e^{-A\delta} - I) f(t)\|$ tends to zero for $\delta \to 0_+$ (in virtue of the property of a C_0-semigroup). So we have proved that $\dot{F}_+(t)$ exists and $\dot{F}_+(t) + A\,F(t) = f(t)$. This right derivative $\dot{F}_+$ is continuous on (t_0, T) as AF and f are continuous (the embedding $X^\beta \to X$ is continuous whenever $\beta > 0$). The continuity of F and its right derivative implies the existence of $\dot{F}$ at any point of the interval (t_0, T) (see e.g. [12], IX. 1.7).

We shall also need a result similar to Lemma 3 for Hölder continuous functions. The following statement is basically Lemma 3.2.1 in [11].

Lemma 4. *Let* (H 1) *be satisfied and let f be a locally Hölder continuous function* *of* $[t_0, T)$ *into* X. *Let* F *be a function given by* (2,3). *Then the statements* (i), (ii) *of Lemma* 3 *are valid for* F *and, moreover,*

$$A\,F(t) = \int_{t_0}^{t} A\mathrm{e}^{-A(t-s)}[f(s) - f(t)]\,\mathrm{d}s + [I - \mathrm{e}^{-A(t-t_0)}]\,f(t)$$

for all $t \in [t_0, T)$.

3. EXISTENCE THEOREM

In this section we shall prove an existence theorem for the equation

$$(3,1) \qquad \dot{u}(t) + A\,u(t) = f(t, u_t) + \int_{t_0}^{t} g(t, s, u(s))\,\mathrm{d}s + h(t)$$

with an initial condition

$$(3,2) \qquad u_{t_0} = \varphi\,.$$

Here u_t denotes the shift of a function u, i.e. $u_t(s) = u(t + s)$ for $s \in (-\infty, 0]$.

Our proof will consist of two steps. First, we prove the existence of a so called mild solution to (3,1), (3,2), i.e. a solution of the integral equation

$$(3,3) \qquad u(t) = \mathrm{e}^{-A(t-t_0)}\,\varphi(0) + \int_{t_0}^{t} \mathrm{e}^{-A(t-s)}\,h(s)\,\mathrm{d}s + $$
$$+ \int_{t_0}^{t} \mathrm{e}^{-A(t-s)} f(s, u_s)\,\mathrm{d}s + \int_{t_0}^{t} \mathrm{e}^{-A(t-s)} \int_{t_0}^{s} g(s, \sigma, u(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}s$$

with the initial condition (3,2). Secondly, we shall prove the regularity of the mild solution.

For the existence of a mild solution it is sufficient to find an appropriate space and conditions on f, g, A under which the map given by the right hand side in (3,3) has a fixed point. Let $\alpha \in [0, 1]$, $\beta > 0$. Note that the most interesting case is $\alpha = 1$. We denote by $Y^\alpha(T)$ the Banach space of all bounded uniformly continuous maps of the interval $(-\infty, T]$ into the space X^α endowed with the norm

$$\|v\|_{Y^\alpha(T)} = \sup_{t \in (-\infty, T]} \|v(t)\|_\alpha\,.$$

The following assumptions will be introduced in the sequel.

(H 3) *There exists an open subset* U_1 *of* $[t_0, +\infty) \times Y^\alpha(0)$ *such that* f *is a continuous map of* U_1 *into* X^β.

(H 4) *There exists an open subset* U_2 *of* $\{[t, s] \in [t_0, +\infty)^2;\ s \leqq t\} \times X^\alpha$ *such that*

(a) *g is a continuous map of* U_2 into X.

(b) *g is locally Hölder continuous in the first variable and locally Lipschitzian in the third variable on* U_2, *i.e. for any* $[t_0, s_0, x_0] \in U_2$ *there are a neighborhood* V *of this point and positive numbers* C, γ *such that the inequality*

$$\|g(t_1, s, x_1) - g(t_2, s, x_2)\| \leqq C[|t_1 - t_2|^\gamma + \|x_1 - x_2\|_\alpha] \tag{3,4}$$

holds for all $[t_i, s, x_i] \in V \cap U_2$, $i = 1, 2$.

(H 5) $[t_0, \varphi] \in U_1$ *and* $[t_0, t_0, \varphi(0)] \in U_2$.

(H 6) *There exists* $\tau > t_0$ *such that* h *maps* $[t_0, \tau)$ *continuously into* X *and is locally Hölder continuous on* $[t_0, \tau)$.

Denote

$$\Psi(t) = \begin{cases} \varphi(t - t_0) & \text{for} \quad t \leqq t_0\,, \\ e^{-A(t-t_0)}\,\varphi(0) + \displaystyle\int_{t_0}^{t} e^{-A(t-s)}\, h(s)\,ds & \text{for} \quad t > t_0\,; \end{cases}$$

$$\Phi_1(u) \colon t \to \begin{cases} 0 & \text{for} \quad t \leqq t_0\,, \\ \displaystyle\int_{t_0}^{t} e^{-A(t-s)} f(s, u_s)\,ds & \text{for} \quad t > t_0\,; \end{cases}$$

$$\Phi_2(u) \colon t \to \begin{cases} 0 & \text{for} \quad t \leqq t_0\,, \\ \displaystyle\int_{t_0}^{t} e^{-A(t-s)} \Big[\int_{t_0}^{s} g(s, \sigma, u(\sigma))\,d\sigma\Big]\,ds & \text{for} \quad t > t_0\,, \end{cases}$$

$$\Phi(u) = \Phi_1(u) + \Phi_2(u) + \Psi\,.$$

By continuity, one can find $r > 0$, $T > t_0$ such that $\Psi \in Y^\alpha(T)$ and $\Phi_i(u)\,t$ exist and belong to X for $i = 1, 2$, $t \in (-\infty, T]$ and $u \in Z(r)$, where

$$Z(r) = \{u \in Y^\alpha(T),\ u_{t_0} = \varphi \text{ and } \|u(t) - \varphi(0)\|_\alpha \leqq r \text{ for } t \in (t_0, T]\}\,.$$

Lemma 5. *Assume* (H 1) – (H 3), (H 5). *Then* Φ_1 *is a compact continuous map of* $Z(r)$ *into* $Y^\alpha(T)$ *for all sufficiently small positive* r *and* T *near to* t_0.

Proof. As $[t_0, \varphi] \in U_1$ and f satisfies (H 3), there exists a neighborhood V_1 of $[t_0, \varphi]$ in U_1 on which f is bounded. Put $M_1 = \sup_{V_1} \|f(s, x)\|_\beta$ and choose $r > 0$, $T > t_0$ such that $[s, u_s] \in V_1$ for $t_0 \leqq s \leqq T$, $u \in Z(r)$. For these r, T we will prove the statement.

(i) $\{\Phi_1(u);\ u \in Z(r)\}$ forms an equicontinuous family in $Y^\alpha(T)$.

We always can choose $\delta > 0$ such that $c = \alpha - \beta + \delta < 1$. By Lemma 1, the following estimates hold for any $t_0 \leqq t_1 \leqq t_2 \leqq T$:

$$\|\Phi_1(u)\, t_2 - \Phi_1(u)\, t_1\|_\alpha \leqq$$

$$\leqq \int_{t_0}^{t_1} \|A_1^{\alpha-\beta}[\mathrm{e}^{-A(t_2-t_1)} - I]\, \mathrm{e}^{-A(t_1-s)} A_1^\beta f(s, u_s)\| \,\mathrm{d}s +$$

$$+ \int_{t_1}^{t_2} \|A_1^{\alpha-\beta} \mathrm{e}^{-A(t_2-s)} A_1^\beta f(s, u_s)\| \,\mathrm{d}s \leqq$$

$$\leqq K_2(\delta)\,(t_2 - t_1)^\delta K_1(c)\, M_1(1-c)^{-1}\,(t_1 - t_0)^{1-c} K_1(\alpha - \beta)\,.$$

$$.\,(1 - \alpha + \beta)^{-1} M_1(t_2 - t_1)^{1-(\alpha-\beta)} \leqq \text{const.}\,(t_2 - t_1)^\delta\,.$$

(ii) For any $t \in (-\infty, T]$ the set $\{\Phi_1(u)\, t;\ u \in Z(r)\}$ is relatively compact in X^α. This is obvious for $t \leqq t_0$, and if $t > t_0$ then it follows from the boundedness of this set in $X^{\alpha+\delta}$ by the assumption (H 2).

Using a generalization of the classical Arzela-Ascoli theorem for vector functions (see e.g. [13], Ch. 7) we conclude that Φ_1 is a compact map. The continuity of Φ_1 follows directly by the Lebesgue dominated convergence theorem.

By the assumption (H 5) and (H 4), there is a neighborhood V_2 of the point $[t_0, t_0, \varphi(0)]$ in U_2 such that g is bounded on V_2 (H 4a) and g satisfies (3,4) on V_2 (H 4b). Put $M_2 = \sup_{V_2} \|g(t, s, x)\|$ and $G(t, u) = \int_{t_0}^t g(t, s, u(s))\,\mathrm{d}s$. We choose $r > 0$, $T > t_0$ so small that $[t, s, u(s)] \in V_2$ for all $t \in [t_0, T]$, $s \in [t_0, t]$, $u \in Z(r)$. Thus for any $t_0 \leqq t_1 \leqq t_2 \leqq T$ and $u_1, u_2 \in Z(r)$ we have

$$\|G(t_1, u_1) - G(t_2, u_2)\| \leqq \int_{t_0}^{t_1} \|g(t_1, s, u_1(s)) - g(t_2, s, u_2(s))\| \,\mathrm{d}s + \tag{3.5}$$

$$+ \int_{t_1}^{t_2} \|g(t_2, s, u_2(s))\| \,\mathrm{d}s \leqq C(T - t_0)\,(t_2 - t_1)^\gamma +$$

$$+ C(T - t_0)\, \|u_1 - u_2\|_{Y^\alpha} + M_2(t_2 - t_1) \leqq$$

$$\leqq C_1(t_2 - t_1)^\gamma + C(T - t_0)\, \|u_1 - u_2\|_{Y^\alpha}\,.$$

This estimate and Lemma 4 imply that $\Phi_2(u) \in Y^1(T)$ for all $u \in Z(r)$ supposing r and T are small enough.

Lemma 6. *Suppose* (H 1), (H 2), (H 4) *and* (H 5). *Then there exist* $r > 0$, $T > t_0$ *such that the operator* Φ_2: $Z(r) \to Y^\alpha(T)$ *is the sum of a compact continuous operator and a contractive one.*

Proof. It is sufficient to prove this assertion for $A\Phi_2$: $Z(r) \to C([t_0, T], X) \equiv Y$. By Lemma 4, we can write

$$A\,\Phi_2(u)\, t = H_1(u)\, t + H_2(u)\, t\,,$$

where

$$H_1(u)\,t = \int_{t_0}^{t} A e^{-A(t-s)}[G(s,u) - G(t,u)]\,ds\,, \quad t \in [t_0, T]\,,$$

and

$$H_2(u)\,t = [I - e^{-A(t-t_0)}]\,G(t,u)\,, \quad t \in [t_0, T]\,.$$

As the inequality

$$\|H_2(u) - H_2(v)\|_Y \leqq 2 \sup_{t\in[0,T-t_0]} \|e^{-At}\|\, C(T - t_0)\, \|u - v\|_{Y^*}$$

holds for all $u, v \in Z(r)$, the map H_2 is a contraction for T sufficiently close to t_0.

We now prove that H_1 is a compact continuous operator from $Z(r)$ into Y. To this end we use the Arzela-Ascoli theorem as in Lemma 5.

(i) $B \equiv \{H_1(u)\,t;\ u \in Z(r)\}$ is a relatively compact set for any fixed $t \in [t_0, T]$. For $t = t_0$ the statement is obvious. Let therefore $t > t_0$. For sufficiently large natural n denote by $B_1^{(n)}$ the set of all elements

$$\int_{t_0}^{t-1/n} A e^{-A(t-s-(2n)^{-1})}[G(s,u) - G(t,u)]\,ds$$

where u goes through $Z(r)$, and by $B_2^{(n)}$ the set of

$$\int_{t-1/n}^{t} A e^{-A(t-s)}[G(s,u) - G(t,u)]\,ds$$

for the same values of u. As $B \subset e^{-A(2n)^{-1}}(B_1^{(n)}) + B_2^{(n)}$, we prove the compactness of B by showing that $B_1^{(n)}$ is a bounded set for any n ($e^{-A(2n)^{-1}}$ is a compact map by (H 2)) and

$$\lim_{n\to\infty} \mu(B_2^{(n)}) = 0 \tag{3,6}$$

for the Kuratowski measure of noncompactness μ (see e.g. [14], [17]). Lemma 1 and the estimate (3,5) immediately imply the boundedness of $B_1^{(n)}$ and the existence of a positive number K such that

$$\left\| \int_{t-1/n}^{t} A e^{-A(t-s)}[G(s,u) - G(t,u)]\,ds \right\| \leqq \frac{K}{n^\gamma}$$

holds. Therefore diam $(B_2^{(n)}) \leqq (2K/n^\gamma)$ and (3,6) is true.

(ii) $H_1(Z(r))$ is an equicontinuous subset of Y. For any $t_0 \leqq t_1 < t_2 \leqq T$, $u \in Z(r)$, we can write

$$H_1(u)\,t_2 - H_1(u)\,t_1 = \int_{t_1}^{t_2} A e^{-A(t_2-s)}[G(s,u) - G(t_2,u)]\,ds\, +$$

$$+ \int_{t_0}^{t_1} A(e^{-A(t_2-t_1)} - 1)\, e^{-A(t_1-s)}[G(s, u) - G(t_1, u)]\, ds +$$

$$+ \int_{t_0}^{t_1} A e^{-A(t_2-s)}[G(t_1, u) - G(t_2, u)]\, ds$$

and therefore we obtain

$$\|H_1(u)\, t_2 - H_1(u)\, t_1\| \leqq \gamma^{-1} K_1(1)\, C_1(t_2 - t_1)^\gamma +$$
$$+ 2/\gamma\, K_2(\gamma/2)\, K_1(1 + \gamma/2)\, C_1(t_2 - t_1)^{\gamma/2} +$$
$$+ K_1(1)\, C_1(t_2 - t_1)^\gamma \log \frac{t_2 - t_0}{t_2 - t_1} \leqq \text{const.}\ (t_2 - t_1)^{\gamma/2}\,.$$

It remains to prove the continuity of H_1. Suppose $u_n \to u$ in $Y^\alpha(T)$ and choose some $\delta > 0$ such that $t_0 < t_0 + \delta < T$. Then

$$\|H_1(u_n) - H_1(u)\|_Y =$$
$$= \max \Bigg\{ \sup_{t \in [t_0, t_0 + \delta]} \left\| \int_{t_0}^{t} A e^{-A(t-s)}[G(s, u_n) - G(t, u_n) + G(t, u) - G(s, u)]\, ds \right\|,$$
$$\sup_{t \in [t_0 + \delta, T]} \Bigg\| \int_{t_0}^{t-\delta} A e^{-A(t-s)}[G(s, u_n) - G(s, u) + G(t, u) - G(t, u_n)]\, ds +$$
$$+ \int_{t-\delta}^{t} A e^{-A(t-s)}[G(s, u_n) - G(t, u_n) + G(t, u) - G(s, u)]\, ds \Bigg\| \Bigg\} \leqq$$
$$\leqq 2\, K_1(1) \max \Bigg\{ \sup_{t \in [t_0, t_0 + \delta]} \gamma^{-1} C_1(t - t_0)^\gamma,\ \sup_{t \in [t_0 + \delta, T]} \Big[C(T - t_0)\, \|u_n - u\|_{Y^\alpha}$$
$$\log \frac{t - t_0}{\delta} + \gamma^{-1} C_1 \delta^\gamma \Big] \Bigg\}.$$

It is now clear that $H_1(u_n) \to H_1(u)$ in Y.

From the definitions of Ψ, Φ_1 and Φ_2 one can easily see that $r > 0$, $T > t_0$ may be chosen so that $\Phi(Z(r)) \subseteq Z(r)$ if the assumption (H 6) is satisfied. By Lemma 5 and 6, the operator Φ is the sum of a compact continuous and a contractive operator and therefore it is a k-set contraction (see e.g. [5], [17]) for $k < 1$. Using the Darbo modification of the Schauder Theorem ([5], [17]) we conclude that Φ has a fixed point in $Z(r)$, i.e. there exists a solution of (3,3) which is an element of $Y^\alpha(T)$.

Now we are coming to the second step of the proof, namely the proof of the regularity of this solution. By the construction, this solution u belongs to $Z(r)$ which implies

(i) $s \to f(s, u_s)$ is a continuous map of the interval $[t_0, T]$ into X^β,

and

(ii) $s \to h(s) + G(s, u)$ is a Hölder continuous map of the interval $[t_0, T]$ into X.

This means that Lemma 3 can be applied to the first map (only here it is substantial $\beta > 0$, i.e. the existence of a mild solution has been proved under a weaker assumption $\alpha - \beta < 1$) and Lemma 4 to the second map. This concludes the proof of the following existence theorem.

Theorem 1. *Let* (H 1)–(H 6) *be satisfied. Then there exist* $T > t_0$ *and a function* $u \in Y^\alpha(T)$ *such that*

(i) *for any* $t \in (t_0, T)$ *the strong derivative* $\dot{u}(t)$ (*in the space* X) *exists,*

(ii) u *satisfies* (3,1) *at all points of the interval* (t_0, T),

(iii) u *satisfies the initial condition* (3,2).

Remark 1. Let $u \in Y^\alpha(t_1)$ be a solution of (3,1), (3,2) on an interval $[t_0, t_1)$ with $h = 0$ and let $[t_1, u_{t_1}] \in U_1$, $[t_1, t_1, u(t_1)] \in U_2$. As the set $\{[t_1, s, u(s)];\ s \in [t_0, t_1]\}$ is a compact subset of U_2, the function

$$h(t) = \int_{t_0}^{t_1} g(t, s, u(s))\,\mathrm{d}s$$

is defined on some interval $[t_1, \tau)$ and is Hölder continuous on it – the last statement follows from (3,4). The existence theorem yields a solution $v \in Y^\alpha(t_2)$ of the equation

$$\dot{v}(t) + A\,v(t) = f(t, v_t) + \int_{t_1}^{t} g(t, s, v(s))\,\mathrm{d}s + h(t),$$

with the initial condition

$$v_{t_1} = u_{t_1}.$$

Putting $w(t) = u(t)$, $t \in (-\infty, t_1]$, $w(t) = v(t)$, $t \in (t_1, t_2)$, it can be checked that this function w belongs to the space $Y^\alpha(t_2)$ and solves the integral equation (3,3). By the regularity argument, it is a solution of the original equation (i.e. with $h = 0$) (3,1) with the initial condition (3,2). Therefore the Zorn maximality lemma yields a maximal, i.e. noncontinuable, solution of (3,1), (3,2). The forthcoming paper will be devoted to the interesting questions of continuous dependence of maximal solutions on φ, f, g.

Remark 2. The property (3,5) has played the crucial role in proving Theorem 1. It is an open question whether G can be replaced by a more general function G_1 which is Hölder continuous in the first variable and satisfies the Lipschitz condition in the second with a sufficiently small constant.

4. APPLICATION

Coleman, Gurtin [3] and Gurtin, Pipkin [8] proposed a theory of heat conduction based on thermodynamics for materials with memories. Now we briefly describe the basic features of this approach (for more details see e.g. [16]). Let q denote the heat flux and e the internal energy and suppose that these quantities depend on a thermal history of a material which is given by an ordered pair $[u_t, g_t]$, where g is the gradient of a temperature function u and the index t denotes the shift of u as above. If we restrict our consideration to thermal histories which are close to an equilibrium and assume the material to be isotropic, we can linearize the constitutive equations $q = q(u_t, g_t)$; $e = e(u_t, g_t)$. In such a way we obtain the equations

$$q(t) = -k(0)\, g(t) - \int_0^{+\infty} k'(s)\, g(t - s)\, \mathrm{d}s\,,$$

$$e(t) = e_0 + \alpha(0)\, u(t) + \int_0^{+\infty} \alpha'(s)\, u(t - s)\, \mathrm{d}s\,,$$

where $k(s)$ is called the heat conduction relaxation function and $\alpha(s)$ is called the energy temperature relaxation function. The quantities q and e obey the energy balance equation

$$\dot{e} = -\operatorname{div} q + r\,,$$

where r denotes the heat supply by the surroundings. The heat capacity $\alpha(0)$ must be positive and if we suppose that the instantaneous conductivity $k(0)$ is also positive we arrive at the linearized heat equation of parabolic type

$$\alpha(0) \frac{\partial u}{\partial t}(x, t) - k(0)\, \Delta u(x, t) = r(x, t) - \tag{4,1}$$

$$- \int_{-\infty}^{t} \alpha'(t - s) \frac{\partial u}{\partial t}(x, s)\, \mathrm{d}s + \int_{-\infty}^{t} k'(t - s)\, \Delta u(x, s)\, \mathrm{d}s\,.$$

This equation has been studied by several authors (see e.g. [15] and the references given there). The stability was also examined by Seifert [20] and the continuous dependence by Chen, Grimmer [1], [2].

A nonlinear version of (4,1) is more complicated. As far as the authors know the first result were obtained via methods of monotone operators by Crandall, Londen, Nohel [4] and by Webb [23] who used an approach similar to ours. As he used the contraction fixed point theorem, his assumptions are more restrictive and his equation is less general then the one given below.

As an example we shall treat the problem

$$\frac{\partial u}{\partial t}(t, x) - a \frac{\partial^2 u}{\partial x^2}(t, x) = F_0\left(u(t, x), \frac{\partial u}{\partial x}(t, x)\right) + \tag{4,2}$$

$$+ \int_{-\infty}^{t} k_1(s)\, F_1\left(u(s, x), \frac{\partial u}{\partial x}(s, x)\right) \mathrm{d}s +$$

$$+ \int_0^t k_2(t, s)\, G\left(\frac{\partial^2 u}{\partial x^2}(s, x)\right) \mathrm{d}s + h(t)(x),$$

$$t > 0, \quad x \in [0, \pi],$$

$$u(t, x) = \varphi(t)(x) \quad \text{for} \quad t \leqq 0, \quad x \in [0, \pi], \tag{4,3}$$

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, \pi) = 0 \quad \text{for} \quad t > 0. \tag{4,4}$$

We require $a > 0$ and impose the following conditions on the functions entering the equation: There is a positive T such that

(i) $k_1 \in L^1(-\infty, T)$,

(ii) k_2 is continuous on the interval $[0, T) \times [0, T)$ and is locally Hölder continuous in the first variable on this interval,

(iii) $F_0, F_1 \in C^1(R^2)$,

(iv) G is locally Lipschitz continuous on R and there are constants k_1, k_2 such that $|G(\xi)| \leqq k_1|\xi| + k_2$ for all real ξ,

(v) h maps the interval $[0, T)$ into $L^2(0, \pi)$ and is locally Hölder continuous on $[0, T)$,

(vi) φ maps the interval $(-\infty, 0]$ into X^1, where $X^1 = \{v \in L^2(0, \pi);\ v$ and v' are absolutely continuous on $[0, \pi]$, $v'(0) = v'(\pi) = 0$, and $v'' \in L^2(0, \pi)\}$. Moreover, φ is bounded and uniformly continuous on the interval $(-\infty, 0]$.

Theorem 2. *Let the assumptions* (i)–(vi) *be satisfied. Then there are* $t_0 > 0$ *and a function* u *on the interval* $(-\infty, t_0) \times [0, \pi]$ *such that*

(1) *the map* $t \to u(t, \cdot)$ *is continuous from* $[0, t_0)$ *into* X^1;

(2) *for each* $t \in (0, t_0)$ *the strong derivative* $(\mathrm{d}/\mathrm{d}t)\, u(t, \cdot)$ *exists and the equation* (4,2) *is satisfied in the sense of the space* $L^2(0, \pi)$.

(3) u *satisfies the initial condition* (4,3).

Proof. Put $A\, u(x) = -a\, \partial^2 u(x)/\partial x^2$ with $\mathscr{D}(A) = X^1$. This operator A is self-adjoint in $L^2(0, \pi)$ and its spectrum consists of $\{an^2\}_{n=0,1,\ldots}$. So A is sectorial and for $A_1 = A + I$ the condition (2,1) is satisfied. The A_1-graph norm $\|\ \|_1$ on X^1 is clearly equivalent to the norm of the Sobolev space $H^2(0, \pi)$. The operator A_1 has the representation

$$A_1\, v(x) = \sum_{n=0}^{\infty} (an^2 + 1)\, v_n \cos nx\,,$$

where

$$v_0 = \pi^{-1} \int_0^\pi v(x)\, \mathrm{d}x\,, \quad v_n = 2\pi^{-1} \int_0^\pi v(x) \cos nx\, \mathrm{d}x\,.$$

Hence

$$A_1^{1/2}\, v(x) = \sum_{n=0}^{\infty} (an^2 + 1)^{1/2}\, v_n \cos nx$$

with

$$X^{1/2} = \mathscr{D}(A_1^{1/2}) = \{v \in X;\ \sum_{n=0}^{\infty} (an^2 + 1)\, v_n^2 = \|v\|_{1/2}^2 < \infty\}\,.$$

It follows that the set $X^{1/2}$ cocincides with the Sobolev space $H^1(0, \pi)$ and the graph norm $\|\ \|_{1/2}$ is equivalent to the Sobolev norm. Therefore $\partial u/\partial x \in X^{1/2}$ for $u \in X^1$ and $\|\partial u/\partial x\|_{1/2} \leqq c\|u\|_1$. Here and in the sequel c denotes any constant (which can depend on z). Let us define

$$f_0(z)(x) = F_0\left(z(0, x), \frac{\partial z}{\partial x}(0, x)\right),$$

$$f_1(s, z)(x) = F_1\left(z(s, x), \frac{\partial z}{\partial x}(s, x)\right)$$

for $z \in Y^1(0)$. By the above argument and the assumption (iii), f_0 is a continuous map of $Y^1(0)$ into $X^{1/2}$. If $z \in Y^1(0)$ then $[z, \partial z/\partial x]$ is bounded on the interval $(-\infty, 0] \times [0, \pi]$. Using the assumption (iii) we obtain the following estimates for any $s_1, s_2 \in (-\infty, 0]$:

$$\|f_1(s_1, z) - f_1(s_2, z)\|_{1/2} \leqq c\left[\|f_1(s_1, z) - f_1(s_2, z)\|_X + \right.$$

$$\left. + \left\|\frac{\partial}{\partial x}[f_1(s_1, z) - f_1(s_2, z)]\right\|_X\right] \leqq c\|z(s_1, \cdot) - z(s_2, \cdot)\|_{C^1} +$$

$$+ c\left[\int_0^\pi \left|F'_{1,1}\left(z(s_1, x), \frac{\partial z}{\partial x}(s_1, x)\right) - F'_{1,1}\left(z(s_2, x), \frac{\partial z}{\partial x}(s_2, x)\right)\right|^2\right.$$

$$\left.\left|\frac{\partial z}{\partial x}(s_2, x)\right|^2 \mathrm{d}x\right]^{1/2} +$$

$$+ c\left[\int_0^\pi \left|F'_{1,1}\left(z(s_1, x), \frac{\partial z}{\partial x}(s_1, x)\right)\right|^2 \left|\frac{\partial z}{\partial x}(s_1, x) - \frac{\partial z}{\partial x}(s_2, x)\right|^2 \mathrm{d}x\right]^{1/2} +$$

$$+ \ldots \leqq c(1 + \|z(s_2, \cdot)\|_{1/2} + \|z(s_2, \cdot)\|_1)\| z(s_1, \cdot) - z(s_2, \cdot)\|_{C^1} +$$

$$+ c\|z(s_1, \cdot) - z(s_2, \cdot)\|_{1/2} + c\|z(s_1, \cdot) - z(s_2, \cdot)\|_1 \leqq c\|z(s_1, \cdot) - z(s_2, \cdot)\|_1\,.$$

It follows that f maps $Y^1(0)$ into $Y^{1/2}(0)$. In a similar way we also prove

$$\|f_1(s, z_1) - f_1(s, z_2)\|_{1/2} \leqq c\|z_1 - z_2\|_{Y^1(0)}\,,$$

which implies the continuity of f_1. Further: the map $s \to k_1(t+s) f_1(s, z)$: $: (-\infty, 0] \to X^{1/2}$ is strongly measurable and

$$\int_{-\infty}^{0} \|k_1(t+s) f_1(s, z)\|_{1/2} \, ds \leqq c \|z\|_{Y^1(0)} \|k_1\|_{L^1(-\infty, T)}$$

for $t \in [0, T)$, $z \in Y^1(0)$. Hence the operator

$$f_2(t, z) = \int_{-\infty}^{0} k_1(t+s) f_1(s, z) \, ds$$

maps $[0, T) \times Y^1(0)$ into $X^{1/2}$ and a simple argument yields that f_2 is also continuous. Therefore the map $f(t, z) = f_0(z) + f_2(t, z)$ satisfies (H 3) for $\beta = 1/2$. Let us denote

$$g(t, s, v)(x) = k_2(t, s) \, G(\partial^2 v / \partial x^2 (s, x)) .$$

The assumption (iv) guarantees that $G(u) \in X$ for $u \in X^1$ and consequently, $g(t, s, v) \in \in X$ for any $[t, s, v] \in [0, T)^2 \times X^1$. The verification of (H 4)–(H 6) is now an easy calculation and Theorem 2 follows.

It is possible to replace the Neumann conditions (4,4) by the Dirichlet ones

$$u(t, 0) = u(t, \pi) = 0 \quad \text{for} \quad t > 0 . \tag{4,5}$$

In this case we put

$$Bu = -a \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad u \in \mathscr{D}(B) = X^1 = H_0^1(0, \pi) \cap H^2(0, \pi) .$$

The operator B is sectorial and satisfies the condition (2,1). But $X^{1/2} \cong H_0^1(0, \pi)$ and thus instead of (iii) we need

(iiia) $F_0, F_1 \in C^1(R^2)$ and $F_0(0, \xi) = F_1(0, \xi) = 0$ for all real ξ.

Further, we suppose

(via) φ maps the interval $(-\infty, 0]$ into $X^1 = H_0^1(0, \pi) \cap H^2(0, \pi)$ and is bounded and uniformly continuous on this interval.

The same argument as in the proof of Theorem 2 yields the following result:

Theorem 2a. *Let the assumption* (i), (ii), (iiia), (iv), (v) *and* (via) *be satisfied. Then the conclusion of Theorem* 2 *holds.*

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