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CONVOLUTION OPERATORS FOR THE ONE-SIDED LAPLACE TRANSFORMATION

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NOTATION

Throughout the paper, we will refer to the classical spaces defined in [7]: $\mathscr{D}$ the space of test functions in R, its dual $\mathscr{D}'$, $\mathscr{E}$ the space of complex C^∞ functions defined in R, $\mathscr{S}$ the space of rapidly decreasing functions in R, its dual $\mathscr{S}'$, and O'_C the space of rapidly decreasing distributions. All the above spaces have their usual topologies. $L^2(R)$ is the Hilbert space of square integrable functions, N the set of all nonnegative integers. For $m \in N$, $D^m = \mathrm{d}^m/\mathrm{d}x^m$ is the distributional derivative of order m. The Fourier Transformation $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is based on the kernel $\mathrm{e}^{-\mathrm{i}xy}$. For $\gamma \in R$ we write $\mathrm{e}_\gamma(x) = \mathrm{e}^{\gamma x}$. τ_β is the translation operator: $\tau_\beta\, \varphi(x) = \varphi(x - \beta)$ for $\varphi \in \mathscr{D}$, and $\langle \tau_\beta f, \varphi \rangle = \langle f, \tau_{-\beta}\varphi \rangle$ for $f \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$. If $f \in \mathscr{D}'$ has support in $[\alpha, \infty)$ and $\mathrm{e}_{-\gamma} f \in \mathscr{S}'$ for some $\alpha, \gamma \in R$, then for $\sigma > \gamma$, $\mathscr{F}(\mathrm{e}_{-\sigma} f)$ is a function and $\sigma + \mathrm{i}\tau \mapsto F(\sigma + \mathrm{i}\tau) = \mathscr{F}(\mathrm{e}_{-\sigma} f)(\tau)$ is a holomorphic function on $\sigma > \gamma$ which is called the Laplace Transform on f and is denoted by $\mathscr{L}f(\sigma + \mathrm{i}\tau)$. From now on γ will be a positive number.

Definition 1. Let $\mathscr{L}^0_{0\gamma} = \{f \in \mathscr{D}' : \operatorname{supp} f \subset [0, \infty),\ \ \mathrm{e}_{-\gamma} f \in L^2(R)\}$. We write $\mathscr{L}^\alpha_{0\gamma} = \tau_\alpha \mathscr{L}^0_{0\gamma}$ for $\alpha \in R$ and $\mathscr{L}^\alpha_{p\gamma} = D^p \mathscr{L}^\alpha_{0\gamma}$ for $p \in N$.

Remark. $\mathscr{L}^0_{0\gamma}$ was denoted $L_{2\gamma}$ in [3]. The space $\mathscr{L}^\alpha_{p\gamma}$ is Hilbert with the inner product:

$$\langle D^p f, D^p g \rangle^\alpha_{p\gamma} = \langle f, g \rangle^\alpha_{0\gamma} = \int_R \mathrm{e}_{-2\gamma} f \bar{g}\, \mathrm{d}x.$$

For $p = 0$ the proof follows from the completeness of $L^2([\alpha, \infty))$. In the general case, notice that $D^p : \mathscr{L}^\alpha_{0\gamma} \to \mathscr{L}^\alpha_{p\gamma}$ is injective.

Definition 2. Let $p \in N$ and $\alpha \in R$. Then we define $H^\alpha_{p\gamma}$ to be the space of all holomorphic functions F on the set $\{\sigma + \mathrm{i}\tau \in C : \sigma > \gamma\}$ for which

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$$\sup_{\sigma>\gamma}\int_R \frac{e^{2\alpha\sigma}|F(\sigma+i\tau)|^2}{(\sigma^2+\tau^2)^p}\,d\tau < \infty\,.$$

Proposition 1. *For any $p \in N$ and $\alpha \in R$, the Laplace Transformation maps $\mathscr{L}^\alpha_{p\gamma}$ onto $H^\alpha_{p\gamma}$.*

Proof. For $\mathscr{L}^0_{0\gamma}$ the proof is in [3]. The general case follows from the formula

$$\mathscr{L}(\tau_\alpha D^p f)(u) = u^p e^{-\alpha u}\mathscr{L} f(u)\,, \quad f \in \mathscr{L}_{0\gamma}\,, \quad Ru > \gamma\,.$$

Lemma 1. *Let $\alpha \in R$ and $p \in N$. Then*

1) *For each $F \in H^\alpha_{p\gamma}$ the limit $\operatorname{Lim}_{\sigma\to\gamma+} e^{\alpha(\sigma+i\tau)}F(\sigma+i\tau)/(\sigma+i\tau)^p$ exists in the topology of $L^2(R)$ (with respect to the variable τ). We denote*

$$\operatorname{Lim}_{\sigma\to\gamma+}\frac{e^{\alpha(\sigma+i\tau)}\,F(\sigma+i\tau)}{(\sigma+i\tau)^p} = \frac{e^{\alpha(\gamma+i\tau)}\,F(\gamma+i\tau)}{(\gamma+i\tau)^p}\,.$$

2) *$H^\alpha_{p\gamma}$ is a Hilbert space with the inner product*

$$(F, G)^\alpha_{p\gamma} = \frac{e^{-2\gamma\alpha}}{2\pi}\int_R \frac{F(\gamma+i\tau)\,\overline{G(\gamma+i\tau)}\,d\tau}{(\gamma^2+\tau^2)^p}\,.$$

With this inner product the Laplace Transformation $\mathscr{L}$ is a unitary mapping of $\mathscr{L}^\alpha_{p\gamma}$ onto $H^\alpha_{p\gamma}$.

Proof. It is an immediate consequence of ([3], Lemma 4) and Proposition 1.

Remark. From Proposition 1 and Lemma 1 we easily see that $H^0_{p\gamma} \subset H^0_{p+1,\gamma}$. Hence $H^\alpha_{p\gamma} \subset H^\alpha_{p+1,\gamma}$ and $\mathscr{L}^\alpha_{p\gamma} \subset \mathscr{L}^\alpha_{p+1\gamma}$, where all the inclusions are continuous. We also have continuous inclusions $\mathscr{L}^\alpha_{p\gamma} \subset \mathscr{L}^\beta_{p\gamma}$ for $\beta \leqq \alpha$. For any $p \in N$ and $\alpha \in R$ we have $H^\alpha_{p\gamma} = u^p e^{-\alpha u} H^0_{0\gamma}$.

Definition 3. We define $\mathscr{L}_{p\gamma}$ and $H_{p\gamma}$ to be the *strict inductive limits*:

$$\mathscr{L}_{p\gamma} = \operatorname{ind\,lim}_{\alpha\to-\infty} \mathscr{L}^\alpha_{p\gamma}\,,$$

$$H_{p\gamma} = \operatorname{ind\,lim}_{\alpha\to-\infty} H^\alpha_{p\gamma}\,.$$

By Lemma 1 we have

Theorem 1. *The Laplace Transformation is a topological isomorphism of $\mathscr{L}_{p\gamma}$ onto $H_{p\gamma}$.*

Proposition 2. *Let $p \in N$. Then $\mathscr{D} \subset \mathscr{L}_{p\gamma} \subset \mathscr{D}'$, where the inclusions are continuous and $\mathscr{D}$ is dense in $\mathscr{L}_{p\gamma}$.*

Proof. For $p = 0$ it is evident. By the remark after Lemma 1 we have $\mathcal{D} \subset H_{p\gamma}$ continuously. It is clear from the case $p = 0$ that the inclusion $\mathcal{L}_{p\gamma} \subset \mathcal{D}'$ is continuous. Finally, if a sequence $\{\varphi_n\}_{n\in N} \subset \mathcal{D}$ converges to f in $\mathcal{L}_{0\gamma}$, then $\{D^p\varphi_n\}$ converges to D^pf in $\mathcal{L}_{p\gamma}$. So $\mathcal{D}$ is dense in $\mathcal{L}_{p\gamma}$.

Definition 3. Let $p, q \in N$, $p \leqq q$. Let C^{γ}_{pq} be the space of continuous linear operators $T: \mathcal{L}_{p\gamma} \to \mathcal{L}_{q\gamma}$ such that $T(\tau_\beta f) = \tau_\beta T(f)$ for any $\beta \in R, f \in \mathcal{L}_{p\gamma}$. The elements of C^{γ}_{pq} are called *convolution operators.*

For $T \in C^{\gamma}_{pq}$ and $m \in N$ we define $D^mT: \mathcal{L}_{p\gamma} \to \mathcal{L}_{q+mp}$ by $D^m T(f) = D^m(T(f))$. Evidently $D^mT \in C^{\gamma}_{pq+m}$. Conversely, if $T \in C^{\gamma}_{0q}$ then for any $f \in \mathcal{L}_{0\gamma}$, there exists a unique $S(f) \in \mathcal{L}_{0\gamma}$ such that $T(f) = D^q(S(f))$. Since the topology of $\mathcal{L}_{p\gamma}$ is copied from $\mathcal{L}_{0\gamma}$ through the oprator D^q we have that $S: \mathcal{L}_{0\gamma} \to \mathcal{L}_{0\gamma}$ is continuous. Further, S is linear and $\tau_\beta S = S\tau_\beta$ for any $\beta \in R$, so we have the following lemma:

Lemma 2. *Let $q \in N$. Then for any $T \in C^{\gamma}_{0q}$ there exists a unique $S \in C^{\gamma}_{00}$ such that $D^qS = T$.*

Lemma 3. *Given $T \in C^{\gamma}_{pq}$ and $\alpha \in R$ there exists $\beta \in R$ such that $T(\mathcal{L}^{\alpha}_{p\gamma}) \subset \mathcal{L}^{\beta}_{q\gamma}$.*

Proof. Suppose the opposite, then there is a sequence $\{f_n\}_{n\in N}$ in $\mathcal{L}^{\alpha}_{p\gamma}$ such that $\operatorname{supp} T(f_n) \cap (-\infty, -n) \neq \emptyset$. Let $g_n = (n\|f_n\|^{\alpha}_{p\gamma})^{-1} f_n$. Then $\lim_{n\to\infty} g_n = 0$ in $\mathcal{L}^{\alpha}_{p\gamma}$ and $\lim_{n\to\infty} T(g_n) = 0$ in $\mathcal{L}_{q\gamma}$ which is impossible since $\mathcal{L}_{q\gamma}$ is a strict inductive limit so there is $\beta \in R$ such that $\lim_{n\to\infty} T(g_n) = 0$ in $\mathcal{L}^{\beta}_{0\gamma}$ which contradicts that $\operatorname{supp} T(g_n) \cap \cap (-\infty, -n) \neq \emptyset$.

Lemma 4. *Let $T \in C^{\gamma}_{pq}$ $p, q \in N$, $p \leqq q$. Then $T(\mathcal{D}) \subset \mathcal{E}$ and $T: \mathcal{D} \to \mathcal{E}$ is continuous.*

Proof. 1) Let $T \in C^{\gamma}_{00}$ and $\varphi \in \mathcal{D}$. Take $\alpha \in R$ such that $\operatorname{supp} \varphi \cup \operatorname{supp} T(\varphi) \cup \cup \operatorname{supp} T(D\varphi) \subset [\alpha, \infty)$.

Consider the function

$$g(x) = \int_{-\infty}^{x} T(D\varphi)\,dy = \begin{cases} \int_{\alpha}^{x} T(D\varphi)\,dy & \alpha < x \\ 0 & \text{otherwise.} \end{cases}$$

Since $e_{-\gamma} T(D\varphi) \in L^2(R)$, $T(D\varphi)$ is locally integrable, and $g(x)$ is well defined. Further, g is absolutely continuous and $Dg = T(D\varphi)$ (the distributional derivative). On the other hand $\lim_{n\to\infty} n[\tau_{-1/n}\varphi - \varphi] = D\varphi$ in $\mathcal{D}$ and hence in $\mathcal{L}_{0\gamma}$. Thus $\lim_{n\to\infty} n[\tau_{-1/n} T(\varphi)] - T(\varphi)] = T(D\varphi)$. Since $\mathcal{L}_{0\gamma}$ is a strict inductive limit, there exists $d \in R$ such that the last limit exists in $\mathcal{L}^{d}_{0\gamma}$. We can assume without loss of generality that $\alpha = d$. Thus

$$e_{-\gamma}[n(\tau_{-1/n} T(\varphi) - T(\varphi))] = e_{-\gamma} T(D\varphi) \quad \text{in} \quad L^2(R).$$

It follows that

$$g(x) = \int_{-\infty}^{x} T(D\varphi)\,dy = \lim_{n\to\infty} \int_{-\infty}^{x} n[\tau_{-1/n} T(\varphi) - T(\varphi)] =$$

$$= \lim_{n\to\infty} n \int_{x}^{x+1/n} T(\varphi)(y)\,dy = T(\varphi)(x) \quad \text{almost everywhere}.$$

We deduce that $T(\varphi)$ is continuous (it can be represented by a continuous function), similarly $T(D\varphi)$, so $g(x)$ is everywhere differentiable, and its classical derivative satisfies

$$D\,g(x) = D(T(\varphi))(x) = T(D\varphi)(x).$$

Inductively we prove that $T(\varphi) \in \mathscr{E}$, and the classical derivative $D^n T(\varphi)(x)$ equals $T(D^n\varphi)(x)$ for any $x \in R$.

2) Now we prove that $T: \mathscr{D} \to \mathscr{E}$ is continuous: Let $K \subset R$ be a compact set. Let $\alpha \in R$ such that $K \subset [\alpha, \infty)$. By Lemma 3, $T(\mathscr{L}^{\alpha}_{0\gamma}) \subset \mathscr{L}^{\beta}_{0\gamma}$ for some $\beta \in R$. So there exists $C > 0$ such that $\int_{\beta}^{\infty} |e_{-\gamma} T(f)|^2\,dy \leqq C \int_{R} |e_{-\gamma} f|^2\,dy$ for any $f \in \mathscr{L}^{\alpha}_{0\gamma}$.

Hence if $M \subset R$ is a compact set and $\varphi \in \mathscr{D}_K = \{\psi \in \mathscr{D} : \operatorname{supp} \psi \in K\}$ then

$$\sup_{x\in M} |T(\varphi)(x)| \leqq \sup_{x\in M} \int_{\beta}^{x} |T(D\varphi)|\,dy \leqq \sup_{x\in M} \left(\int_{\beta}^{x} |e_{\gamma}|^2\,dy\right)^{1/2} \left(\int_{\beta}^{x} |e_{-\gamma} T(D\varphi)|^2\,dy\right)^{1/2} \leqq$$

$$\leqq C' \left(\int_{R} |e_{-\gamma} D\varphi|^2\,dy\right)^{1/2} \leqq C'' \sup_{x\in K} |D\,\varphi(x)|.$$

Thus $T: \mathscr{D}_K \to \mathscr{E}$ is continuous and the lemma is proved for C^{γ}_{00}.

3) For $T \in C^{\gamma}_{0q}$ we have that $T = D^q S$ where $S \in C^{\gamma}_{00}$, so we can apply 1 and 2 to S. Finally, since $\mathscr{L}_{0\gamma} \subset \mathscr{L}_{p\gamma}$ continuously we have $C^{\gamma}_{pq} \subset C^{\gamma}_{0q}$ and the proof is complete.

Let $T \in C^{\gamma}_{pq}$. By the previous lemma $T: \mathscr{D} \to \mathscr{E}$ is a continuous linear operator commuting with translations. Then there exists a unique distribution u_T satisfying $T(\varphi) = u_T * \varphi$ for any $\varphi \in \mathscr{D}$ (see [6] p. 158). Further, if $m \in N$, $D^m T(\varphi) = D^m(T(\varphi)) = D^m(u_T * \varphi) = (D^m u_T) * \varphi$. So $u_{D^m T} = D^m u_T$.

Let us show now that $\operatorname{supp} u_T \subset [\alpha, \infty)$ for some $\alpha \in R$. It is sufficient to do so for $T \in C^{\gamma}_{00}$. Let $\{\varphi_n\}_{n\in N} \subset \mathscr{D}$ with $\operatorname{supp} \varphi_n \subset [-1, 1]$ and $\lim_{n\to\infty} \varphi_n = \delta$ in the topology of $\mathscr{D}'$, where δ is the Dirac distribution. Then $\{\varphi_n\}_{n\in N} \subset \mathscr{L}^{-1}_{0\gamma}$ and there is $\alpha \in R$ such that $\operatorname{supp} T(\varphi_n) \subset [\alpha, \infty)$ for any $n \in N$. If $\psi \in \mathscr{D}$ and $\operatorname{supp} \psi \subset (-\infty, \alpha)$ then

$$\langle u_T, \psi \rangle = \lim_{n\to\infty} \langle u_T * \varphi_n, \psi \rangle = \lim_{n\to\infty} \langle T(\varphi_n), \psi \rangle = 0$$

which proves that $\operatorname{supp} u_T \subset [\alpha, \infty)$. Since any $T \in C^\gamma_{pq}$ is a derivative of some $S \in C^\gamma_{00}$, the proof is complete. Given $p, q \in N$, $p \leqq q$ and $\alpha \in R$, let $O^\alpha_{pq\gamma}$ be the space of continuous linear operators $T: \mathscr{L}^0_{p\gamma} \to \mathscr{L}^\alpha_{q\gamma}$. $O^\alpha_{pq\gamma}$ is a Banach space with the usual norm

$$\|T\|^\alpha_{pq\gamma} = \sup_{\|f\|^0_{pp} \leq 1} \|T(f)\|^\alpha_{q\gamma} .$$

The spaces $O^\alpha_{pq\gamma}$ have the following properties:

1) $O^\alpha_{pq\gamma} \subset O^\beta_{pq\gamma}$ for $\alpha \leqq \beta$. The inclusion is continuous.

2) $O^\alpha_{pq\gamma} \subset O^\alpha_{0q\gamma}$ continuously.

3) $D^m: O^\alpha_{pq\gamma} \to O^\alpha_{pq+m\gamma}$ is an isometry.

4) If $T \in O^\alpha_{pq\gamma}$ then the composition $T \circ D^p \in O^\alpha_{0q\gamma}$ and the mapping $T \mapsto T \circ D^p$ is an isometry of $O^\alpha_{pq\gamma}$ onto $O^\alpha_{0q\gamma}$.

Now we define the strict inductive limit

$$O_{pq\gamma} = \operatorname{ind} \lim_{\beta \to -\infty} O^\beta_{pq\gamma} .$$

By Lemma 3, $C^\gamma_{pq} \subset O_{pq\gamma}$, thus we can equip C^γ_{pq} with the topology of $O_{pq\gamma}$.

Lemma 5. *If $T \in O^\beta_{00\gamma} \cap C^\gamma_{00}$ and $f \in \mathscr{L}^\alpha_{0\gamma}$, then*

$$T(f) \in \mathscr{L}^{\alpha+\beta}_{0\gamma} \quad \textit{and} \quad \|T(f)\|^{\alpha+\beta}_{0\gamma} \leqq \|T\|^\beta_{00\gamma} \|f\|^\alpha_{0\gamma} .$$

Proof. $T(f) = \tau_\alpha T(\tau_{-\alpha} f)$, so

$$\left(\int_R |e_{-\gamma} T(f)|^2 \, dx\right)^{1/2} = e^{-\alpha\gamma}\left(\int_R |e_{-\gamma} T(\tau_{-\alpha} f)|^2 \, dy\right)^{1/2} \leqq$$

$$\leqq e^{-\alpha\gamma} \|T\|^\beta_{0\gamma} \left(\int_R |e_{-\gamma} T(\tau_{-\alpha} f)|^2 \, dy\right)^{1/2} = \|T\|^\beta_{00\gamma} \|f\|^\alpha_{0\gamma} .$$

Theorem 2. *Given $p, q \in N$, $p \leqq q$, the mapping*

$$(T, f) \mapsto T(f) : C^\gamma_{pq} \times \mathscr{L}_{p\gamma} \to \mathscr{L}_{q\gamma} \quad \textit{is hypocontinuous}.$$

Proof. For $q = 0$.

1) Let B be a bounded set in $\mathscr{L}_{0\gamma}$. Then B is bounded in some $\mathscr{L}^\alpha_{0\gamma}$, so $\|f\|^\alpha_{0\gamma} \leqq M$ for any $f \in B$ and some $M > 0$. Let V be a neighbourhood of zero in $\mathscr{L}_{0\gamma}$. For each $\delta \in R$ let ε_δ such that $\{f \in \mathscr{L}^\delta_{0\gamma} : \|f\|^\delta_{0\gamma} < \varepsilon_\delta\} \subset V \cap \mathscr{L}^\delta_{0\gamma}$. Let $\omega_\beta = \{T \in O^\beta_{00\gamma} \cap C^\gamma_{00} : \|T\|^\beta_{00\gamma} \leqq \varepsilon_{\alpha+\beta}/M\}$. Then if $T \in \omega_\beta$ and $f \in B$, we have by Lemma 5 that $\|T(f)\|^{\alpha+\beta}_{0\gamma} \leqq \varepsilon_{\alpha+\beta}$, hence $T(f) \in V$.

2) Let $B \subset C^\gamma_{00}$ be a bounded set, then $B \subset O^\beta_{00\gamma}$ for some $\beta \in R$ and is bounded there. Let V and $\{\varepsilon_\delta\}_{\delta \in R}$ be as in 1. Let $M > 0$ such that $\|T\|^\beta_{00\gamma} \leqq M$ for any $T \in B$. Then if $\omega_\alpha = \{f \in \mathscr{L}^\alpha_{0\gamma} : \|f\|^\alpha_{0\gamma} \leqq \varepsilon_{\alpha+\beta}/M\}$ we have $\|T(f)\|^{\alpha+\beta}_{0\gamma} \leqq \varepsilon_{\alpha+\beta}$, so $T(f) \in V$. The proof for the case $q = 0$ is now complete. The general case is a consequence of 1, 2 and properties 3 and 4 of $O^\alpha_{pq\gamma}$.

Theorem 3. *$\mathscr{D} \subset C^{\gamma}_{pq} \subset \mathscr{D}'$ and each inclusion is continuous.*

Proof. Let $\varphi \in \mathscr{D}$ and $f \in \mathscr{L}_{0\gamma}$. It follows from the inequality $\|e_{-\gamma}\varphi * e_{-\gamma}f\|_{L^2(R)} \leqq \leqq \|e_{-\gamma}\varphi\|_{L^1(R)} \|e_{-\gamma}f\|_{L^2(R)}$ that the inclusion $\mathscr{D} \subset C^{\gamma}_{00}$ is continuous. Let us prove the continuity of $C^{\gamma}_{00} \cap O^{\beta}_{00\gamma} \subset \mathscr{D}'$. Take a sequence $\{T_n\}_{n\in N}$ converging to zero in $C^{\gamma}_{00} \cap O^{\beta}_{00\gamma}$ and $B \subset \mathscr{D}$ bounded. Then there exists a compact set $K \subset R$ and $\alpha \in R$ such that $B \subset \mathscr{D}_K \subset \mathscr{L}^{\alpha}_{0\gamma}$. Further, we can assume that for any $\varphi \in B$ the function $\check{\varphi}(x) = \varphi(-x)$ is also in $\mathscr{D}_K$. Clearly $\{D\check{\varphi} : \varphi \in B\}$ is a bounded set in $\mathscr{L}^{\alpha}_{0\gamma}$. Let $M > 0$ such that $\|D\check{\varphi}\|^{\alpha}_{0\gamma} \leqq M$ for any $\varphi \in B$. The given $\varphi \in B$ we have

$$|\langle u_T, \varphi\rangle| = |u_{T_n} * \check{\varphi}(0)| \leqq \int_{-\infty}^{0} |u_{T_n} * D\check{\varphi}| \, dy = \int_{\alpha+\beta}^{0} |u_{T_n} * D\check{\varphi}| \, dx \leqq$$

$$\leqq \left(\int_{\alpha+\beta}^{0} |e_{\gamma}|^2 d\, y\right)^{1/2} \left(\int_{\alpha+\beta}^{0} |e_{-\gamma}(u_{T_n} * D\check{\varphi})|^2 \, dy\right)^{1/2} \leqq C(\alpha, \beta) M \|T\|^{\alpha}_{00\gamma} .$$

So $C^{\gamma}_{00} \subset \mathscr{D}'$ is continuous. The rest of the proof is a simple application of properties 2 and 3 of $O^{\alpha}_{pq\gamma}$.

For the following results it is helpful to recall that if $f, g \in \mathscr{D}'$ have supports in $[\alpha, \infty)$ for some $\alpha \in R$, then for any $\sigma \in R$, $e_{-\sigma}(f * g) = e_{-\sigma}f * e_{-\sigma}g$. In particular we have $(e_{-\sigma}u_T) * \varphi \in L^2(R)$ for any $T \in C^{\gamma}_{00}$, $\varphi \in \mathscr{D}$ and $\sigma \geqq \gamma$.

Lemma 6. *Given $0 \leqq p \leqq q$ integers and $T \in C^{\gamma}_{pq}$, then $e_{-\sigma}u_T \in O'_C$ for any $\sigma > \gamma$.*

Proof. Let $T \in C^{\gamma}_{00}$, for any $\sigma > \gamma$ and $\varphi \in \mathscr{D}$, the function $(e_{-\sigma}u_T) * \varphi$ is integrable. Hence $(e_{-\sigma}u_T) * \varphi(x) = \int_{-\infty}^{x} e_{-\sigma}u_T * D\,\varphi(g)\, dy$ is a bounded function. This implies that any regularization $(e_{-\sigma}u_T) * \varphi$ is rapidly decreasing at infinity. It follows that $e_{-\sigma}u_T \in O'_C$. We have to show now that $e_{-\sigma}D^m u_T \in O'_C$ for any $m \in N$, $\sigma > \gamma$ and $T \in \in C^{\gamma}_{00}$. For $m = 1$ we have $e_{-\gamma}Du_T = D(e_{-\sigma}u_T) - \sigma e_{-\sigma}u_T$ which is an element of O'_C. We complete the proof by induction.

Proposition 3. *Let $p, q \in N$, $T \in C^{\gamma}_{pq}$, $f \in \mathscr{L}_{p\gamma}$. Then $T(f) = u_T * f$.*

Proof. It is easily seen that the mapping $f \mapsto e_{-\sigma}f : \mathscr{L}_{p\gamma} \to \mathscr{S}'$ is continuous for each $p \in N$ and $\sigma \geqq \gamma$. Let $f \in \mathscr{L}_{p\gamma}$ and $\{\varphi_n\}_{n\in N}$ converging to f in $\mathscr{L}_{p\gamma}$. Since $e_{-\sigma}u_T \in \in O'_C$ for $\sigma > \gamma$, we have in $\mathscr{S}'$

$$e_{-\sigma}\, T(f) = e_{-\sigma}\, T(\lim_{n\to\infty} \varphi_n) = \lim_{n\to\infty} e_{-\sigma}\, T(\varphi_n) = \lim e_{-\sigma}(u_T * \varphi_n) =$$

$$= \lim_{n\to\infty} e_{-\sigma}u_T * e_{-\sigma}\varphi_n = e_{-\sigma}u_T * e_{-\sigma}f .$$

Hence $T(f) = u_T * f$.

Theorem 4. *Given $p, q \in N$, $p \leqq q$, $f \in \mathscr{L}_{p\gamma}$ and $T \in C^{\gamma}_{pq}$, then $\mathscr{L}\, T(f) = \mathscr{L}u_T \,.\, \mathscr{L}f$.*

Proof. By Proposition 3,

$$\mathscr{F}(\mathrm{e}_{-\sigma}\, T(f)) = \mathscr{F}(\mathrm{e}_{-\sigma}u_T * \mathrm{e}_{-\sigma}f) = \mathscr{F}(\mathrm{e}_{-\sigma}u_T)\, \mathscr{F}(\mathrm{e}_{-\sigma}f)$$

or

$$\mathscr{L}(T(f))\,(\sigma + \mathrm{i}\tau) = \mathscr{L}u_T(\sigma + \mathrm{i}\tau)\, \mathscr{L}f(\sigma + \mathrm{i}\tau)$$

for $\sigma > \gamma$.

Definition 4. Let $p, q \in N$, $p \leqq q$. We denote by M^{γ}_{pq} the space of all functions G holomorphic on $\operatorname{Re} z > \gamma$ such that the mapping $F \mapsto GF$: $H_{p\gamma} \to H_{q\gamma}$ is well defined and continuous.

We notice that $\mathscr{L}(C^{\gamma}_{pq}) \subset M^{\gamma}_{pq}$. Conversely if $G \in M^{\gamma}_{pq}$ it is easy to see that the mapping $T(f) = \mathscr{L}^{-1}(G\, \mathscr{L}(f))$ is in C^{γ}_{pq} and $\mathscr{L}u_T = G$. Thus $\mathscr{L}(C^{\gamma}_{pq}) = M^{\gamma}_{pq}$. From the properties of C^{γ}_{0q} we conclude that M^{γ}_{0q} is the space of holomorphic functions $G = u^q \mathrm{e}^{-\alpha u} G'$ where $\alpha \in R$, $G' \in M^{\gamma}_{00}$ and $F \mapsto G'F : H^0_{0\gamma} \to H^0_{0\gamma}$ is continuous. On the other hand, it is not difficult to prove that $M^{\gamma}_{pq} = u^{q-p} M^{\gamma}_{00}$. So it is enough to know M^{γ}_{00} to characterize M^{γ}_{pq} and then C^{γ}_{pq}.

Theorem 5. *A holomorphic function G in $\operatorname{Re} u > \gamma$ is in M^{γ}_{pq} if and only if $G = u^{q-p}\mathrm{e}^{-\alpha u} G'$ for some $\alpha \in R$ and G' is a bounded holomorphic function on $\operatorname{Re} u > \gamma$.*

Proof. If G' is holomorphic and bounded on $\operatorname{Re} u > \gamma$ we clearly have that $G' \in M^{\gamma}_{00}$, moreover G' as a linear operator maps $H^0_{0\gamma}$ into itself. Conversely let G' be a function with these properties. For some $C \in R$, $C^{-1}G'$ has a norm 1 (as an automorphism of $H^0_{0\gamma}$). Since $f(u) = u^{-1}$ is a function in $H^0_{0\gamma}$ we have

$$\|C^{-1}G'u^{-1}\|^0_{0\gamma} \leqq \|u^{-1}\|^0_{0\gamma}\,.$$

We deduce that $\|(C^{-1}\, G')^n\;\; u^{-1}\|^0_{0\gamma} \leqq \|u^{-1}\|^0_{0\gamma}$ for any $n \in N$. We claim that $|C^{-1}\, G'(u)| \leqq 1$ for $\operatorname{Re} u > \gamma$. Suppose the opposite so $|C^{-1}\, G'(\sigma_0 + \mathrm{i}\tau_0)| > M > 1$ for some $\sigma_0 > \gamma$, τ_0, $M \in R$. Then for some $\varepsilon > 0$, $|G'(\sigma_0 + \mathrm{i}\tau)| > M$ whenever $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$.

Hence

$$\|(C^{-1}G')^n\, u^{-1}\|^0_{0\gamma} \geqq \left(\int_R \frac{|C^{-1}\, G'(\sigma_0 + \mathrm{i}\tau)|\, 2^n}{\sigma_0^2 + \tau^2}\, \mathrm{d}\tau\right)^{1/2} \geqq M^n \int_{\sigma_0-\varepsilon}^{\sigma_0+\varepsilon} \frac{\mathrm{d}\tau}{\sigma_0^2 + \tau^2}\,,$$

which is a contradiction. Thus $|G'(\sigma + \mathrm{i}\tau)| < C$ for $\sigma > \gamma$. The rest of the proof follows from the comment after Definition 4.

Remark. We have considered all the distributions f with $\operatorname{supp} f \subset [\alpha, \infty)$ for some $\alpha \in R$ and $\mathrm{e}_{-\gamma}f \in \mathscr{S}'$ where $\gamma > 0$ is a real number. Actually for such f there exists a continuous slowly increasing function g such that $\mathrm{e}_{-\gamma}f = D^p g$. Let $l \in \mathscr{E}$

be 1 in a neighborhood of $[\alpha, \infty)$ and supp $l \subset [\beta, \infty)$ where $\beta < \alpha$. It is easy to prove that

$$f = \sum_{n=1}^{p} D^n(e_\gamma l_n g)$$

where each l_n is a linear combination of derivatives of l. Thus $f \in \mathscr{L}_{p\delta}$ for any $\delta > \gamma$.

We define $\mathscr{L}_\gamma = \operatorname{ind} \lim_{p \to \infty} \mathscr{L}_{p\gamma}$, and notice that $\mathscr{L}_\gamma = \operatorname{ind} \lim_{p \to \infty} \mathscr{L}_{p\gamma}^{-p}$.

Theorem 6. *The largest space of distributions T for which the convolution $f \mapsto \mapsto T * f: \mathscr{L}_\gamma \to \mathscr{L}_\gamma$ is continuous, is precisely $C^\gamma = \bigcap_{p \in N} C_p^\gamma$, where $C_p^\gamma = \bigcup_{q \geq p} C_{pq}^\gamma$.*

Proof. Let $T: \mathscr{L}_\gamma \to \mathscr{L}_\gamma$ be a continuous linear operator such that $\tau_\beta T = T\tau_\beta$ for every $\beta \in R$. Given $p \in N$, the operator $T: \mathscr{L}_{p\gamma}^{-p} \to \mathscr{L}_\gamma$ is continuous and linear. If we denote B_p the unit ball in $\mathscr{L}_{p\gamma}^{-p}$, we have that $T(B_p)$ is a bounded set in $\mathscr{L}_\gamma$. Hence there exists $q \in N$ such that $T(B_p)$ is a bounded set in $\mathscr{L}_{q\gamma}^{-q}$. (See 5). Thus T: $\mathscr{L}_{p\gamma}^{-p} \to \mathscr{L}_{q\gamma}^{-q}$ is continuous and that proves that $T \in C_{pq}^\gamma$. It follows that $T \in C^\gamma$. That every element of C^γ defines a convolution operator for $\mathscr{L}_\gamma$ is trivial.

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